Probability of Escape of Electrons across the Surface of Photosensitive Material

PETR BECKMANN
Electrical Engineering Department, University of Colorado, Boulder, Colorado
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The paper derives the probability of escape from a half-space of photosensitive material for an electron at a given distance from the surface, assuming that after being energized it can only experience either elastic scattering or absorption. From this, the electron yield is calculated for both the two- and three-dimensional problems. The cases when absorption is much more probable than collision (and vice versa) are investigated in detail.

I. INTRODUCTION

IN the study of photoemission, the following model applies in certain cases. A photosensitive medium extends over the half-space $x<0$ (Fig. 1). The light impinging on it parallel to the x axis penetrates into the medium with intensity

$$\phi=\phi_0, \quad \text{for} \quad x\geq 0; \qquad =\phi_0 e^{-\alpha|x|}, \quad \text{for} \quad x\leq 0. \tag{1}$$

We assume that starting at a distance h from the surface $x=0$, the electron may experience one of three possibilities: (a) it may reach the surface $x=0$ in a straight line without collisions, and hence escape, or it may collide with atoms of the medium, in which case it may be either (b) absorbed or (c) elastically scattered without loss of energy, with all directions equally probable. In the latter case the three possibilities prevail, so that the process may continue as indicated in Fig. 2.

The basic problem to be solved is the calculation of the probability that an electron energized at a distance h from the surface will eventually escape into the half-space $x>0$. Once this problem is solved, additional points, such as the presence of a potential barrier at the surface, are taken care of comparatively easily and the electron yield, the probability distribution of the escape angle and other quantities of interest may be obtained as a function of ϕ_0 and α in (1). The problem will first be solved for the two-dimensional case, where the electron's random walk is restricted to the plane $z=0$; the three-dimensional case does not introduce difficulties of principle, though the numerical evaluation becomes more involved.

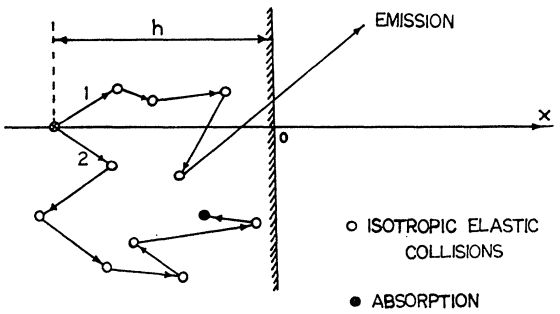

FIG. 1. Electron starts at $x=-h$. Shown are a possible path leading to emission (1) and a possible path leading to absorption (2).

It is theoretically possible to solve the problem by working through the process indicated in Fig. 2; however, this procedure leads to very involved expressions rapidly increasing in complexity with each step, since the probability distribution of the distance of the electron from the surface, and hence also the three probabilities, change after each step. We evade this difficulty by first solving an easier problem and then correcting the solution for the original problem.

II. HOMOGENEOUS MEDIUM

We first consider a situation which differs from the original problem, but will serve as a stepping stone to its solution. We let the entire space be filled by the same medium (so that all electrons must eventually be absorbed) and ask for the probability that an electron originally energized at $x=-h$ will be absorbed in the half-space $x>0$; this differs from the original problem in that the electron may be absorbed in the space $x<0$ after having "escaped" to the space $x>0$, but being scattered back into the space $x<0$, whereas in the original problem the escape across the surface $x=0$ is an irreversible journey.

Let the probability of an electron being elastically scattered on an element dr of its path be $k_1 dr$ and the probability of being absorbed $k_2 dr$; thus the probability of a collision (either elastic or absorbing) on an element of path dr is $k dr$, where

$$k=k_1+k_2. \tag{2}$$

If, as we assume, this probability is the same at all points of the path, then the length r of the path from one collision to the next has an exponential probability density

$$p(r)=ke^{-kr} \quad (r\geq 0). \tag{3}$$

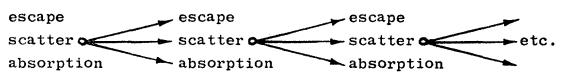

FIG. 2. Each step takes place at a different distance from the surface and hence the probabilities of the three possibilities are different for each step.

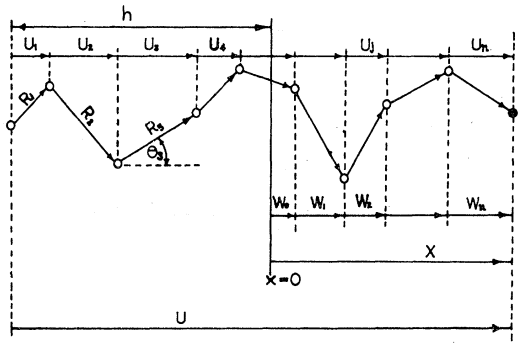

FIG. 3. For clarity of the figure, all U_j have been drawn positive; in reality they are equally likely to be positive or negative.

It is evident from (3) that $1/k$ is the mean free path between collisions; from (2) it follows that $1/k_1$ would be the mean free path between collisions if no absorptions occurred, and $1/k_2$ is the mean free path to absorption (a broken line with segments corresponding to the path between elastic collisions). The parameters k_1 and k_2 are all that we will require to describe the medium in which collisions occur.

If a collision occurs, it may be either an elastic one or it may be an absorption. The respective probabilities of these two events are obviously

$$p_1=k_1/k, \quad \text{and} \quad p_2=k_2/k. \tag{4}$$

In the present modification of the problem every electron must eventually be absorbed. Since the absorbing collision must be the last in a series of n collisions, the probability of $(n-1)$ elastic collisions before the final absorbing one, i.e., the probability of a total of n collisions is

$$P(n)=p_1^{n-1}p_2=(k_1/k)^{n-1}(k_2/k) \quad (n=1, 2, \cdots). \tag{5}$$

Consider now the random variable X, the x coordinate of the point of absorption of the electron starting at $x=-h$, or more conveniently (Fig. 3), the random variable

$$U=h+X, \tag{6}$$

i.e., the difference of the x coordinates of the absorption and starting points of the electron. We are looking for the probability that the electron will be absorbed in the space $x>0$, i.e.,

$$P(X>0)=P(U>h). \tag{7}$$

Let U_j be the difference in x coordinates between the jth and $(j-1)$th collision (Fig. 3); then the shift of the electron along the x axis to absorption after n collisions (of which $n-1$ are elastic) is

$$S_n=\sum_{j=1}^{n} U_j. \tag{8}$$

In the following we use capitals for random variables (U), lower case letters for the values they assume (u) and $p(\ )$ for the probability density of the corresponding variable, e.g., $p(u)$. [Thus $p(u)$ and $p(x)$ are two different functions.]

Then by the theorem of total probability, the density of U is

$$p(u)=\sum_{n=1}^{\infty} P(n)p(s_n), \tag{9}$$

where $P(n)$ is given by (5). Thus, if the density $p(u_j)$ is known, $p(s_n)$ may [by (8)] be obtained by an n-fold convolution of $p(u_j)$ with itself (since the U_j are independent and identically distributed); relation (9) then yields $p(u)$ and the problem is solved through

$$P(U>h)=\int_h^\infty p(u)du=\sum_{n=1}^{\infty} P(n)\int_h^\infty p(s_n)ds_n. \tag{10}$$

This is an outline of the method of solution and we start on its detailed execution by finding $p(u_j)$. Since, from Fig. 3, $U_j=R_j\cos\theta_j$, with θ_j uniform from 0 to 2π, we have

$$U_j=R_jC_j,$$

where $p(r_j)$ is given by (3) and the density of $C_j=\cos\theta_j$ is

$$p(c_j)=1/\pi(1-c_j^2)^{1/2} \qquad (-1\leq c_j\leq 1). \tag{11}$$

The density of U_j is then found as that of the product of the two independent random variables R_j and C_j:

$$p(u_j)=\frac{k}{\pi}\int_{u_j}^{\infty}\frac{e^{-kr_j}dr_j}{(r_j^2-u_j^2)^{1/2}}.$$

Substituting $r_j=|u_j|\cosh t$, this yields

$$p(u_j)=\frac{k}{\pi}\int_0^\infty e^{-k|u_j|\cosh t}dt=(k/\pi)K_0(k|u_j|) \qquad (-\infty<u_j<\infty), \tag{12}$$

where K_0 is the modified Bessel function of the second kind (Neumann's function of imaginary argument) of order zero, for which tables are available.[1] We shall call (12) the "basic" distribution for the two-dimensional case which we are now treating; for the three-dimensional case this distribution will be different, though the rest of the procedure remains the same.

The density of S_n in (8) might now be found by an n-fold convolution of (12). To bypass this difficulty, we find the characteristic function of (12),

$$\chi_j(v)=\int_{-\infty}^{\infty}p(u_j)e^{ivu_j}du_j=\frac{2k}{\pi}\int_0^\infty K_0(ku_j)\cos vu_j du_j =k/(k^2+v^2)^{1/2}, \tag{13}$$

[1] *Handbook of Mathematical Functions*, edited by M. Abramowitz and I. A. Steyun (U. S. Department of Commerce, National Bureau of Standards, Washington, D. C., 1965), pp. 388–389, bibliography.

whence by (8), the characteristic function of S_n is

$$(v)=k^n/(k^2+v^2)^{n/2} \tag{14}$$

and therefore the density of S_n is, by the inverse Fourier transform of (14)[2],

$$p(s_n)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{k^n\cos s_n v}{(k^2+v^2)^{n/2}}dv$$

$$=\frac{k^{2\nu}}{2(\nu-1)!}\left|\frac{d^{\nu-1}}{d(k^2)^{\nu-1}}\left(\frac{e^{-k|s_n|}}{k}\right)\right| \quad \text{for} \quad n=2\nu; \tag{15}$$

$$=\frac{k^{\nu+1}|s_n{}^{\nu}|K_{\nu}(k|s_n|)}{(\sqrt{\pi})2^{\nu}(\nu-\tfrac{1}{2})!} \quad \text{for} \quad n=2\nu+1,$$

where ν is an integer. Substituting (15) in (10) we have

$$P(U>h)=\frac{k_2}{k}\sum_{\nu=0}^{\infty}\left(\frac{k_1}{k}\right)^{\nu}\frac{k^{\nu+1}}{(\sqrt{\pi})2^{\nu}(\nu-\frac{1}{2})!}\int_h^{\infty}s^{\nu}K_{\nu}(ks)ds$$

$$+\frac{k_2}{k}\sum_{\nu=1}^{\infty}\left(\frac{k_1}{k}\right)^{2\nu-1}\frac{k^{2\nu}}{2(\nu-1)!}\left|\frac{d^{\nu-1}}{d(k^2)^{\nu-1}}\frac{e^{-kh}}{k^2}\right|. \tag{16}$$

Let

$$B_{\nu}(x)=\int_x^{\infty}t^{\nu}K_{\nu}(t)dt. \tag{17}$$

This function is not tabulated (except for $\nu=0$, when it reduces to Ki_1), but can be reduced to a sum of tabulated functions K_ν and Ki_1 by repeated integration by parts and application of the recurrence formulas for Bessel functions; however, it may be more advantageous to compute (17) directly by numerical integration. Then (16) may, after a little manipulation, be expressed as

$$P(U>h)=\frac{k_2}{k}\sum_{\nu=0}^{\infty}\left(\frac{k_1}{k}\right)^{2\nu}\frac{B_{\nu}(kh)}{(\sqrt{\pi})2^{\nu}(\nu-\frac{1}{2})!}$$

$$+k_2\sum_{\nu=1}^{\infty}\frac{k_1{}^{2\nu-1}}{2(\nu-1)!}\left|\frac{d^{\nu-1}}{d(k^2)^{\nu-1}}\frac{e^{-kh}}{k^2}\right|, \tag{18}$$

or, in expanded form,

$$P(U>h)=\frac{k_2}{k\pi}\mathrm{Ki}_1(kh)+\frac{k_1k_2}{2k^2}e^{-kh}+\frac{k_1{}^2k_2}{k^3\pi}B_1(kh)$$

$$+\frac{k_1{}^3k_2}{2k^4}(1+kh/2)e^{-kh}+\frac{k_1{}^4k_2}{3k^5\pi}B_2(kh)+\cdots \tag{19}$$

and this is the solution of our intermediate problem.

[2] I. S. Gradshteyn and I. M. Ryzhik, *Tables of Integrals, Series and Products* (Academic Press Inc., New York, 1965), 4th ed., Nos. 3.773.5 and 3.773.6.

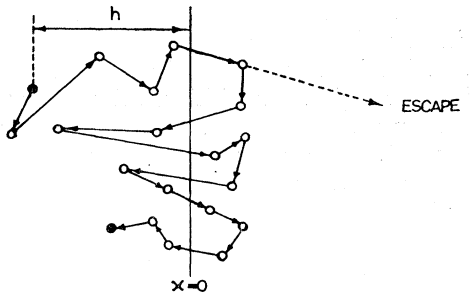

FIG. 4. In the original problem, the electron escapes as soon as it reaches the boundary $x=0$ (dotted arrow); in the "intermediate" problem, it may cross the boundary $x=0$ any number of times before being absorbed.

III. ESCAPE ACROSS BOUNDARY

The intermediate problem of the preceding section differs from the original problem owing to possible "excursions" of the electron across the boundary $x=0$ and back into the half-space $x<0$ (Fig. 4). In the original problem such excursions are impossible, since there are no collisions in the half-space $x>0$ and the electron will, once it has crossed the boundary $x=0$, escape, as shown by the dashed arrow in Fig. 4. Thus the probability $P(U>h)$ for the intermediate problem, as given by (18) or (19), is smaller than the probability $P(0)$ of the electron reaching the boundary $x=0$ and hence escaping, since in the former case the number of absorptions in the left half-space is in part due to "returning" electrons which in the latter case become escaping electrons. We will solve the original problem by correcting the intermediate problem for the fraction of "returning" electrons.

Let "0" stand for the event that the electron will cross the boundary $x=0$, and let "+" stand for the event that it will be absorbed in the half-space $x>0$ under the conditions of the preceding section (intermediate problem). Then from Bayes' theorem

$$P(0)=P(0|+)P(+)/P(+|0). \tag{20}$$

Now $P(0|+)=1$, since the electron cannot be absorbed in the right half-space unless it crossed the boundary $x=0$. Hence

$$P(0)=P(+)/P(+|0)=P(U>h)/P(+|0). \tag{21}$$

But $P(+)=P(U>h)$ is given by (18) or (19); hence we only have to find $P(+|0)$ to solve the problem completely. The required quantity $P(+|0)$ is the probability that an electron entering the space $x>0$ under the conditions of the preceding section will be absorbed in it (and will not return to $x<0$), which, as is evident from Fig. 3, is equivalent to the probability that the sum of the W_j will be positive. This is a problem quite similar to the one solved in the preceding section: The W_j have, for $j\neq 0$, the same distribution as the U_j, i.e., the density (12), and the distribution of the number

n of collisions is again given by (5) (though this distribution now includes the value $n=0$); the essential difference lies in the presence of the random variable W_0, since the angle under which the electron enters the right half-space is now distributed uniformly from $-\pi/2$ to $\pi/2$, i.e., over a range of π, and not 2π. The path length corresponding to W_0 is exponential, as in (3), for it can be shown that if the path length (from one collision to the next) is exponential, then the remaining path length (from any point to the next collision) must be exponential with the same parameters also. Proceeding as in (9) through (12), we now find

$$p(w_0)=(2k/\pi)K_0(kw_0) \quad (w_0\geq 0). \tag{22}$$

Let

$$\sum_{j=1}^{n} W_j = T_n. \tag{23}$$

Since the W_j are for $j\neq 0$ distributed identically with the U_j, the distribution of T_n is given by (15) with t_n substituted for s_n. The distribution of

$$L_n = W_0 + T_n \tag{24}$$

is then given by the convolution

$$p(l_n)=\int_0^\infty p(w_0)p_{T_n}(l_n-w_0)dw_0. \tag{25}$$

If W is the distance from $x=0$ to the point of absorption, we therefore have

$$p(w)=\sum_{n=0}^{\infty} P(n)p(l_n), \tag{26}$$

where n is now the number of elastic collisions before absorption, so that

$$P(n)=(k_2/k)(k_1/k)^n. \tag{27}$$

Thus, in analogy to (10),

$$P(+\,|\,0)=P(W>0)=\int_0^\infty p(w)dw$$

$$=\frac{k_2}{k}\sum_{u=0}^{\infty}\left(\frac{k_1}{k}\right)^n\int_0^\infty p(l_n)dl_n. \tag{28}$$

Substituting for $p(l_n)$ from (25) and interchanging the integrations over l_n and w_0, we have

$$P(+\,|\,0)=\frac{k_2}{k}\sum_{n=0}^{\infty}\left(\frac{k_1}{k}\right)^n\int_0^\infty p(w_0)$$

$$\times\left[\int_0^\infty p_{T_n}(l_n-w_0)dl_n\right]dw_0. \tag{29}$$

From (15) we find after some manipulation the integral[3]

$$\int_0^\infty p_{T_n}(l_n-w_0)dl_n$$

$$=\frac{k^{2\nu}}{2(\nu-1)!}\left|\frac{d^{\nu-1}}{d(k^2)^{\nu-1}}\frac{2-e^{-kw_0}}{k^2}\right| \quad \text{for} \quad n=2\nu$$

$$=\tfrac{1}{2}+\frac{1}{(\sqrt{\pi})2^\nu(\nu-\tfrac{1}{2})!}\int_0^{w_0k} t^\nu K_\nu(t)dt \quad \text{for} \quad n=2\nu+1. \tag{30}$$

Substituting (30) in (29) and performing the integration, we obtain

$$P(+\,|\,0)=\frac{k_2}{k}+\left(1-\frac{1}{\pi}\right)k_2\sum_{\nu=1}^{\infty}\left(\frac{k_1}{k}\right)^{2\nu}\frac{k^{2\nu}}{(\nu-1)!}$$

$$\times\left|\frac{d^{\nu-1}}{d(k^2)^{\nu-1}}\frac{1}{k^3}\right|+\frac{k_2}{2k}\sum_{\nu=0}^{\infty}\left(\frac{k_1}{k}\right)^{2\nu+1}$$

$$+\frac{2k_2}{k\pi^{3/2}}\sum_{\nu=0}^{\infty}\left(\frac{k_1}{k}\right)^{2\nu+1}\frac{1}{2^\nu(\nu-\frac{1}{2})!}$$

$$\times\int_0^\infty dw_0\int_0^{w_0k} t^\nu K_\nu(t)K_0(w_0k)dt. \tag{31}$$

The second infinite series is a geometric progression; in the first we substitute

$$\left|\frac{d^{\nu-1}}{d(k^2)^{\nu-1}}\frac{1}{k^3}\right|=\frac{(2\nu-1)!}{2^{2\nu-2}k^{2\nu+1}(\nu-1)!}.$$

On substituting this in the first series in (31), simplifying, setting $k_1^2/k_2^2=q$, and comparing the resulting series with the one for $q(1-q)^{-3/2}$, it will be found that the series can be summed and yields

$$\sum_{\nu=1}^{\infty}\left(\frac{k_1}{k}\right)^{2\nu}\frac{k^{2\nu}}{(\nu-1)!}\left|\frac{d^{\nu-1}}{d(k^2)^{\nu-1}}\frac{1}{k^3}\right|=\frac{kk_1^2}{k_2^{3/2}(k+k_1)^{3/2}}. \tag{32}$$

The double integral in (31) may be simplified as follows. Setting $w_0k=z$, we have

$$\int_0^\infty dz\int_0^z K_0(z)K_\nu(t)t^\nu dt=\int_0^\infty dt\int_t^\infty K_0(z)K_\nu(t)t^\nu dz$$

$$=\int_0^\infty t^\nu K_\nu(t)\,\mathrm{Ki}_1(t)dt, \tag{33}$$

[3] No. 6.561.16 of Ref. 2.

where both $K_\nu(t)$ and

$$\mathrm{Ki}_1(t)=\int_t^\infty K_0(u)du \tag{34}$$

are tabulated functions.[4]

Substituting (32) and (33) in (31) and summing the resulting geometric series we finally obtain

$$P(+|0)=\frac{k_2}{k}+\left(1-\frac{1}{\pi}\right)\frac{k_1^2}{(\sqrt{k})(k+k_1)^{3/2}}+\frac{k_1}{2(k+k_1)}$$
$$+\frac{k_2}{\pi^{3/2}k}\sum_{\nu=0}^{\infty}\left(\frac{k_1}{k}\right)^{2\nu+1}\frac{1}{2^{\nu-1}(\nu-\frac{1}{2})!}$$
$$\times\int_0^\infty t^\nu K_\nu(t)\,\mathrm{Ki}_1(t)dt \tag{35}$$

and this expression substituted in (21) together with (18) for $P(U>h)$ gives the general solution $P(0)$.

IV. SPECIAL CASES

By first considering a homogeneous medium and then correcting for the fact that no collisions occur in the half-space $x>0$, a more complicated form of the solution (resulting, for example, from the procedure suggested by Fig. 2) has been avoided; nevertheless, the solution given by (21) after substituting (18) and (35) is far from simple and in general calls for a computer to be numerically evaluated.

However, there are some special cases for which the general solution strongly simplifies.

A. Absorption Much More Likely Than Elastic Collision

If $k_1\ll k_2$, so that $k\to k_2$, $k_1\to 0$, then from (35) we find $P(+|0)=1$ (as was to be expected) and hence, from (21) and (19),

$$P(0)=(1/\pi)\,\mathrm{Ki}_1(k_2h)\quad(k_1\ll k_2), \tag{36}$$

which attains a maximum value of 0.5 for $h=0$ (as is also obvious from physical considerations). Figure 5 shows a plot of the relation (36).

The electron yield may be computed as follows. Assuming the number of electrons energized per second proportional to (1), the emission current made up of electrons originally energized at a distance h from the surface is

$$dI=I_0P(0)e^{-\alpha h}dh, \tag{37}$$

where I_0 is a constant of proportionality. Hence the

[4] Cf. bibliography, pp. 490–491 of Ref. 1.

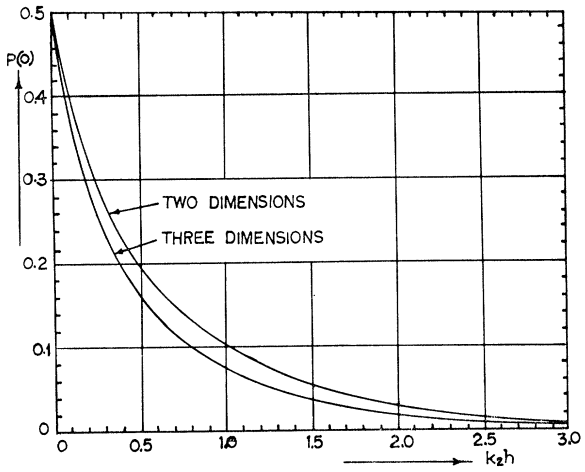

FIG. 5. The probability $P(0)$ that an electron energized at a distance h from the surface will escape from a medium with mean free path to absorption equal to k_2^{-1}, assuming a probability of collision negligible compared with probability of absorption. No energy barrier at the surface is considered. The two-dimensional case is given by (36) and three-dimensional case by (54).

total emission current across the surface $x=0$ is

$$I=I_0\int_0^\infty P(0)e^{-\alpha h}dh=\frac{I_0}{\pi}\int_0^\infty \mathrm{Ki}_1(k_2h)e^{-\alpha h}dh. \tag{38}$$

Substituting for Ki_1 from (34), interchanging the order of integration, setting

$$\beta=\alpha/k_2, \tag{39}$$

and using integral No. 6.611.9 of Gradshteyn and Ryzhik,[2] we find

$$I=[(I_0/\alpha)\tfrac{1}{2}-(1/\pi)F(\beta)]\quad(k_1\ll k_2), \tag{40}$$

where

$$F(\beta)=\frac{\arccos\beta}{(1-\beta^2)^{1/2}}\qquad\text{for}\quad\beta<1$$
$$=1\qquad\text{for}\quad\beta=1 \tag{41}$$
$$=\frac{\arg\cosh(\beta^2-1)^{1/2}}{(\beta^2-1)^{1/2}}\qquad\text{for}\quad\beta>1.$$

For $\alpha\to 0$ we similarly find from (38)

$$\lim_{\alpha\to 0}I=I_0/k_2\pi. \tag{42}$$

Curves of (40) are shown in Fig. 6.

B. Elastic Collision Much More Likely Than Absorption

In this case we have $k_2\ll k_1$, $k\to k_1$, and (35) yields

$$P(+|0)=\tfrac{1}{4}+(\pi-1)/2\pi\approx 0.59\quad(k_2\to 0). \tag{43}$$

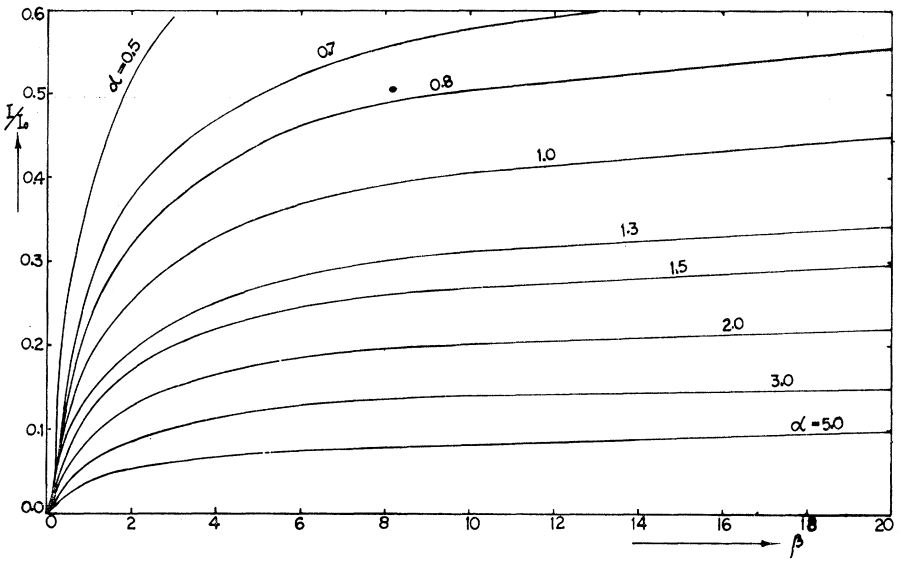

FIG. 6. Electron yield as a function of α and β.

However, substituting $k_2=0$ in (18) would yield $P(U>h)=0$ (as it should, since this gives the probability that the electron will be absorbed in the half-space $x>0$, but for $k_2=0$ there will be no absorptions anywhere). However, we may attack the problem from another angle. For $k_2\to 0$, there will be very many elastic collisions before an absorption, so that the mean n will be large and we may apply the central limit theorem to (8), since all the U_j are independent and identically distributed with density (12); the mean of U_j is $\langle U_j\rangle=0$ and its variance is

$$D(U_j)=\frac{k}{\pi}\int_{-\infty}^{\infty}u_jK_0(k|u_j|)du_j=\frac{1}{k^2}. \tag{44}$$

Hence for large n, the distribution of S_n in (8) is

$$p(s_n)=\frac{k_1}{(2\pi n)^{1/2}}\exp(-k_1^2s_n^2/2n)\quad(k\to k_1). \tag{45}$$

From (27) the mean n is

$$\langle n\rangle=\sum_{n=1}^{\infty}nP(n)=\frac{k_2}{k}\sum_{n=1}^{\infty}n\left(\frac{k_1}{k}\right)^{n-1}=\frac{k}{k_2}. \tag{46}$$

As a somewhat coarse approximation, we replace (9) by (45) with $n=\langle n\rangle$. This is a good approximation if for large n we have $(\langle n^2\rangle)^{1/2}\to\langle n\rangle$. Unfortunately, this is not the case here, since it may be shown that for $k_2\to 0$ we have $\langle n^2\rangle\to\langle n\rangle\sqrt{2}$ and we use the procedure only for lack of a better method to obtain an orientational formula.

Following this procedure, we have from (45)

$$P(S_n>h)=\int_n^{\infty}p(s_n)ds_n=\tfrac{1}{2}\operatorname{erfc}\frac{kh}{(2n)^{1/2}}$$
$$=\tfrac{1}{2}\operatorname{erfc}[h(k_1k_2/2)^{1/2}] \tag{47}$$

whence from (43) and (21) the probability of escape is

$$P(0)\approx 0.848\operatorname{erfc}[h(k_1k_2/2)^{1/2}], \tag{48}$$

which should be regarded as a formula of only orientational value.

V. THREE-DIMENSIONAL MODEL

If the energized electron is permitted to perform its random walk in three dimensions (i.e., also in the z direction), the above procedure is equally applicable; the essential difference lies in the basic distribution of the U_j, which will differ from (12).

By considering a sphere with center at $u=0$ with all points on its surface equally likely to be reached by the electron before collision, we again find $U_j=R_jC_j=R_j\cos\theta_j$, but C_j this time is distributed, not as in (11), but uniformly from -1 to $+1$. In analogy to (12) we now obtain

$$p(u_j)=\frac{k}{2}\int_{|u_j|}^{\infty}e^{-kr}r^{-1}dr, \tag{49}$$

which is easily converted to an incomplete gamma function or expressed by the generalized exponential integral

$$E_n(x)=\int_1^{\infty}t^{-n}e^{-xt}dt, \tag{50}$$

for which tables are available.[5] Using (54), we obtain from (49)

$$p(u_j)=(k/2)E_1(k|u_j|)=-(k/2)\,\mathrm{Ei}(-k|u_j|) \quad (-\infty<u_j<\infty) \tag{51}$$

as the basic distribution for the three-dimensional case. Its characteristic function is[6]

$$\chi_j(v)=k\int_0^\infty E_1(ku_j)\cos vu_j du_j=(k/v)\arctan(v/k) \tag{52}$$

and hence the density of S_n is in analogy to (15),

$$\begin{aligned} p(s_n)&=\frac{1}{2\pi}\int_{-\infty}^{\infty}(k/v)^n\arctan^n(v/k)\cos s_n v dv \\ &=\frac{k}{2\pi}\int_{-\infty}^{\infty}\left(\frac{\arctan t}{t}\right)^n\cos s_n k t dt, \end{aligned} \tag{53}$$

but here the analytical treatment ends, for (53) cannot be evaluated in terms of tabulated functions and the calculations require a computer to be brought in earlier than for the two-dimensional case.

However, the two special cases of Sec. IV may be evaluated as for the two-dimensional case. For case (a) we have $k_1\to 0$, $k_2\to k$, so that the electron is capable only of a single step U_1, which will take it either to absorption or across the surface $x=0$. Hence from (51) we obtain[7]

$$\begin{aligned} P(0)=P(U>h)&=\frac{k}{2}\int_h^\infty E_1(k_2u)du \\ &=\tfrac{1}{2}[e^{-k_2h}-k_2hE_1(k_2h)] \quad (k_1\ll k_2). \end{aligned} \tag{54}$$

[5] Cf. bibliography, pp. 236–237 of Ref. 1.
[6] No. 6.232.2 of Ref. 2.
[7] No. 6.223 and 6.221 of Ref. 2.

This expression[8] is plotted in Fig. 5. In analogy to (38) we have[9]

$$\begin{aligned} I&=\frac{I_0}{2}\int_0^\infty[e^{-k_2h}-k_2hE_1(k_2h)]e^{-\alpha h}dh \\ &=\frac{I_0}{2\alpha}\left[\frac{\beta}{1+\beta}-k_2h\ln(1+\beta)\right] \quad (k_2\gg k_1), \end{aligned} \tag{55}$$

where β is given by (39).

For case (b), $k_2\ll k_1$, we may use the same method as in the two-dimensional case (with the same reservations). Since $\langle U_j\rangle=0$ and[10]

$$D(U_j)=k\int_0^\infty u^2E_1(ku)du=\tfrac{2}{3}k^2, \tag{56}$$

we obtain in analogy to (47)

$$P(S_n>h)=\tfrac{1}{2}\,\mathrm{erfc}[h(3k_1k_2)^{1/2}/2] \quad (k_1\gg k_2) \tag{57}$$

and $P(0)$ is proportional to this expression; but the constant of proportionality $1/P(+|0)$ is now not as easily obtained as in the two-dimensional case, since the corresponding integrations must be performed by computer.

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[8] The same expression may be obtained from Eq. (1) of C. N. Berglund and W. E. Spicer, Phys. Rev. **136**, A1030 (1964), by substituting $\sec\theta=t$ and converting the resulting incomplete gamma function to E_1.
[9] No. 6.224.1 of Ref. 2.
[10] No. 6.223 of Ref. 2.