New Mechanism for Superconductivity*

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A new mechanism which leads to superconductivity even for particles with purely repulsive forces between them is described. The general theory is given for weak short-ranged pair forces between the particles and for arbitrary (convex) Fermi surfaces. It is shown that the transition temperature due to this mechanism is in general quite low, though it may be enhanced somewhat by very special Fermi-surface geometry.

I. INTRODUCTION

IN a recent brief note[1] a new and rather general mechanism for superconductivity has been indicated. This mechanism has nothing to do with the conventional electron-phonon attractive interaction, and is present even in the case of weak purely repulsive forces between pairs of particles. It is connected with the (assumed) sharpness of the Fermi surface for a normal system.

To understand what is involved, we first take an oversimplified view of the effect. It has long been known[2] that if a change is placed in a metal, the screening is such that there remains a long-range oscillatory potential of the form $\cos(2k_F r+\varphi)/r^3$ (for the case where the Fermi surface is a sphere of radius k_F). These are the so-called "Friedel" oscillations. This leads to a long-ranged interaction between pairs of such external charges embedded in the metal. Formally, the source of this long-range force is the singularity of the dielectric constant as a function of the momentum transfer q, when $q=2k_F$. This singularity in the Fourier transform of the interaction gives rise to a long-ranged oscillatory force in ordinary space. All that is necessary for this effect is a sharp Fermi surface; a rounding of the Fermi surface due to (say) finite temperature or impurities will give rise to an interaction which drops off more rapidly than any power of the distance, at large distances.

It is plausible to suppose that similarly the effective interaction between the electrons themselves will have a long-range oscillatory part. By taking advantage of the attractive regions we might expect Cooper pairs to form and therefore superconductivity to occur. We shall show in what follows that this indeed can be the case, although the picture above is a little oversimplified.

It is clear that since the screening itself is a result of electron-electron interactions, the effect which we are discussing must be one which comes from higher order corrections to the bare electron-electron interaction. Therefore, to analyze the problem rigorously we must investigate the general criterion for the onset of Cooper pair formation (i.e., the equation for the transition temperature to a superconducting state) beyond the lowest order in the interaction between the particles.

In Sec. II, this is done for the case of an isotropic gas of fermions of spin $\frac{1}{2}$ (no periodic potential, i.e., no band effects) with weak short-ranged pair forces between them. In Sec. III it is shown that for this system at low enough temperatures there is in general a transition to the superconducting state, independent of the exact nature of the original forces as long as they are sufficiently weak and well behaved at infinity. In Sec. IV, the results are extended to the case where a periodic potential is present. Finally, in Sec. V we consider some quantitative aspects of the problem. We also discuss the possibility of enhancing the effect treated in this paper by considering metals of a somewhat exotic Fermi surface geometry.

II. GENERAL FORMULATION

We need a general criterion for a system to become superconducting. Such a criterion (depending only on normal-state properties) has been used by Thouless in a special case, and has been formulated quite generally by Gorkov and Pitaevski.[3] It involves the appearance of a pole (as a function of temperature) in a certain particle scattering vertex.

Let us define the two-particle Green's function as follows[4]:

$$G_{l_1 l_2, l_1' l_2'}(v_1 v_2; v_1' v_2') \equiv \langle T(a_{l_1}(v_1) a_{l_2}(v_2) \times a_{l_2'}^{\dagger}(v_2') a_{l_1'}^{\dagger}(v_1')) \rangle, \quad (2.1)$$

where $l=(\mathbf{k},s)$, $\mathbf{k}$ being the momentum of a state and s its spin (units such that $\hbar=1$ are used throughout this paper), and for any operator A we define

$$A(v) \equiv e^{v(\mathcal{H}-\mu N)} A e^{-v(\mathcal{H}-\mu N)}, \quad (2.2)$$

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[1] W. Kohn and J. M. Luttinger, Phys. Rev. Letters **15**, 524 (1965).

[2] J. Friedel, Advan. Phys. **3**, 446 (1954); Nuovo Cimento, Suppl. **2**, 287 (1958); W. Kohn and S. H. Vosco, Phys. Rev. **119**, 912 (1960); J. S. Langer and S. H. Vosco, J. Phys. Chem. Solids **12**, 196 (1959).

[3] D. J. Thouless, Ann. Phys. **10**, 553 (1960); L. P. Gor'kov and T. K. Melik-Barkhudarov, Zh. Eksperim. i Teor. Fiz. **40**, 1452 (1961) [English transl: Soviet Phys.—JETP **13**, 1018 (1961)]; L. P. Gor'kov and L. P. Pitaevski, Zh. Eksperim. i Toer. Fiz. **46**, 600 (1962) [English transl.: Soviet Phys.—JETP **15**, 417 (1962)].

[4] A good general reference for the Greens' functions we use is A. A. Abrikosov, L. P. Gorkov, and I. E. Dzyaloshinski, *Methods of Quantitative Field Theory in Statistical Physics* (Prentice Hall Inc., Englewood Cliffs, New Jersey, 1963). Our definitions differ, however, very slightly from theirs.

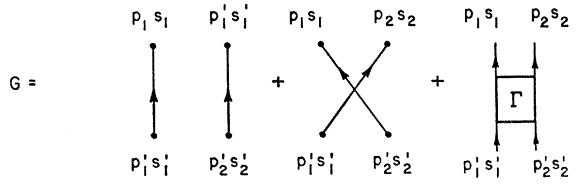

FIG. 1. General decomposition of the two-particle Green's function.

$\mathcal{H}$ being the Hamiltonian of the system. T is the usual Wick ordering operator (applied to ordering in v), and the angular brackets represent a thermal equilibrium average over the grand canonical distribution. It follows from the definition that G has the expansion

$$G_{l_1l_2,l_1'l_2'}(v_1v_2;\, v_1'v_2') = \frac{1}{\beta^2}\sum_{\substack{m_1,m_2=-\infty \\ m_3,m_4}}^{\infty} G_{s_1s_2,s_1's_2'}(p_1p_2,p_1'p_2')\exp -(\omega_1v_1+\omega_2v_2-\omega_3v_3-\omega_4v_4), \quad (2.3)$$

where

$$\omega_j=\pi i(2m_j+1)/\beta,$$
$$\beta=1/k_BT,$$
$$p=\mathbf{k},\omega.$$

The quantity $G_{s_1s_2,s_1's_2'}(p_1p_2;\, p_1'p_2')$ has a simple diagrammatic expansion, i.e., it consists of all connected diagrams with $p_1's_1'$, $p_2's_2'$ entering and p_1s_1, p_2s_2 leaving. Therefore, we may write (see Fig. 1)

$$G_{s_1s_2,s_1's_2'}(p_1p_2;\, p_1'p_2') = G(p_1)G(p_2)\times[\delta_{p_1p_1'}\delta_{p_2p_2'}\delta_s-\delta_{p_1p_2'}\delta_{p_2p_1'}\tilde{\delta}_s] -(1/\beta)G(p_1)G(p_2)\Gamma_{s_1s_2,s_1's_2'}\times(p_1p_2,p_1'p_2')G(p_1')G(p_2'), \quad (2.4)$$

where $G(p)$ is the exact single-particle propagator. Γ is called the *vertex part*. Further, $\delta_s\equiv\delta_{s_1s_1'}\delta_{s_2s_2'}$, $\tilde{\delta}_s=\delta_{s_1s_2'}\delta_{s_2s_1'}$.

If we define the *irreducible vertex part* I as the collection of all those diagrams of Γ which cannot be cut into two parts by cutting two internal lines, then one easily sees that Γ satisfies the integral equation

$$\Gamma_{s_1s_2,s_1's_2'}(p_1p_2,p_1'p_2') = I_{s_1s_2,s_1's_2'}(p_1p_2,p_1'p_2') -\frac{1}{2\beta}\sum_{\substack{p_1''p_2'' \\ s_1''s_2''}}\Gamma_{s_1s_2,s_1''s_2''}(p_1p_2,p_1''p_2'')G(p_1'')G(p_2'')\times I_{s_1''s_2'',s_1's_2'}(p_1''p_2'',p_1'p_2'). \quad (2.5)$$

If one iterates this equation, one obtains the Γ terms coming from a single irreducible vertex, from two irreducible vertices, etc.

The vertex part Γ corresponds essentially to the scattering of a pair of quasiparticles. The object which determines the transition temperature is Γ for equal and opposite momenta and ω (i.e., equal and opposite p) entering and leaving. A singularity of Γ for these variables indicates the formation of Cooper pairs with equal and opposite momenta and infinitesimal binding ($\omega_1+\omega_2=0$ means that the energy of the pair of quasiparticles is 2μ). Putting $p_1=-p_2=p$ and $p_1'=-p_2'=+p'$, we obtain

$$\Gamma_{s_1s_2,s_1's_2'}(p,p') = I_{s_1s_1',s_2s_2'}(p,p') -\frac{1}{2\beta}\sum_{p''s_1''s_2''}\Gamma_{s_1s_2,s_1''s_2''}(p,p'')G(p'')\times G(-p'')I_{s_1''s_2''s_1's_2'}(p'',p'). \quad (2.6)$$

The criterion mentioned above for the onset of superconductivity is that Γ—viewed as a function of temperature—has a singularity as T approaches T_c from above.

Symbolically, call p, s_1, $s_2=j$ and suppress j'. Then (2.6) becomes (where the temperature dependence is made explicit)

$$\Gamma_j=I_j+\sum_{j'}\Gamma_{j'}K_{j'j}. \quad (2.7)$$

Now these linear equations always have a solution inversely proportional to the determinant of $(\delta_{jj'}-K_{j'j})$. When the homogeneous equations

$$\Gamma_j^H=\sum_{j'}K_{jj'}\Gamma_{j'}^H \quad (2.8)$$

have a solution, this determinant vanishes. Therefore, for T slightly greater than T_c [the temperature at which (2.8) has a solution], this determinant is proportional to $T-T_c$ and therefore in general Γ_j will behave like

$$\Gamma_j\sim A_j(T-T_c). \quad (2.9)$$

(Actually, A_j would vanish at $T=T_c$ if Γ_j^H were orthogonal to I_j, and under these conditions there would be no singularity. It is easy to verify that in our case this does not occur.) Therefore, the criterion for a pole of Γ is that the homogeneous equation (2.8) has a solution. In more explicit notation,

$$\psi_{s_1s_2}(p)=-\frac{1}{2p}\sum_{p',s_1',s_2'}I_{s_1s_2,s_1's_2'}\times(p,p')G(p')G(-p')\psi_{s_1's_2'}(p') \quad (2.10)$$

has a solution at $T-T_c$.

From the original definition of Γ one easily sees that

$$\Gamma_{s_1s_2,s_1's_2'}(p_1p_2,p_1'p_2')=-\Gamma_{s_2s_1,s_1's_2'}(p_2p_1,p_1'p_2'). \quad (2.11)$$

Therefore, we must choose $\psi_{s_1s_2}(p)$ such that

$$\psi_{s_1s_2}(p)=-\psi_{s_2s_1}(-p). \quad (2.12)$$

The criterion (2.10) may also be proven directly by means of the general pairing theory of Migdal and

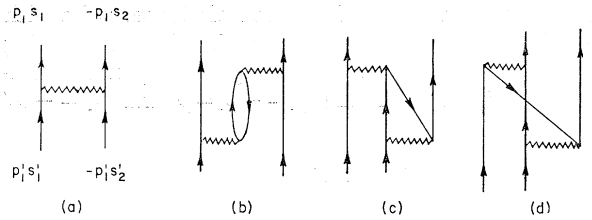

FIG. 2. Diagrams for I to the second order (a) represents the direct interaction, (b) represents the effect we have considered in the Introduction, i.e., the screened direct interaction, (c) is from a wave function modification of the particles and (d) is from exchange.

Larkin.[5] We shall not give the proof here. (A version of it will appear shortly in my lectures at the Summer School in Oiso, Japan.)

We now consider (2.10) for a weak short-ranged potential. We need to know I. We shall assume that the interaction is weak enough so that we need only to go to the second order in the interaction (this is the lowest order in which our effect occurs). The diagrams for I are given in Fig. 2. It is easy to evaluate the contributions of these diagrams. Specializing to the case of interest ($p_1=-p_2=p$, $p_1'=-p_2'=p$) and using unperturbed propagators in the evaluation[6] one finds

$$I_{s_1s_2,s_1's_2'}(p,p')=(K(p,p')\delta_s-K(p,-p')\tilde{\delta}_s), \tag{2.13}$$

$$K(p,p')=K^{(a)}(p,p')+K^{(b)}(p,p') +K^{(c)}(p,p')+K^{(d)}(p,p'), \tag{2.14}$$

$$K^{(a)}(p,p')=u(\mathbf{k}-\mathbf{k}'), \tag{2.15}$$

$$K^{(b)}(p,p')=2\sum_{\mathbf{k}''}u^2(\mathbf{k}-\mathbf{k}')Q_{\mathbf{k}''}(\mathbf{k}-\mathbf{k}',\omega-\omega'), \tag{2.16}$$

$$K^{(c)}(p,p')=-\sum_{\mathbf{k}''}u(\mathbf{k}-\mathbf{k}')[u(\mathbf{k}''+\mathbf{k}') +u(\mathbf{k}''-\mathbf{k})]Q_{\mathbf{k}''}(\mathbf{k}-\mathbf{k}',\omega-\omega'), \tag{2.17}$$

$$K^{(d)}(p,p')=-\sum_{\mathbf{k}''}u(\mathbf{k}-\mathbf{k}'')u(\mathbf{k}'-\mathbf{k}'') \times Q_{\mathbf{k}''}(\mathbf{k}+\mathbf{k}',\omega+\omega'), \tag{2.18}$$

$$Q_{\mathbf{k}''}(q,z)\equiv\frac{f_{\mathbf{k}''}^{-}-f_{\mathbf{k}''-\mathbf{q}}^{-}}{\epsilon_{\mathbf{k}''}-\epsilon_{\mathbf{k}''-\mathbf{q}}-z}.$$

Here $u(\mathbf{k})$ is the $\mathbf{k}$th Fourier component of the bare particle-particle interaction, $\epsilon_{\mathbf{k}}$ is the unperturbed energy of a particle, $f_{\mathbf{k}}^{-}$ the ordinary Fermi distribution, and we have taken a unit volume for the system.

[5] See, for example, A. I. Larkin and A. B. Migdal, Zh. Eksperim. i Teor. Fiz. 44, 1703 (1963) [English transl.: Soviet Phys.—JETP 17, 1146 (1963)]. There only the $T=0$ case is treated but the analysis for $T\neq 0$ is almost identical.

[6] In calculating to the second order we need only use propagators correct to zeroth order since the lowest order diagram contains no propagators.

Inserting (2.13) in (2.10), we have

$$\psi_{s_1s_2}(p)=-\frac{1}{2\beta}\sum_{p';\,s_1s_2}\{K(p,p')\delta_s-K(p,-p')\tilde{\delta}_s\} \times G(p')G(-p')\psi_{s_1's_2'}(p') =-\frac{1}{\beta}\sum_{p'}K(p,p')G(p')G(-p')\psi_{s_1s_2}(p') \tag{2.19}$$

on using (2.12). The solutions of (2.19) may be obtained from those of

$$\chi(p)=-\frac{1}{\beta}\sum_{p'}K(p,p')G(p')G(-p')\chi(p'), \tag{2.20}$$

where, if $\chi(p)$ is even in p, we must multiply $\chi(p)$ by an antisymmetric spin function; if $\chi(p)$ is odd in p it must be multiplied by a symmetric spin function.

In (2.20) we may use the lowest order propagator. (The propagator correct to the first order contains first a trivial Hartree-Fock shift in the single-particle energy and is easily seen to change nothing in the argument given below.) The unperturbed propagator is

$$G(p')=1/(\omega'-\tilde{\epsilon}_{k'}),$$
$$\tilde{\epsilon}_{k'}=(k'^2-k_F^2)/2m,$$
$$\omega'=[\pi i(2m'+1)]/\beta,$$

so that

$$G(p')G(-p')=-\frac{1}{\omega'^2-\tilde{\epsilon}_{k'}^2}=\frac{1}{|\omega'|^2+\tilde{\epsilon}_{k'}^2}. \tag{2.21}$$

This function has its maximum as a function of ω' for $\omega'=0$; for $m'\sim 1$ (which are typical values in the sum) $\omega'\sim k_BT$. Since K varies slowly with ω (the scale of variation being ϵ_F), and since $k_BT\ll\epsilon_F$, we can take $\omega'=0$ in $\chi(p')$ and $K(p,p')$. Setting $\omega=0$ also, we obtain

$$\chi(\mathbf{k})=-\sum_{\mathbf{k}'}K(\mathbf{k},\mathbf{k}')\frac{\tanh(\beta\tilde{\epsilon}_{k'}/2)}{2\tilde{\epsilon}_{k'}}\chi(\mathbf{k}') \tag{2.22}$$

[where $\chi(\mathbf{k})=\chi(\mathbf{k},\omega=0)$, etc.], since

$$\frac{1}{\beta}\sum_{m'}\frac{1}{\omega'^2-\epsilon_{k'}^2}=\frac{\tanh(\beta\epsilon_{k'}/2)}{2\tilde{\epsilon}_{k'}}. \tag{2.23}$$

When the system is isotropic, $K(\mathbf{k},\mathbf{k}')$ must be a rotational invariant. Therefore, the solutions of (2.22) must have the angular dependence of spherical harmonics in k space. If we put

$$\chi(\mathbf{k})=g_l(k)Y_l^m(\theta,\varphi) \tag{2.24}$$

and

$$K(\mathbf{k},\mathbf{k}')=K(k,k',\cos\vartheta) =\sum_{l=0}^{\infty}\frac{2l+1}{2}K_l(k,k')P_l(\cos\vartheta), \tag{2.25}$$

where ϑ is the angle between $\mathbf{k}$ and $\mathbf{k}'$, (2.22) becomes

$$g_l(k)=-\frac{1}{(2\pi)^2}\int_0^\infty dk'k'^2\frac{\tanh(\beta\tilde{\epsilon}_{k'}/2)}{2\tilde{\epsilon}_{k'}}\times K_l(k,k')g_l(k'). \quad (2.26)$$

Consider for a moment (2.26) in the limit $T=0$ (or $\beta=\infty$). Then $\tanh(\beta\tilde{\epsilon}_{k'}/2)=\mathrm{sgn}\tilde{\epsilon}_{k'}$, and the integral in (2.26) will diverge logarithmically near $\tilde{\epsilon}_{k'}=0$, or $k'=k_F$. For finite β, $1/\beta$ provides a cutoff (since $\tanh(\beta x)$ approaches zero as x approaches zero), and makes the integral finite. Therefore, the leading term on the right-hand side of (2.26) comes from the region $k'\sim k_F$ and goes as $\ln\beta$. We may write

$$g_l(k)=-\frac{1}{(2\pi)^2}K_l(k,k_F)g_l(k_F)\times\int dk'k'^2\frac{\tanh(\beta\tilde{\epsilon}_{k'}/2)}{2\tilde{\epsilon}_{k'}}+\cdots, \quad (2.27)$$

where the integral goes over the neighborhood of $k'=k_F$, and the dots indicate smaller terms.

$$\int dk'k'^2\frac{\tanh(\beta\tilde{\epsilon}_{k'}/2)}{2\tilde{\epsilon}_{k'}}=mk_F(\ln\beta+\cdots). \quad (2.28)$$

Using (2.28), (2.27) becomes

$$g_l(k)=-\frac{mk_F}{(2\pi)^2}K_l(k,k_F)g_l(k_F)(\ln\beta+\cdots). \quad (2.29)$$

Writing the correction as $\ln\epsilon_l$, (2.29) becomes (also putting $k=k_F$)

$$1=-\frac{mk_F}{(2\pi)^2}K_l(k_F,k_F)\ln(\beta\epsilon_l). \quad (2.30)$$

It is not difficult to obtain an expression for ϵ_l, but as we shall see it is not necessary for our purposes. (See Gorkov and Pitaevskii for an example of how this is done.) The only thing about ϵ_l that we need to know is that it is in general of the order of ϵ_F, having nothing to do with the temperature.

Equation (2.30) is an expression for the transition temperature. If for any l there exists a temperature such that (2.30) has a solution, then the system will become superconducting at that temperature. If there are several l's for which there are solutions, then the one with the highest T is the transition temperature. As long as $K_l(k_F,k_F)$ is *negative* for some l, no matter how small it is in absolute value, the equation for T has a solution for low enough temperature since $\ln(\beta\epsilon_l)$ approaches infinity as T approaches zero.

III. TRANSITION TEMPERATURE FOR LARGE l

The expression for $K_l(k_F,k_F)(\equiv K_l)$ is given from (2.25) by

$$K_l=\int_0^\pi d\vartheta(\sin\vartheta)K(k_F,k_F,\cos\vartheta)P_l(\cos\vartheta)=\int_{-1}^1 dxP_l(x)K(x). \quad (3.1)$$

We want to discuss (3.1) for large l. Consider first $K^{(a)}(x)$ [see (2.151)], which is just the Born approximation to the scattering amplitude. It is well known[7] that if the potential is a superposition of Yukawa potentials, the first singularity of $K^{(a)}(x)$ occurs on the real axis for $x>1$. Under these circumstances it is not difficult to prove that $K_l^{(a)}$ decreases (for large l) at least exponentially with l. A proof is outlined in Appendix A.

On the other hand, if $K(x)$ *does* have a singularity on $[-1,1]$, K_l will drop off with some power of l. (We shall see this on specific examples below.) Therefore, if $K^{(b)}$, $K^{(c)}$, or $K^{(d)}$ has a singularity on $[-1,1]$, at sufficiently large l they will dominate no matter how small the potential is. Now, in fact, the second-order K does have a log type of singularity at $x=\pm1$ for a pure metal at zero temperature. For our purposes, it is sufficient to consider the zero-temperature expression for K. The following qualitative argument can be verified by using the explicit temperature-dependent expressions for K. The fact that the temperature is not zero rounds off the Fermi surface and therefore the "Friedel oscillations" are cut off at same distance, say R_2. If the Fermi surface is rounded off over a range of momenta Δk, then one would expect

$$R_2\sim 1/\Delta k. \quad (3.2)$$

owing to temperature the Δk is given by

$$k_F\Delta k/m\sim k_BT \quad (3.3)$$

so that

$$R_2\sim k_F/mk_BT. \quad (3.4)$$

On the other hand, if the particles are in an angular momentum state l (with $l\gg1$) and they have the Fermi momentum k_F, we expect on semiclassical grounds that the important contributions to the pair wave functions come from distances R_1 given by

$$k_FR_1\sim l$$

or

$$R_1\sim l/k_F. \quad (3.5)$$

If $R_2\gg R_1$, the effect of temperature will be

[7] See, for example, R. Omnes and M. Froissart, *Mandelstam Theory and Regge Poles* (W. A. Benjamin, Inc., New York, 1963), p. 29 ff.

negligible.

$$\frac{R_2}{R_1}\sim\frac{k_F^2/m}{lk_BT}\sim\frac{\epsilon_F}{lk_BT}. \tag{3.6}$$

In fact ϵ_F/k_BT_l (for the formation of superconductivity in the lth state) goes as e^{cl^4} as we shall see below. Therefore, for large l, $R_2\gg R_1$ and the pairs are formed long before the Friedel oscillations are washed out.

The singularities in the second order arise, just as the Friedel oscillations do, in the summation $\sum_{\mathbf{k}''}Q_{k''}(\mathbf{q},0)$ from those regions where $q\cong 2k_F$ and $\mathbf{k}''$ is near k_F and directed opposite to $\mathbf{q}$. It is very easy to see that the singular parts of K are given by

$$\begin{aligned}K_s^{(b)}&=2u^2(2k_F)Q(x),\\ K_s^{(c)}&=-2u(0)u(2k_F)Q(x),\\ K_s^{(d)}&=-u^2(0)Q(-x),\end{aligned} \tag{3.7}$$

where

$$Q(x)=(mk_F/16\pi^2)(1+x)\ln(1+x). \tag{3.8}$$

The necessary integrals to obtain K_l are elementary (one only needs to use Rodrigues' formula for P_l and integrate by parts) and we obtain, for sufficiently large l,

$$\begin{aligned}K_l^{(b)}&=2\left(\frac{mk_F}{4\pi^2}\right)u^2(2k_F)\frac{(-1)^l}{l^4},\\ K_l^{(c)}&=-2\left(\frac{mk_F}{4\pi^2}\right)u(0)u(2k_F)\frac{(-1)^l}{l^4},\\ K_l^{(d)}&=-\left(\frac{mk_F}{4\pi^2}\right)u^2(0)\frac{1}{l^4}.\end{aligned} \tag{3.9}$$

Therefore, for large l, since $K_l^{(a)}$ is negligible

$$K_l=-\frac{mk_F}{4\pi^2}\frac{1}{l^4}[u^2(0)+2(-1)^l\times(u(0)u(2k_F)-u^2(2k_F))]. \tag{3.10}$$

For l *odd* this becomes

$$K_l=-\frac{mk_F}{4\pi^2}\frac{1}{l^4}[(u(0)-u(2k_F))^2+u^2(2k_F)]<0. \tag{3.11}$$

Therefore, large enough odd l, $K_l<0$, and the equation (2.30) for the transition temperature always has a solution.

Thus, we have established the following result: A gas of spin-$\frac{1}{2}$ fermions interacting with a sufficiently weak short-ranged potential will always become superconducting at sufficiently low temperatues if no other transition occurs at higher temperatures to destroy the normality.

It might be argued that we have not shown that even for a normal fermion system, the interactions do not destroy the sharpness of the Fermi surface. What has been shown[8] is that to arbitrary order in the interaction the Fermi surface remains sharp. Therefore, if there is some rounding of the Fermi surface Δk due to interactions it would have a form like

$$\Delta k\sim k_Fe^{-c/\alpha}, \tag{3.12}$$

where c is a constant and α is some dimensionless measure of the strength of the interaction. Therefore, by

$$R_2\sim(1/k_F)e^{c/\alpha} \tag{3.13}$$

and

$$R_2/R_1\sim(1/l)e^{c/\alpha}. \tag{3.14}$$

For fixed l we can make this as large as we please by choosing the interaction (or α) as small as we please. Therefore, for a sufficiently weak potential, particle-particle interaction cannot destroy the effect discussed here.

There is no other phase transition known which, for an arbitrarily weak potential, can give rise to the destruction of the normal state of a fermion gas. If one existed it might occur at a higher temperature than the transition discussed above and therefore invalidate our result.

IV. THE BAND CASE

We shall now generalize the results of the previous section to the case where the electrons are also in a periodic potential, so that the energy levels form bands. To make the notation as simple as possible we shall consider the case where the conduction electrons fall into a single simple band and the Fermi surface is convex. (This is by no means an essential limitation, but it reduces enormously the number of special circumstances one must discuss.)

From the discussion of the previous section, we see that the essential feature that we need for our effect is the presence in the second-order K of an analytical singularity as a function of $(\mathbf{k}-\mathbf{k}')$ or $(\mathbf{k}+\mathbf{k}')$, both $\mathbf{k}$ and $\mathbf{k}'$ being on the Fermi surface. Such singularities also exist in the band case. We do not want to write down here the full formulas equivalent to (2.15)–(2.18) for the band case because they are quite complicated. However, it is very easy to see that the only parts which can have singular behavior of the type we need come from electrons within the conduction band. Then (2.22) is still valid, the summation being over the Brillouin zone, the conduction band index being suppressed, and (2.15)–(2.18) being replaced by

$$\begin{aligned}K^{(b)}=2\sum_{\mathbf{k}''}&\langle\mathbf{k},\mathbf{k}''-(\mathbf{k}-\mathbf{k}')|u|\mathbf{k}',\mathbf{k}''\rangle\\ &\times\langle\mathbf{k}'',-\mathbf{k}|u|\mathbf{k}''-(\mathbf{k}-\mathbf{k}'),-\mathbf{k}'\rangle Q_{\mathbf{k}''}(\mathbf{k}-\mathbf{k}'),\end{aligned} \tag{4.1}$$

$$\begin{aligned}K^{(c)}=-\sum_{\mathbf{k}''}&[\langle\mathbf{k},\mathbf{k}''-(\mathbf{k}-\mathbf{k}')|u|\mathbf{k}',\mathbf{k}''\rangle\\ &\times\langle-\mathbf{k},\mathbf{k}''|u|\mathbf{k}''-(\mathbf{k}-\mathbf{k}'),-\mathbf{k}'\rangle\\ &+\langle\mathbf{k},\mathbf{k}''-(\mathbf{k}-\mathbf{k}'|u|\mathbf{k}'',\mathbf{k}'\rangle\\ &\times\langle-\mathbf{k},\mathbf{k}''|u|-\mathbf{k}',\mathbf{k}''-(\mathbf{k}-\mathbf{k}')\rangle]Q_{\mathbf{k}''}(\mathbf{k}-\mathbf{k}'),\end{aligned} \tag{4.2}$$

[8] J. M. Luttinger, Phys. Rev. **119**, 1153 (1960).

$$K^{(d)}=-\sum_{\mathbf{k}''}\langle\mathbf{k},\mathbf{k}''-(\mathbf{k}+\mathbf{k}')\,|u|\,\mathbf{k}'',-\mathbf{k}'\rangle \times\langle\mathbf{k}'',-\mathbf{k}|u|\mathbf{k}',\mathbf{k}''-\mathbf{k}-\mathbf{k}'\rangle Q_{\mathbf{k}''}(\mathbf{k}+\mathbf{k}'). \quad (4.3)$$

In (4.1)–(4.3)

$$Q_{\mathbf{k}''}(\mathbf{q})=\frac{f_{\mathbf{k}''}-f_{\mathbf{k}''-\mathbf{q}}}{\epsilon_{\mathbf{k}''}-\epsilon_{\mathbf{k}''-\mathbf{q}}}$$

and

$$\langle k_1k_2|u|k_1'k_2'\rangle=\int\psi_{\mathbf{k}_1}^*(\mathbf{r}_1)\psi_{\mathbf{k}_2}^*(\mathbf{r}_2)u(\mathbf{r}_1-\mathbf{r}_2) \times\psi_{\mathbf{k}_1'}(\mathbf{r}_1)\psi_{\mathbf{k}_2'}(\mathbf{r}_2)d\mathbf{r}_1d\mathbf{r}_2, \quad (4.4)$$

where $\psi_{\mathbf{k}}(\mathbf{r})$ is the Bloch function for electrons in the conduction band, $\epsilon_{\mathbf{k}}$ the corresponding energy.

The singular behavior of expressions like (4.1)–(4.3) have been investigated by Kohn.[9] For a convex Fermi surface (for $\mathbf{k}$ and $\mathbf{k}'$ on the Fermi surface) (4.1) and (4.2) have a singularity for $\mathbf{k}'=-\mathbf{k}$, the singularity coming from $\mathbf{k}''$ near $\mathbf{k}$. For (4.3) the singularity is for $\mathbf{k}'=\mathbf{k}$, the contributing region of integration for $\mathbf{k}''$ again being near $\mathbf{k}$. We shall skip the somewhat tedious evaluation of these singular terms and only quote the results for the singular parts of $K^{(2)}$.

$$K_s^{(b)}=\frac{1}{32\pi^2}\frac{2M_1^2(\mathbf{k})}{v_n^2(\mathbf{k})\sqrt{K(\mathbf{k})}}A(\mathbf{k},-\mathbf{k}')\ln A(\mathbf{k},-\mathbf{k}'), \quad (4.5)$$

$$K_s^{(c)}=\frac{1}{32\pi^2}\frac{2M_1(\mathbf{k})M_2(\mathbf{k})}{v_n^2(\mathbf{k})\sqrt{K(\mathbf{k})}}A(\mathbf{k},-\mathbf{k}')\ln A(\mathbf{k},-\mathbf{k}'), \quad (4.6)$$

$$K_s^{(d)}=\frac{1}{32\pi^2}\frac{M_2^2(\mathbf{k})}{v_n^2(\mathbf{k})\sqrt{K(\mathbf{k})}}A(\mathbf{k},\mathbf{k}')\ln A(\mathbf{k},\mathbf{k}'). \quad (4.7)$$

Here $v_n(\mathbf{k})$ is the velocity normal to Fermi surface at the point $\mathbf{k}$, $K(\mathbf{k})$ is the Gaussian curvature[10] of the Fermi surface at the point $\mathbf{k}$, M_1, and M_2 are the matrix elements

$$M_1(\mathbf{k})=\langle\mathbf{k},-\mathbf{k}|u|-\mathbf{k},\mathbf{k}\rangle, \quad (4.8)$$

$$M_2(\mathbf{k})=\langle\mathbf{k},-\mathbf{k}|u|\mathbf{k},-\mathbf{k}\rangle. \quad (4.9)$$

(It is easy to see that M_1 and M_2 are real.)

$$A(\mathbf{k},\mathbf{k}')=\sum_{\alpha,\beta}\frac{\partial^2\epsilon_{\mathbf{k}}}{\partial k_\alpha\partial k_\beta}(k_\alpha-k_\alpha')(k_\beta-k_\beta'). \quad (4.10)$$

When the Fermi surface is convex, the quadratic form (4.10) is positive definite, since a displacement of a point on the Fermi surface in a direction perpendicular to the normal at that point always leads to a point outside the Fermi surface where the energy is higher. Therefore,

$$A(k,k')\geqslant 0. \quad (4.11)$$

[9] W. Kohn, Phys. Rev. Letters 2, 393 (1959).

[10] See, for example, D. J. Struick, *Differential Geometry* (Addison-Wesley Publishing Company, Inc., Reading, Massachusetts, 1950), p. 83.

For a general Fermi surface the integral equation (2.22) is not separable, and therefore instead of getting a simple expression like (2.30) for the transition temperawe are left with an integral equation over the Fermi surface. This has the following form:

$$\chi(\mathbf{k})=\frac{1}{(2\pi)^5}\frac{\ln(\beta\epsilon_c)}{8v_n^3(\mathbf{k})\sqrt{K(\mathbf{k})}}\int_{\text{F.S.}}ds'\{M_2^2(\mathbf{k})A(\mathbf{k},\mathbf{k}') \times\ln A(\mathbf{k},\mathbf{k}')+2(M_1(\mathbf{k})M_2(\mathbf{k})-M_1^2(\mathbf{k})) \times A(\mathbf{k},-\mathbf{k}')\ln A(\mathbf{k},-\mathbf{k}')\}\chi(\mathbf{k}'). \quad (4.12)$$

In (4.12), $\mathbf{k}$, $\mathbf{k}'$ are points on the Fermi surface, and ds is an element of area on the Fermi surface. The quantity ϵ_c is a cutoff [like ϵ_l in (2.30)] which depends on the solution of (4.12) which we consider, but is in general of the order of the Fermi energy.

One sees at once that the solutions of (4.12) are either odd or even functions of $\mathbf{k}$. Let us consider the odd solutions $\chi^-(k)$ (analogous to considering l odd in the previous section). For these [changing $\mathbf{k}'$ into $-\mathbf{k}'$ in the last term of Eq. (4.12)] the integral equation becomes

$$\chi^-(\mathbf{k})=\frac{(\ln\beta\epsilon_c)[(M_2(\mathbf{k})-M_1(\mathbf{k}))^2+M_2^2(\mathbf{k})]}{(2\pi)^5 8v_n^3(\mathbf{k})\sqrt{K(\mathbf{k})}} \times\int_{\text{F.S.}}ds'A(\mathbf{k},\mathbf{k}')\ln A(\mathbf{k},\mathbf{k}')\chi^-(\mathbf{k}'). \quad (4.13)$$

The solution of (4.13) for highly excited states (corresponding to large l in the previous section) can be carried out by a method analogous to the usual WKB method for the Schrödinger equation. This method also shows that for large quantum numbers the eigenvalues of the kernel in (4.13) go as the fourth power of the quantum numbers, rather than exponentially with them as is the case for a regular kernel.

To see this, it is convenient to introduce parameters (ξ,η) which characterize a point on the Fermi surface. As is well known,[11] the differential geometry of a surface is described by two quadratic forms in these parameters

$$I=Ed\xi^2+2Fd\xi d\eta+Gd\eta^2, \quad (4.14)$$

$$II=ed\xi^2+2fd\xi d\eta+gd\eta^2. \quad (4.15)$$

The coefficients are all functions of ξ, η. Here I represents an element of distance squared on the surface, and II gives essentially how much the direction of the normal changes when we move a small distance on the surface. In terms of these quantities the element of surface area and Gaussian curvature are given by

$$ds=(EG-F^2)^{1/2}d\xi d\eta, \quad (4.16)$$

$$K=(eg-f^2)/(EG-F^2). \quad (4.17)$$

For $\mathbf{k}$ very near $\mathbf{k}'$, we have (ξ,η) very near (ξ',η')

and therefore, on using the expressions found in footnote 11),

$$A(\mathbf{k},\mathbf{k}')=v_n(\xi,\eta)[e(\xi-\xi')^2+2f(\xi-\xi')(\eta-\eta')+g(\eta-\eta')^2]. \quad (4.18)$$

A particularly useful set of parameters ξ, η for our problem are the so-called *isometric* ones with respect to the form II. These are the parameters for which (4.15) takes the form

$$II=\mu(\xi,\eta)(d\xi^2+d\eta^2). \quad (4.19)$$

It is always possible to introduce such coordinates for a positive definite quadratic form in two variables.[12] Assuming that ξ, η are isometric parameters, the singular part of (5.13) becomes

$$\chi^-(\xi,\eta)=\frac{\alpha^2}{16\pi}U^2(\xi,\eta)\int d\xi' d\eta' R^2 \ln R^2 \chi^-(\xi',\eta'), \quad (4.20)$$

where

$$\alpha=+\sqrt{\frac{\ln(\beta\epsilon_c)}{(2\pi)^4}}, \quad (4.21)$$

$$U(\xi,\eta)=+\sqrt{\frac{[(M_2-M_1)^2+M_2^2](EG-F^2)}{v_n^2}}, \quad (4.22)$$

$$R^2=(\xi-\xi')^2+(\eta-\eta')^2. \quad (4.23)$$

To use the WKB approximation on (4.20), we make an ansatz of the form

$$\chi^-(\xi,\eta)=B(\xi,\eta)e^{iQ(\xi,\eta)}. \quad (4.24)$$

For highly excited states Q is proportional to some large parameter (since Q is assumed to be a smooth function of ξ, η). Then again assuming that the main contribution for highly excited states comes from the neighborhood of the singularity $\xi',\eta'=\xi,\eta$, we may write (4.20) as

$$B(\xi,\eta)e^{iQ(\xi,\eta)}=\frac{\alpha^2}{16\pi}U^2(\xi,\eta)\int d\xi' d\eta' R^2 \times \ln R^2 B(\xi',\eta')e^{iQ(\xi',\eta')}$$

$$\cong\frac{\alpha^2}{16\pi}U^2 B(\xi,\eta)e^{iQ(\xi,\eta)}\int d\xi' d\eta' R^2 \times \ln R^2 e^{i[Q(\xi'\eta')-Q(\xi\eta)]}$$

or

$$1\cong\frac{\alpha^2}{16}U^2\int d\xi' d\eta' R^2(\ln R^2)e^{i[Q_\xi(\xi'-\xi)+Q_\eta(\eta'-\eta)\eta]}, \quad (4.25)$$

where

$$Q_\xi=\frac{\partial Q(\xi,\eta)}{\partial\xi}, \quad Q_\eta=\frac{\partial Q(\xi,\eta)}{\partial\eta}.$$

Changing variables to $\xi'-\xi=x$, $\eta'-\eta=y$ and integrating over a small circle in the xy plane in the neighborhood of the origin, one sees at once that the leading contribution for large Q is

$$1=\frac{\alpha^2}{16\pi}U^2\frac{16\pi}{(Q_\xi^2+Q_\eta^2)^2} \quad (4.26)$$

or

$$Q_\xi^2+Q_\eta^2-\alpha U(\xi,\eta)=0. \quad (4.27)$$

Q is any solution of this partial differential equation which makes the (odd part of the) wave function single-valued. This gives us the desired quantum conditions for calculating the allowed values of α (i.e., temperature) for which we have solutions of the original integral equation.

Since U^2 is positive and so is $(Q_\xi^2+Q_\eta^2)^2$, it is clear that the solutions of (4.26) will always lead to positive values of α^2 and therefore to real finite transition temperatures. That is, as soon as the singular part dominates then pairs of electrons will be able to form Cooper pairs in these odd excited states and a transition to a superconducting state will occur. This is the analog of the result of the previous section for odd-l states.

It is convenient in thinking about (4.27) to make a mechanical analogy to it. Equation (4.27) is the WKB equation for a mass point of mass $\frac{1}{2}$ with coordinates ξ, η, moving in a potential $-\alpha U(\xi,\eta)$ and having zero energy. What we are asking is: What are the allowed values of α (the strength of the potential) such that some state of the system has exactly zero energy? The smallest α corresponds to the ground state having zero energy, the next to the first excited state having zero energy, etc. We only take the α's for odd states.

The quantum conditions for a particle with a non-separable potential were first given by Einstein[13] and refined somewhat by Keller.[14] In their extreme form (very high quantum numbers) they may be written as follows. Let $Q(\xi,\eta)$ be any complete integral[15] of the Hamilton-Jacobi equation (4.27). The quantity

$$J=\int_c (Q_\xi d\xi+Q_\xi d\eta)$$

[11] See Struick, Ref. 16, p. 75ff. We have

$$E=\mathbf{k}_\xi\cdot\mathbf{k}_\xi, \quad F=\mathbf{k}_\xi\cdot\mathbf{k}_\eta, \quad G=\mathbf{k}_\eta\cdot\mathbf{k}_\eta,$$

$$e=\frac{\mathbf{k}_{\xi\xi}\cdot(\mathbf{k}_\xi\times\mathbf{k}_\eta)}{(EG-F^2)^{1/2}} \quad f=\frac{\mathbf{k}_{\xi\eta}\cdot(\mathbf{k}_\xi\times\mathbf{k}_\eta)}{(EG-F^2)^{1/2}} \quad g=\frac{\mathbf{k}_{\eta\eta}\cdot(\mathbf{k}_\xi\times\mathbf{k}_\eta)}{(EG-F^2)^{1/2}}$$

where a subscript means differentiation with respect to the corresponding variable.

[12] A particularly simple proof requiring no results of differential geometry may be found in G. Springer, *Theory of Riemann Surfaces* (Addison-Wesley Publishing Company, Inc., Reading, Massachusetts, 1957), p. 19ff. This proof also gives formulas for constructing infinitely many sets of isometric parameters.

[13] A. Einstein, Verh. Deut. Phys. Ges. (1917). See also L. Landau and E. Lifschitz, *Quantum Mechanics* (Pergamon Press, Ltd., London, 1958), p. 183ff.

[14] J. B. Keller, Ann. Phys. (N. Y.) 4, 180 (1958).

[15] That is, any solution with two arbitrary nonadditive constants.

will not necessarily vanish when c is a closed curve in a (ξ,η) plane, because the function Q is not necessarily single-valued. In general, though, there will only be two closed curves c_1 and c_2 which give independent values of J, for any other closed curve J is just a linear combination of these "action" variables J_1 and J_2, with integer coefficients. The Einstein quantization conditions are then

$$J_1=\oint_{c1}(Q_\xi d\xi+Q_\eta d\eta)=2\pi n_1,$$
$$J_2=\oint_{c2}(Q_\xi d\xi+Q_\eta d\eta)=2\pi n_2, \tag{4.28}$$

where n_1 and n_2 are integers. It is clear from this that if one changes the scale of Q

$$Q=n_1\bar{Q}, \tag{4.29}$$

that $\bar{Q}$ is a function of order unity of (n_2/n_1). Therefore, from (4.27), the allowed values of $\alpha(\alpha_{n_1n_2})$ have the form

$$\alpha_{n_1n_2}=n_1{}^4f(n_2/n_1). \tag{4.30}$$

This is the analog of the l^4 dependence of the previous section. It shows that the contribution to $\alpha_{n_1n_2}$ of the singular part of the kernel goes as the quantum numbers to the fourth power.

To complete our proof we really need to show that for a kernel with no analytic singularity on the Fermi surface the contribution to the integral equation is exponentially small. In the case of a spherical Fermi surface this was provided for by the theorem (see Appendix A) on integrals over Legendre polynomials. What we must discuss is the integral

$$L=\int d\xi' d\eta' F(\xi\eta,\xi'\eta')\chi^-(\xi',\eta'), \tag{4.31}$$

where χ^- is given by (4.24) or

$$\chi^-(\xi,\eta)=B(\xi,\eta)e^{in_1Q(\xi,\eta)} \tag{4.32}$$

and F is an analytic function of $\xi'\eta'$ on the Fermi surface. We want to show that for sufficiently large n_1, L decreases more rapidly than any power of n_1. Intuitively this is clear. The behavior of L for large n_1 is determined by the singular points of F, the boundary points or the points of stationary phase, as is well known.[16] F is analytic by assumption. χ^-, being a wave function, will also be analytic on the Fermi surface. (However, the WKB solution for χ^- will not be, because of artificial singularities at the classical turning points, so to really evaluate L one must use a better solution.) Finally because of (4.27), $Q_\xi{}^2+Q_\eta{}^2>0$ except at pathological points (where v_n is infinite, which we exclude.) Therefore, there are no points of stationary phase (points at which $Q_\xi=Q_\eta=0$). Under these circumstances the argument of Focke shows that L decreases with n_1 more rapidly than any power of n_1.

V. OBSERVABILITY

In this section we discuss some aspects of the question of whether one may realistically hope to observe superconductivity in some metal, as a consequence of the mechanism proposed in this paper. First consider a simple isotropic metal like Na or K. We treat the bare interaction as a screened Thomas-Fermi Coulomb potential. This gives

$$u(q)=\frac{4\pi e^2}{q^2+2\xi k_F{}^2},\quad \xi=\frac{e^2k_F}{\pi\epsilon_F}\cong\frac{r_s}{3}. \tag{5.1}$$

For a "typical" alkali metal $r_s\sim4.5$, which we shall take from now on when we need a value for it, so

$$u(q)=4\pi e^2/(q^2+3k_F{}^2). \tag{5.2}$$

Even with a potential as simple as this the second-order terms (2.16)–(2.18) are prohibitively difficult to evaluate. To get a very rough idea of what is happening, we replace $u(q)$ in (2.16)–(2.18) by a δ-function potential with the correct average value $u(0)(=\pi^2/mk_F)$. Then the full second-order term ($K^{(2)}$) becomes

$$K^{(2)}=-\left(\frac{\pi^2}{mk_F}\right)^2\sum_{\mathbf{k}''}\frac{f_{\mathbf{k}''}{}^-f_{\mathbf{k}''-\mathbf{k}-\mathbf{k}'}{}^-}{\epsilon_{\mathbf{k}''}-\epsilon_{\mathbf{k}''-\mathbf{k}-\mathbf{k}'}}. \tag{5.3}$$

This integral is elementary (for free electrons at zero temperature) and gives on the Fermi surface

$$K^{(2)}(x)=\frac{\pi^2}{8mk_F}\frac{1}{t}\left\{2t+(1-t^2)\ln\left(\frac{1+t}{1-t}\right)\right\}, \tag{5.4}$$
$$t\equiv\left(\frac{1+x}{2}\right).$$

In the same notation

$$K^{(a)}(x)=\frac{3\pi^2}{2(mk_F)}\frac{1}{\frac{5}{2}-x}. \tag{5.5}$$

The $K_l{}^{(a)}$ are easy to obtain. We find[17]

$$K_l{}^{(a)}=(3\pi^2/mk_F)Q_l(\tfrac{5}{2}). \tag{5.6}$$

where Q_l is the Legendre function of the second kind.

$$K_l{}^{(2)}=(\pi^2/4mk_F)[\delta_{l,0}+R_l] \tag{5.7}$$

$$R_l\equiv2\int_0^1 dt(1-t^2)\ln\frac{1+t}{1-t}P_l(2t^2-1). \tag{5.8}$$

[16] G. Focke, Verh. Sächs. Akad. d. Wiss. (Leipzig) **101**, 1 (1954). This paper extends the usual stationary phase asymptotic treatment of integrals to the two-dimensional case.

[17] *Higher Transcendental Functions, Bateman Manuscript Project*, edited by A. Erdelyi (McGraw-Hill Book Company, Inc., 1953), Vol. I, p. 154.

The R_l are all elementary, though a little tedious to obtain.

From (2.30), we obtain an expression for the transition temperature (for the cases where the K_l turn out negative)

$$k_B T_l = \epsilon_l e^{-1/a_l}, \tag{5.9}$$

$$a_l \equiv -\frac{mk_F}{(2\pi)^2} K_l = a_l^{(1)} + a_l^{(2)}, \tag{5.10}$$

$$a_l^{(1)} = -\tfrac{3}{4} Q_l(2.5), \tag{5.11}$$

$$a_l^{(2)} = -\tfrac{1}{16}(\delta_{l,0} + R_l). \tag{5.12}$$

Numerical values of these quantities for the first seven l are given in Table I. From these we see that

TABLE I. First- and second-order contributions to a_l, which determines the transition temperature by means of (5.9)

l	$a_l^{(1)}$	$a_l^{(2)}$	$a_l^{(2)}$(asym)	$a_l^{(2)}$(WKB)	a_l
0	−0.30	−0.069	⋯	⋯	negative
1	−0.048	−0.019	0.063	0.012	negative
2	−0.0074	−0.0018	0.0039	0.0016	negative
3	−0.0013	−0.00044	0.00077	0.00042	negative
4	−0.00027	−0.00016	0.00024	0.00015	negative
5	−0.000051	−0.000068	0.00010	0.000068	0.000017
6	−0.0000098	−0.000035	0.000048	0.000035	0.000025
7	−0.00000019	−0.000020	0.000026	0.000020	0.000018

through $l=4$ the repulsive part of $K^{(a)}$ dominates the attractive part of $K^{(2)}$. From $l=5$ onwards the K_l is negative. The largest transition temperature actually comes from $l=6$ and

$$T_6 \sim T_F e^{-40\,000}. \tag{5.13}$$

T_F is of the order of 10^4 °K, so this is really quite low.

We have listed in Table I the asymptotic values of $a_l^{(2)} = 1/(2l)^4$. This expansion becomes accurate only at quite high l. We have also listed the value of $a_l^{(2)}$ calculated from the WKB proceedure of the last section. This gives

$$a_l^{(2)}(WKB) = 1/(2l+1)^4. \tag{5.14}$$

(This is discussed below. We need the Bohr-Sommerfeld quantization with the usual $n+\frac{1}{2}$ in the θ variable instead of n as we used in the previous section. It may be shown to be valid in this case.) The WKB approximation agrees remarkably well with the exact calculation by $l=2$, and for $l=1$, it is only about 35% off. From this we may hope to use the WKB answer as a guide in more complicated cases which cannot be treated exactly.

It is clear that if there is to be any hope of finding this type of superconductivity, one must find metals in which the effect is very much enhanced over a typical case. From the above discussion, this could come about if the coefficient of the singular part is enhanced. A relative increase[18] of factor of 10 would seem about enough. This would not only give a larger attractive part (and hence a higher transition temperature) but would also enable the transition to set in at lower l.

The singular part of $K^{(2)}$ is given by (4.5)–(4.7). Again, let us replace the potential by a δ-function potential. This makes $M_1(k) = M_2(k)$ from (4.8) and (4.9). Consequently, $K_s^{(b)}$ cancels $K_s^{(c)}$. Further,

$$M_2(k) = \lambda \int |w_{\mathbf{k}}|^2 |w_{-\mathbf{k}}|^2 d\mathbf{r} \tag{5.15}$$

where $w_{\mathbf{k}}(\mathbf{r})$ is the periodic part of the Bloch function, and λ is the strength of the δ function. Since the normalization tells us that the integral of $|w_{\mathbf{k}}|^2$ is unity and since generally $w_{\mathbf{k}}$ does not vary very strongly over most of the unit cell, it is an overestimate but not too bad a one to replace the integral by unity. For λ we previously took π^2/mk_F which is just the reciprocal of the density of states per unit volume. Introducing g_F to be the density of states at the Fermi surface we have finally

$$K_s^{(2)} \cong -\frac{1}{32\pi^2 g_F^2} \frac{1}{v_n^2(\mathbf{k})\sqrt{K(\mathbf{k})}} \times A(\mathbf{k},\mathbf{k}')[\ln A(\mathbf{k},\mathbf{k}')]. \tag{5.16}$$

We are interested in the case where $K_s^{(2)}$ dominates $K^{(1)}$, so we shall neglect the latter in estimating the effect of (5.16). This will again be an overestimate, but probably not a serious one.

Therefore, the integral equation we must study is [analogous to (4.13)]

$$\chi(\mathbf{k}) = \frac{\alpha^2}{16\pi v_n^3(\mathbf{k})(K(\mathbf{k})g_F^2)^{1/2}} \int ds' A(\mathbf{k},\mathbf{k}') \times [\ln A(\mathbf{k},\mathbf{k}')]\chi(\mathbf{k}'), \tag{5.17}$$

where α is given by (4.21).

The WKB method of the previous section gives the same result as (4.27), where U is now given by

$$U = +\sqrt{\left(\frac{EG-F^2}{g_F^2 v_n^2(\xi,\eta)}\right)}. \tag{5.18}$$

Suppose the Fermi surface is a surface of revolution. Choose for a moment the parametric equations

$$\begin{aligned} k_x &= \rho\cos\varphi, \\ k_y &= \rho\sin\varphi, \\ k_z &= f(\rho). \end{aligned} \tag{519}$$

[18] That is, between the singular part of the second-order term and the first-order term. Increasing the density of states or number of electrons increases them both in roughly the same way, and we shall therefore not consider this possibility here. This leaves the geometry of the Fermi surface as our main variable.

Direct calculation gives

$$e=-\frac{\rho f'(\rho)}{(1+f'^2)^{1/2}}, \quad \bar{E}=\rho^2$$
$$g=-\frac{f''(\rho)}{(1+f'^2)^{1/2}}, \quad \bar{G}=1+f'^2 \tag{5.20}$$
$$f=0, \quad \bar{F}=0.$$

($\bar{E}$, $\bar{G}$, $\bar{F}$ are the coefficients of the first quadratic form for the parameters ρ, φ)

$$(\bar{E}\bar{G}-\bar{F}^2)^{1/2}=\rho^+(1+f'^2), \tag{5.21}$$
$$II=\frac{-f'\rho}{(1+f'^2)^{1/2}}\left(d\varphi^2+\frac{f''}{f'\rho}d\rho^2\right). \tag{5.22}$$

Therefore, isometric coordinates can be obtained by choosing φ for one of them and ξ for the other, where

$$\left(\frac{d\xi(\rho)}{d\rho}\right)^2=\frac{f''}{f'\rho}. \tag{5.23}$$

The Hamilton-Jacobi equation (4.27) is now separable. Clearly we must choose

$$Q(\xi,\eta)=Q(\xi)+m, \tag{5.24}$$

where m is any positive or negative integer.

Then (4.27) becomes (U is a function of ξ alone)

$$\left(\frac{dQ(\xi)}{d\xi}\right)^2=p_\xi^2=\alpha U-m^2. \tag{5.25}$$

The WKB method gives

$$\oint p_\xi d_\xi=2\pi(n+\tfrac{1}{2}). \tag{5.26}$$

Actually we do not need to find ξ, since by (5.23) we may write this as

$$\oint(\alpha U-m^2)^{1/2}\left(\frac{f''}{\rho f'}\right)^{1/2}d\rho=2\pi(n+\tfrac{1}{2}), \tag{5.27}$$

where

$$U=+\left(\frac{\rho^3(1+f'^2)f'}{g_F^2 v_n^2(\rho)f''}\right)^{1/2} \tag{5.28}$$

on making use of the general result

$$(EG-F^2)^{1/2}d\xi d\varphi=(\bar{E}\bar{G}-\bar{F}^2)^{1/2}d\rho d\varphi. \tag{5.29}$$

The density of states is given by

$$g_F=\frac{1}{4\pi^3}\int dS\frac{1}{v_n(\rho)}=\frac{1}{2\pi^2}\int d\rho\rho(1+f'^2)^{1/2}\frac{1}{v_n(\rho)}$$
$$=\frac{1}{\pi^2}\int_0^{\rho_m}d\rho(\rho 1+f'^2)^{1/2}\frac{1}{v_n(\rho)} \tag{5.30}$$

(see Fig. 3).

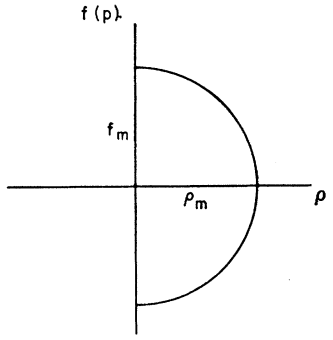

FIG. 3. Profile of Fermi surface in convex case.

It is rather clear that for a given n the "weakest potential" necessary for binding will come from $m=0$. The weakest potential corresponds to the smallest α, or to the largest transition temperature. We consider only this case, which leaves us with

$$\ln(\beta_n\epsilon_c)=(1/C)(2n+1)^4 \tag{5.31}$$

$$C\equiv\left(\frac{2}{\pi}\int_0^{\rho_m}\left[\frac{\rho(1+f'^2)f''}{v_n^2 f'}\right]^{1/4}d\rho\right)^4\Bigg/$$
$$\left(\int_0^{\rho_m}d\rho\rho\frac{(1+f'^2)^{1/2}}{v_n}\right)^2. \tag{5.32}$$

For the spherical case, C is easily calculated to give unity. This corresponds to the formula (5.14). [In fact, if one does not put $m=0$, one easily sees that we have (5.14) with $l=n+|m|$.] An enhancement of our effect due to variation in the shape of the Fermi surface would occur if we found C greater than unity.

We next have to choose $v_n(\rho)$. This is largely dependent on the energy function. It is possible that by a fortuitous combination of geometrical factors (f) and v_n, C can be considerably enhanced. However, it seems difficult to investigate this possibility systematically. We shall choose v_n in the following very simple way:

$$v_n(\rho)=(1/m_0)f(1+f'^2)^{1/2}, \tag{5.33}$$

where m_0 has the dimensions of a mass. This choice gives the correct answer for spherical and ellipsoidal Fermi surfaces, and corresponds to choosing $\epsilon_k=F(k_z^2-f^2(\rho))$. It is chosen to make further integrals as simple as possible. With (5.33), (5.32) becomes

$$C=\left(\frac{2}{\pi}\int_0^{\rho_m}d\rho\left(\frac{\rho f''}{f^2 f'}\right)^{1/4}\right)^4\Bigg/\left(\int_0^{\rho_m}d\rho\frac{\rho}{f}\right)^2. \tag{5.34}$$

The expression (5.34) for C has a remarkable property: If we multiply f by any factor λ', and ρ by any factor λ', then C is invariant. Let us change scale so that

$$\rho=\rho_m t, \quad f=f_m g(t). \tag{5.35}$$

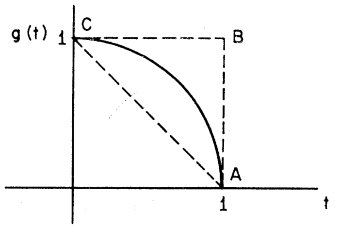

FIG. 4. Convex Ferm. surface in reduced variables.

Then

$$C=\left(\frac{2}{\pi}\int_0^1 dt\left(\frac{tg}{g^2g}\right)^{1/4}\right)^4\Big/\left(\int_0^1 dt\frac{t}{g(t)}\right)^2. \quad (5.36)$$

This expression is actually not bounded. If one takes

$$g(t)=(1-t^\nu)^{1/\nu}, \quad (5.37)$$

C is expressible in Γ function.[6] For large ν we find

$$C\sim\frac{64}{\pi^4}\nu=\frac{\nu}{1.52}. \quad (5.38)$$

Thus, in principle, the enhancement can be as large as one pleases. [Of course, for ν too large, the formula (5.36) is only valid for very large n since the Fermi surface "bends" very sharply. Further, the Fermi surface corresponding to very large ν has very flat portions—portions with small Gaussian curvature—which requires a reinvestigation of the validity of the original expressions (4.5)–(4.7) for the singular part of $K^{(2)}$, cf. below.]

We have investigated (5.36) numerically for a number of Fermi surfaces. Since they must lie in the triangle ABC (see Fig. 4) and be convex, there is not too much freedom. We have found remarkably little variation in C for different "reasonable" choices of g. Only when g practically "hugs" the triangle ABC do we get an appreciable enhancement, and in this case, the formulas are not strictly applicable.

In conclusion, it would seem that, at least for the case of "reasonable" convex Fermi surfaces, the geometry of the Fermi surface can produce very little enhancement of our effect.

The case where the Fermi surface has concave portions is much more complex. The singularity in $K^{(2)}$ can have the opposite sign, and other singularities (than at $\mathbf{k}=\mathbf{k}'$) are also possible. We hope to return to this question at a latter date.

Finally, we shall discuss the case where the Fermi surface has very flat portions. In this case (4.5)–(4.7) seem to get very large since the Gaussian curvature goes to zero. Therefore, one needs to make the expansion leading to (4.5)–(4.7) more carefully. (Of course, as long as the Gaussian curvature is not *exactly* zero these formulas will be valid when we are dealing with sufficiently highly excited states, but those are uninteresting in trying to maximize the transition temperature.) The nature of the singularities for "almost flat" Fermi surfaces is known.[19] We shall not try to discuss the problem in all generality, but shall only consider a simple model. We assume that the electrons are free to move in one direction (say the z direction) and are tightly bound in the plane perpendicular to this direction.

This corresponds to the energy expression

$$\epsilon_{\mathbf{k}}=(k_0^2/2m_0)[(k_z/k_0)^2+\gamma h(\boldsymbol{\kappa})]. \quad (5.39)$$

Here k_0 is of the order of the Fermi momentum, m_0 of the electronic mass. $\boldsymbol{\kappa}$ is the vector (k_x,k_y), $h(\boldsymbol{\kappa})$ is a function of order unity of reciprocal lattice periodicity in the $\boldsymbol{\kappa}$ plane, and $\gamma\ll 1$. The Fermi energy is given by $k_0^2/2m_0$. Therefore, the Fermi surface consists of two essentially flat pieces at $k_z=\pm k_0$, and extending to the zone walls.

The singularities of (4.1)–(4.3) occur at (for $K^{(b)}$ and $K^{(c)}$)

$$k_z''=k_z=-k_z'=\pm k_0 \quad (5.40)$$

and

$$(\text{for } K^{(d)}) \quad k_z''=k_z=k_z'=\pm k_0. \quad (5.41)$$

It is very simple[19] to obtain the leading term (for small γ) of $K^{(2)}$ for this case. Write

$$K^{(2)}(\mathbf{k},\mathbf{k}')=\frac{m_0\ln(8/\gamma)}{(2\pi)^3k_0}I^{\pm,\pm}(\boldsymbol{\kappa},\boldsymbol{\kappa}'), \quad (5.42)$$

$\mathbf{k}=(\boldsymbol{\kappa}\pm k_0)$, $\mathbf{k}'=(\boldsymbol{\kappa}',\pm k_0)$. Then

$$I^{++}(\boldsymbol{\kappa},\boldsymbol{\kappa}')=\int d\boldsymbol{\kappa}''M_{\kappa''}(\boldsymbol{\kappa},\boldsymbol{\kappa}')M_{\kappa''}{}^*(\boldsymbol{\kappa}',\boldsymbol{\kappa}), \quad (5.43)$$

$$I^{+-}(\boldsymbol{\kappa},\boldsymbol{\kappa}')=+\int d\boldsymbol{\kappa}''[\tilde{M}_{\kappa''}(\boldsymbol{\kappa},\boldsymbol{\kappa}')M_{\kappa''}{}^*(-\boldsymbol{\kappa}',\boldsymbol{\kappa}) + \tilde{M}_{\kappa''}{}^*(-\boldsymbol{\kappa}',\boldsymbol{\kappa})M_{\kappa''}(\boldsymbol{\kappa},-\boldsymbol{\kappa}') - 2\tilde{M}_{\kappa''}(\boldsymbol{\kappa},-\boldsymbol{\kappa}')\tilde{M}_{\kappa''}{}^*(-\boldsymbol{\kappa}',+\boldsymbol{\kappa})], \quad (5.44)$$

$$I^{--}(\boldsymbol{\kappa},\boldsymbol{\kappa}')=I^{++}(-\boldsymbol{\kappa},-\boldsymbol{\kappa}'), \quad (5.45)$$

$$I^{-+}(\boldsymbol{\kappa},+\boldsymbol{\kappa}')=I^{+-}(-\boldsymbol{\kappa},-\boldsymbol{\kappa}'), \quad (5.46)$$

where

$$M_{\kappa''}(\boldsymbol{\kappa},\boldsymbol{\kappa}')\equiv\langle\boldsymbol{\kappa},k_0;\boldsymbol{\kappa}''-\boldsymbol{\kappa}-\boldsymbol{\kappa}',-k_0| \times u|\boldsymbol{\kappa}'',k_0;-\boldsymbol{\kappa}',-k_0\rangle, \quad (5.47)$$

$$\tilde{M}_{\kappa''}(\boldsymbol{\kappa},\boldsymbol{\kappa}')\equiv\langle\boldsymbol{\kappa},k_0;\boldsymbol{\kappa}''-\boldsymbol{\kappa}-\boldsymbol{\kappa}',-k_0| \times u|-\boldsymbol{\kappa}',-k_0;\boldsymbol{\kappa}'',k_0\rangle. \quad (5.48)$$

[19] A. M. Afanas'ev and Y. Kagan, Zh. Eksperim. i Teor. Fiz. 43, 1456 (1962) [English transl.: Soviet Phys.—JETP 16, 1030 (1963)]. The nature of the singularities for flat and cylindrical and "almost" flat and cylindrical Fermi surfaces are dealt with in some detail in connection with anomalies in the phonon spectra of metals.

The integral equation (2.22) becomes (assuming again that the second-order terms dominate)

$$\chi^+(\kappa)=-\nu\int d\kappa'(I^{++}(\kappa,\kappa')\chi^+(\kappa') +I^{+-}(\kappa,\kappa')\chi^-(\kappa')), \quad (5.49)$$

$$\chi^-(\kappa)=-\nu\int d\kappa'(I^{-+}(\kappa,\kappa')\chi^+(\kappa') +I^{--}(\kappa,\kappa')\chi^-(\kappa')), \quad (5.50)$$

where

$$\chi^\pm(\kappa)=\chi(\kappa,\pm k_0) \quad (5.51)$$

and

$$\nu=\frac{2m_0^2\ln(\beta\epsilon_0)\ln(8/\gamma)}{(2\pi)^6k_0^2}>0. \quad (5.52)$$

ϵ_0 being of the order of the Fermi energy.

It is very easy to see, using (5.45) and (5.46), that (5.49)–(5.50) possess either even or odd solutions

$$\chi^+(\kappa)=\chi^-(-\kappa) \quad \text{(even)},$$
$$\chi^+(\kappa)=-\chi^-(-\kappa) \quad \text{(odd)}. \quad (5.53)$$

Therefore,

$$\chi^+(\kappa)=-\nu\int d\kappa'(I^{++}(\kappa,\kappa')\pm I^{+-}(\kappa,-\kappa'))\chi^+(\kappa'), \quad (5.54)$$

the plus going with the even solution, the minus with the odd one.

We shall only consider (5.54) for the tight binding limit. Then the wave functions are

$$\psi_{\mathbf{k}}(\mathbf{r})=e^{i\mathbf{k}\cdot\mathbf{r}}w(\varrho), \quad (5.55)$$

$$w(\varrho)=\frac{1}{\sqrt{n_2}}\sum_{\mathbf{l}}e^{i\kappa\cdot(\mathbf{l}-\rho)}\phi(\varrho-\mathbf{l}). \quad (5.56)$$

$\varrho=(x,y)$, n_2 is the number of atoms per unit area, $\mathbf{l}$ is a lattice vector in the x y plane, and $\phi(\rho)$ is the bound state from which the tight-binding band is formed. We have not written an index κ on w, since in the tight-binding limit ϱ must be very close to $\mathbf{l}$ because of the short range of $\phi(\varrho-\mathbf{l})$. Thus, the matrix elements are (for $\mathbf{k}_1-\mathbf{k}_1'=\mathbf{k}_2-\mathbf{k}_2'$)

$$\langle\mathbf{k}_1\mathbf{k}_2|u|\mathbf{k}_1'\mathbf{k}_2'\rangle=\int d\mathbf{r}_1d\mathbf{r}_2e^{-i(\mathbf{k}_1-\mathbf{k}_1')(\mathbf{r}_1-\mathbf{r}_2)} \times w^2(\varrho_1)w^2(\varrho_2)u(\mathbf{r}_1-\mathbf{r}_2), \quad (5.57)$$

This is clearly real (interchange $\mathbf{r}_1$ and $\mathbf{r}_2$), so in this case the matrix elements are real. Write

$$U(\kappa,k_z)\equiv\frac{1}{m_2}\sum_{\mathbf{l}}\int dk_ze^{-ik_zz}e^{-i\kappa\cdot\mathbf{l}}u(\mathbf{l},z),$$

then, in the extreme tight-binding limit,

$$\langle\mathbf{k},\mathbf{k}_2|u|\mathbf{k}_1'\mathbf{k}_2'\rangle\cong U(\kappa_1-\kappa_1',k_{1z}-k_{1z}'). \quad (5.58)$$

Since the matrix elements are real, the solution of (5.54) can be taken real. Multiply both sides of (5.54) by $\chi^+(\kappa)$ and integrate over κ over the Fermi surface. The left-hand side is then positive, and since ν is positive we can only have a solution if

$$Q^\pm\equiv\int d\kappa d\kappa'\chi^+(\kappa)(I^{++}(\kappa,\kappa')\pm I^{+-}(\kappa,\kappa'))\chi^+(\kappa') \quad (5.59)$$

is *less* than zero. Using the definitions of I^{++} and I^{+-}, we find at once

$$Q^\pm=\int d\kappa''\,\mathrm{Tr}\{A_{\kappa''}{}^2\pm2(A_{\kappa''}\tilde{A}_{\kappa''}-\tilde{A}_{\kappa''}{}^2)\}, \quad (5.60)$$

where the matrices $A_{\kappa''}$ and $\tilde{A}_{\kappa''}$ are defined by

$$A_{\kappa''}(\kappa,\kappa')=\chi^+(\kappa)M_{\kappa''}(\kappa,\kappa'), \quad (5.61)$$

$$\tilde{A}_{\kappa''}(\kappa,\kappa')=\chi^+(\kappa)\tilde{M}_{\kappa''}(\kappa,+\kappa'). \quad (5.62)$$

For the *odd* solutions, this may be written

$$Q^-\equiv\int d\kappa''\,\mathrm{Tr}\{(A_{\kappa''}-\tilde{A}_{\kappa''})^2+\tilde{A}_{\kappa''}{}^2\}>0. \quad (5.63)$$

Therefore, for the *odd* solutions, there is never superconductivity. This is just the opposite of the spherical case, where the odd solutions (l odd) were (at least for sufficiently large l) always superconducting. Thus in the case of the almost flat Fermi surface, the effect of our mechanism is to *suppress* the possibility of superconductivity in the odd states rather than to enhance it.

For the even states, one cannot say anything quite as conclusive. If again the effective potential is a delta function, then $M=\tilde{M}$, and $Q^\pm$ are both positive. We never get superconductivity from this mechanism. For long-ranged potentials, we would expect M to be larger than $\tilde{M}$ (less momentum transfer) and again $Q^\pm>0$. It is possible however that for some intermediate case $Q^\pm$ will have negative values and superconductivity will be possible.

To realistically assess this possibility is almost impossible. We need to simultaneously satisfy the condition that the second-order terms are attractive and dominate the first-order ones. Leaving out details, the correct equation (for the even case) is

$$\chi^+(\kappa)=-\frac{2m_0\ln(\beta\epsilon_0)}{(2\pi)^3k_0}\int d\kappa'\Big\{U(\kappa-\kappa',0)+U(\kappa-\kappa',2k_0) +\frac{m_0\ln(8/\gamma)}{(2\pi)^3k_0}\int d\kappa''[U(\kappa-\kappa'',0)U(\kappa''-\kappa',0) +2U(\kappa'',0)U(\kappa-\kappa',2k_0) -U^2(\kappa-\kappa',2k_0)]\Big\}\chi^+(\kappa'). \quad (5.64)$$

This integral equation is solved immediately by an ansatz of the form

$$\chi^+(\kappa)=e^{i\kappa\cdot\mathbf{l}}, \quad (5.65)$$

since the kernel depends only on $\kappa-\kappa'$ and has a reciprocal lattice periodicity. Substituting (5.65) in (5.64) and using (5.58), we find

$$1=\frac{+2m_0\ln(\beta\mathbf{l}\epsilon_0)}{(2\pi)k_0}\left\{-(u_0(\mathbf{l})+u_2(\mathbf{l}))\right.$$
$$\left.+\frac{m_0(\ln(8/\gamma))}{(2\pi)k_0}\Big(\sum_{\mathbf{l}'\neq 0,\mathbf{l}} u_2(\mathbf{l}')u_2(\mathbf{l}'-\mathbf{l})-u_0^2(l)\Big)\right\}, \quad (5.66)$$

where

$$u_0(\mathbf{l})=\int dz\, u(\mathbf{l},z),$$
$$u_2(\mathbf{l})=\int dz\, u(\mathbf{l},z)e^{-2k_0zi}, \quad (5.67)$$

and we have made use of the relationship

$$\text{Area of Fermi surface}=\text{Cross-sectional area of Brillouin zone}=(2\pi)^2n_2.$$

Both terms in (5.66) go down rapidly with $\mathbf{l}$. If there is a transition at reasonable temperature it must come for small $\mathbf{l}$. For weak potentials then the first term will always dominate and we will never get a transition. On the other hand, if the potential is not weak, but we use (5.66) anyway, a transition is possible.

Since $u(\mathbf{l})$ has the dimensions of a momentum over a mass, it certainly seems like a generous overestimate to put typically, for a small $\mathbf{l}$,

$$u_{0,2}(\mathbf{l})\sim k_0/m_0. \quad (5.68)$$

This gives, again very roughly estimated,

$$1\cong\frac{\ln(\beta\epsilon_0)}{2\pi}\left\{\frac{\ln(8/\gamma)}{2\pi}-2\right\}. \quad (5.69)$$

If we take roughly

$$\frac{\ln(8/\gamma)}{(2\pi)}\sim\tfrac{5}{2}. \quad (5.70)$$

this gives a $\gamma\sim10^{-5}$, which is a phenomenally flat Fermi surface. The transition temperature is then of the order

$$T_c\sim T_Fe^{-4\pi}\sim10^{-1}\,^\circ\mathrm{K}, \quad (5.71)$$

still quite small.

It should also be mentioned that one can show that at such flatnesses the cutoff γ in (5.69) is replaced by one of the order of $1/\beta\epsilon_F$. If we take $\epsilon_0\cong\epsilon_F$, (5.69) gives

$$T_c\sim10^{-2}\,^\circ\mathrm{K}, \quad (5.72)$$

a still smaller value.

In conclusion, it seems rather unlikely that a very flat Fermi surface can help much in producing a large transition temperature.

ACKNOWLEDGMENTS

I would like to express my thanks to the Bell Telephone Laboratories, Murray Hill, New Jersey for their hospitality during the summer of 1965, when part of this work was carried out.

I would also like to express my debt to Professor W. Kohn, with whom all these ideas were discussed and with whom many originated.

APPENDIX A

On using Rodrigues' formula

$$P_l(x)=\frac{1}{2^ll}\frac{d^l}{dx^l}(x^2-1) \quad (A1)$$

in (3.1) we obtain, on integrating by parts,

$$K_l^{(a)}=\frac{1}{2^ll}\int_{-1}^{1}(1-x^2)^l\frac{d^lK^{(a)}(x)}{dx^l}dx, \quad (A2)$$

since by assumption the singularities of $K^{(a)}(x)$ lie outside $[-1,1]$. However, since $K^{(a)}(x)$ is analytic on a circle of radius greater than unity, we have

$$\frac{d^lK^{(a)}(x)}{dx^l}=\frac{l!}{2\pi i}\oint_c\frac{K^{(a)}(z)}{(z-x)^{l+1}}dz, \quad (A3)$$

where c is a circle of radius greater than unity on which $K^{(a)}(z)$ is analytic. Therefore,

$$\left|\frac{d^lK^{(a)}(x)}{dx^l}\right|<\frac{l!M2\pi c}{2\pi(c-x)^{l+1}}=\frac{l!Mc}{(c-x)^{l+1}}, \quad (A4)$$

where $c(>1)$ is the radius of the circle, and M is the maximum of $K^{(a)}(z)$ on the circle. From (A2)

$$K_l^{(a)}\leq\frac{Mc}{2^l}\int_{-1}^{1}(1-x^2)^l\frac{dx}{(c-x)^{l+1}}$$
$$=\frac{Mc}{2^l}\int_{-1}^{1}\frac{dx}{c-x}\left(\frac{1-x^2}{c-x}\right)^l. \quad (A5)$$

For large l this integral is at once evaluated by the method of stationary phase, and gives

$$|K_l^{(a)}|<\alpha\frac{1}{\sqrt{l}}\left(\frac{1}{c^2+(c^2-1)^{1/2}}\right)^l. \quad (A6)$$

Therefore, since $c+(c^2-1)^{1/2}>1$, $|K_l^{(a)}|$ decreases at least exponentially.

Actually a much stronger result can be obtained, applying to the case where $K(x)$ is analytic inside an arbitrary ellipse with foci at ±1.[20] The P_l are Jacobi polynomials for $\alpha=\beta=0$.

[20] Cf., C. Szegö, Am. Math. Soc. Coll. Publ. 23, 238 (1939), Theorem 9.1.1.