squares fits (together with the sum rules). The resulting parameters are not appreciably affected by the presence of the new data. This seems to mean that the parameters cannot be modified in such a way as to fit the data on the real part without destroying the accord with the sum rules and with the data on the total and charge-exchange cross sections. The introduction of a third Pomeranchuk pole does not improve the situation. More accurate experimental data on the real part of the amplitude are needed for a conclusive analysis.

Concluding, we think that our best fit, No. 8, gives the most reliable values of the parameters which can be obtained from the Regge-pole hypothesis and the dispersion relations without using the cross sections at nonvanishing momentum transfer. The values obtained for the parameters a_i are given in Table III.

Note added in proof. Note, however, that J. Scanio [University of California Radiation Laboratory Report No. UCRL 16766 (unpublished)] obtains a large value for $\alpha_{P'}(0)$, namely, 0.69.

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Sum Rules and a Scalar Unitary Singlet

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The sum rules relating the axial-vector coupling constant to $\pi\pi$ and πN total cross sections are derived using the commutation relations between the chiral currents $\chi^+(t)$ and $\chi^3(t)$. The sum rules obtained are the same as those obtained by other authors using the commutation relation between $\chi^+(t)$ and $\chi^-(t)$. It is also shown that the assumption of the existence of the σ meson as a scalar unitary singlet helps in saturating the $\pi\pi$ and $K\bar{K}$ sum rules. A definite conclusion cannot be arrived at about the existence of σ meson from the $\pi\pi$ sum rule, since the sum rule can equally well be saturated by postulating a large low-energy $\pi\pi$ scattering in $I=0$ state, but the $K\bar{K}$ sum rule does seem to require the presence of a particle with properties similar to those of the σ meson. $SU(3)$ symmetry is assumed in the latter case.

I. INTRODUCTION

In the recent past, some interesting sum rules for hadrons have been derived by various workers[1–4] on the assumption of the partially conserved-axial-vector-current hypothesis (PCAC).[5,6] These sum rules, as in the case of Adler[1] and Weisberger,[2] have had spectacular success in the calculation of g_A, in remarkable agreement with experiments. Here the information on the total π^+p and π^-p scattering cross section is used as input. In other cases, like $\pi\pi$ scattering, reliable information on total cross section does not exist, and one can use the sum rules with the experimental value of g_A as input to deduce the size of the $\pi\pi$ scattering cross section. Such sum rules have been investigated by authors in Refs. 1–4. The purpose of the present paper is twofold: First, to show that the same sum rules as those of Adler[1] can be derived using current commutation relations which have not been exploited so far (the reasons will be evident later); second, to see if a scalar unitary singlet makes sense when put in the sum rules for the $\pi\pi$ and $K\bar{K}$ scattering.

[1] S. L. Adler, Phys. Rev. Letters 14, 1051 (1965); Phys. Rev. 140, B736 (1965).
[2] W. I. Weisberger, Phys. Rev. Letters 14, 1047 (1965); Phys. Rev. 143, 1302 (1966).
[3] L. K. Pandit and J. Schechter, Phys. Letters 19, 56 (1965).
[4] V. S. Mathur and L. K. Pandit, Phys. Rev. 143, 1216 (1966).
[5] M. Gell-Mann and M. Lévy, Nuovo Cimento 16, 705 (1960).
[6] Y. Nambu, Phys. Rev. Letters 4, 380 (1960).

Sections II and III are devoted to the derivation of $\pi\pi$ and $K\bar{K}$ sum rules, respectively. In Sec. IV we have looked into the implications of the existence of a scalar unitary singlet (the σ meson) with regard to the sum rules. In Sec. V the πN sum rule of Adler[1] and Weisberger[2] has been rederived using a different current commutation relation.

II. $\pi\pi$ SUM RULE

The notations for the currents and "charges" will be

$$A_j^i(t)=i\int d^3x[V_j^i(\mathbf{x},t)]_0\,, \tag{1}$$

$$B_1^k(t)=i\int d^3x[P_1^k(\mathbf{x},t)]_0\,, \tag{2}$$

where V_j^i and P_1^k are the vector and the pseudo-scalar octet of currents. The commutation relations assumed are,[3,7]

$$[A_j^i(t),A_1^k(t)]=\delta_j^kA_1^i(t)-\delta_1^iA_j^k(t)\,, \tag{3}$$

$$[B_j^i(t),B_1^k(t)]=\delta_j^kA_1^i(t)-\delta_1^iA_j^k(t)\,, \tag{4}$$

with

$$\begin{gathered}A_2^1(t)=I^+(t)\,,\quad B_2^1(t)=\chi^+(t)\,,\\ A_1^2(t)=I^-(t)\,,\quad B_1^2(t)=\chi^-(t)\,,\\ A_1^1(t)-A_2^2(t)=2I^3(t)\,,\quad B_1^1(t)-B_2^2(t)=2\chi^3(t)\,,\end{gathered} \tag{5}$$

[7] M. Gell-Mann, Physics 1, 63 (1964).

we have

$$[\chi^+(t),\ \chi^-(t)]=2I^3(t), \quad (6)$$

$$[\chi^3(t),\ \chi^+(t)]=I^+(t), \quad (7)$$

$$[\chi^-(t),\ \chi^3(t)]=I^-(t). \quad (8)$$

The PCAC hypothesis implies that[8]

$$\frac{d}{dt}\chi^\pm(t)=\frac{\sqrt{2}M_NM_\pi g_A}{gK_{\pi NN}(0)}\int d^3x \quad \varphi^\pm(x), \quad (9)$$

$$\frac{d\chi^3(t)}{dt}=\frac{M_NM_\pi^2g_A}{gK_{\pi NN}(0)}\int d^3x \quad \varphi^3(x). \quad (10)$$

Adler[1] (using the technique of Fubini and Furlan[9]) and Weisberger[2] used the commutation relations of Eq. (6). It has many virtues, mainly those of simplicity and convenience. We have, in this paper, used the commutation relation of Eq. (7). The ensuing calculation is not as straightforward as that with Eq. (6), but the same sum rules can still be obtained.

We take the matrix element of the two sides of the equation between $\langle\pi^3|$ and $|\pi^-\rangle$ states. Note that with

$$\pi^\pm=(1/\sqrt{2})(1,\pm i,0) \text{ and } \pi^3=(0,0,1),$$

we have

$$I^+(t)|\pi^-\rangle=\sqrt{2}|\pi^3\rangle,$$
$$I^+(t)|\pi^0\rangle=-\sqrt{2}|\pi^+\rangle. \quad (11)$$

With the relativistic normalization of single-particle states, the right-hand side of Eq. (7) gives

$$\langle\pi^3(q')|I^+(t)|\pi^-(q)\rangle=\sqrt{2}(2\pi)^32q^0\delta^3(\mathbf{q}'-\mathbf{q}). \quad (12)$$

Treating the left-hand side in the same manner as done by Adler,[1] we get, in the limit $q_0\to\infty$,

$$\sqrt{2}(2\pi)^32q^0\delta^3(\mathbf{q}'-\mathbf{q})\frac{M_N^2g_A^2}{g^2K^2_{\pi NN}(0)}\int_{2M_\pi}^\infty dW\sum_{\substack{\text{int}\\ \mathbf{q}=\mathbf{q}'}}$$
$$\times\frac{\delta(W-M_n)}{(M_n^2-M_\pi^2)^2}\{\langle\pi^3(q')|J^3(0)|n\rangle\langle n|J^+(0)|\pi^-(q)\rangle$$
$$-\langle\pi^3(q')|J^+(0)|n\rangle\langle n|J^3(0)|\pi^-(q)\rangle\}. \quad (13)$$

The second term in the brackets of expression (13) is *not* the imaginary part of the $\pi^+\pi^-$ scattering amplitude in the forward direction because it is not π^- which has momentum $\mathbf{q}$ in the final state but rather π^3. What we have is in fact the imaginary part of the *backward* scattering amplitude. Consequently, all partial waves will not enter the sum rule with the same sign but we except a factor $(-1)^L$ to enter the sum rule with the partial cross sections $\sigma^L(W)$. This can be seen better by realizing that

$$\sum\langle\pi^3(q')|J^+(0)|n\rangle\langle n|J^3(0)|\pi^-(q)\rangle$$

is related to the imaginary part of the scattering amplitude $\langle\pi^3(q'),\ \pi^-(k'),\ \text{out}\ |\pi^3(k),\ \pi^-(q),\ \text{in}\rangle$ by contraction over the π^3-field in the "in" state and the π^--field in the "out" state. The imaginary part of this amplitude can be written as a sum of partial cross sections. Due to the fact that, in the center of mass, we are dealing with a π^- thrown backwards, the partial cross section $\sigma^L(W)$ will appear with a factor $(-1)^L$.

A simple partial-wave analysis with π^3 off the mass shell gives, for the second bracket of Eq. (13),

$$\sum_{\text{int}}\langle\pi^3(q)|J^+(0)|n\rangle\langle n|J^3(0)|\pi^-(q)\rangle\delta(W-M_n)$$
$$=\frac{2W}{\pi}\text{Im}T_{-3,-3}(\theta=180°,W)$$
$$\approx\frac{(W^2-M_\pi^2)2W}{\pi}\frac{4\pi}{q^2}(\sin^2\delta_s^{(2)}-3\sin^2\delta_p^{(1)}), \quad (14)$$

where $\delta^{(I)}$ represents the phase shift in isobaric spin state I and the partial-wave summation is restricted to the S and the P waves only.

As for the first term in the bracket of Eq. (13), it can be related to the imaginary part of $\pi^-\pi^+\to\pi^3\pi^3$ in the "forward" direction. One may note here that because of the indistinguishibility of the final state particles there is no distinction between the "forward" and the "backward" scattering amplitudes. Consequently, only even partial waves will enter the sum rule due to this term. In fact

$$\text{Im}T_{-+,33}(\theta=0,W)=\pi\sum_{\text{int}}\frac{1}{2M_n}\langle\pi^3(q)|J^3(0)|n\rangle$$
$$\times\langle n|J^+(0)|\pi^-(q)\rangle\delta(W-M_n). \quad (15)$$

A partial-wave analysis gives

$$\text{Im}T_{-+,33}(\theta=0,W)\approx\frac{8\pi(W^2-M_\pi^2)}{3q^2}$$
$$\times(\sin^2\delta_s^{(0)}-\sin^2\delta_s^{(2)}). \quad (16)$$

The sum rule finally reads

$$1=\frac{M_N^2g_A^2}{g^2K^2_{\pi NN}(0)}\frac{1}{\pi}\int_{4M_\pi^2}^\infty\frac{dW^2}{W^2-M_\pi^2}\frac{4\pi}{q^2}$$
$$\times[3\sin^2\delta_p^{(1)}+\tfrac{2}{3}\sin^2\delta_s^{(0)}-\tfrac{5}{3}\sin^2\delta_s^{(2)}]. \quad (17)$$

This is Adler's sum rule.[10]

III. $K\bar{K}$ SUM RULE

A sum rule for $K\bar{K}$ scattering can be made if one makes the hypothesis that

$$\partial_\mu(P_3{}^1)_\mu=C_K\varphi_{K^+}, \quad (18)$$

[8] S. L. Adler, Phys. Rev. **137**, B1022 (1965).

[9] S. Fubini and G. Furlan, Physics **1**, 229 (1965).

[10] Adler's equations (73) and (77) [see Ref. 1] are in error insofar as he has $\frac{2}{3}\sigma^{1,2}$ instead of $(5/3)\sigma^{1,2}$. This does not change his conclusions. I wish to thank Dr. S. L. Adler for a private communication.

where φ_{K^+} is the K^+ field. The value of C_K determined by working out $\langle\Lambda|\partial_\mu(P_3{}^1)_\mu|p\rangle$ in the same way as done by Adler[8] is

$$C_K=\frac{g_A{}^\Lambda(M_N+M_\Lambda)M_K{}^2}{G_{\Lambda KP}}. \tag{19}$$

Now taking the expectation value of

$$\begin{aligned}[B_3{}^1(t),\ B_1{}^3(t)]&=A_1{}^1(t)-A_3{}^3(t)\\ &=Y+Q\end{aligned} \tag{20}$$

between K^+ states, and following Adler's procedure, one gets a sum rule

$$2=\frac{[g_A{}^\Lambda(M_N+M_\Lambda)M_K{}^2]^2}{G_{\Lambda KP}{}^2}\frac{1}{\pi}\int_{4M_K{}^2}^\infty\frac{dW^2}{(W^2-M_K{}^2)}\times\{\sigma_{K^+K^-}(W)-\sigma_{K^+K^+}(W)\}+(\text{single particle contributions below } K\bar{K} \text{ threshold}). \tag{21}$$

First we shall work out the single-particle contributions. Here we invoke $SU(3)$ symmetry as an additional assumption. The well-established resonances that can contribute to the summation below $K\bar{K}$ threshold are the ρ and the ω mesons. The physical ω and φ states, labeled $|\omega\rangle$ and $|\varphi\rangle$, are mixtures of a pure unitary singlet $|\omega^0\rangle$ and a pure unitary octet state $|\varphi^0\rangle$. This mixture is written as

$$\begin{aligned}|\omega^0\rangle&=|\omega\rangle\cos\theta-|\varphi\rangle\sin\theta,\\ |\varphi^0\rangle&=|\varphi\rangle\cos\theta+|\omega\rangle\sin\theta.\end{aligned} \tag{22}$$

The coupling of the ω and φ states to the $K\bar{K}$ system is given, in $SU(3)$ scheme, as[11]

$$\begin{aligned}g_{\omega K\bar{K}}&=\sqrt{3}g_{\rho K\bar{K}}\sin\theta,\\ g_{\varphi K\bar{K}}&=\sqrt{3}g_{\rho K\bar{K}}\cos\theta,\\ g_{\rho K\bar{K}}&=\tfrac{1}{2}g_{\rho\pi\pi}.\end{aligned} \tag{23}$$

In the limit ${}_\pi{}^0\to\infty$ (we have put the K meson on its mass shell, and we hope that the off-mass-shell factors for the $\rho K\bar{K}$ and the $K\Lambda P$ vertices largely cancel each other out), the contribution of the ω and ρ meson is

$$C_K{}^2g_{\rho K\bar{K}}{}^2\left[\frac{M_\rho{}^2-2M_K{}^2}{(M_\rho{}^2-M_K{}^2)^2}+3\sin^2\theta\frac{M_\omega{}^2-2M_K{}^2}{(M_\omega{}^2-M_K{}^2)^2}\right]. \tag{24}$$

Above the $K\bar{K}$-threshold with K meson on its mass shell and in the center of mass, we have

$$\sigma_{K^+K^-}(W)=\frac{\pi}{q^2}\left[\sum_L\sigma_L{}^{I=0}+\sum_L\sigma_L{}^{I=1}\right]. \tag{25}$$

We shall retain only the S and the P waves. Further we shall approximate the P-wave scattering in $I=0$ state by the φ-meson pole and neglect the P-wave scattering in $I=1$ state. We get, with K meson off the mass shell,

$$\sigma_{K^+K^-}(W)\approx\frac{\pi}{q^2}[\sigma_s{}^{I=0}(W)+\sigma_s{}^{I=1}(W)+F(W)\sigma_p{}^{I=0}(W)], \tag{26}$$

where $F(W)$ is the off-the-mass-shell correction factor. From threshold arguments similar to those of Adler's,

$$F(W)=K^2{}_{\Lambda KP}(0)q^2(0)/q^2, \tag{27}$$

where $q(0)$ is the magnitude of the 3-momentum in the center-of-mass system with $M_K=0$. An estimate of $F(W)$ gives

$$F(M_\varphi)\approx 8K_{\Lambda KP}{}^2(0). \tag{28}$$

Assuming φ meson dominance in the $I=0$, p state, we have

$$\begin{aligned}\varphi_p{}^{I=0}(W)&\approx\frac{12\pi\gamma_\varphi{}^2q^4/(q^2+M_K{}^2)}{(M_\varphi{}^2-W^2)+\gamma_\varphi{}^2q^6/(q^2+M_K{}^2)}\\ &\approx 12\pi^2\gamma_\varphi\frac{q}{(q^2+M_K{}^2)^{1/2}}\delta(W^2-M_\varphi{}^2)\end{aligned} \tag{29}$$

in the narrow resonance approximation. The relation between the reduced width γ_φ and $g_{\varphi K\bar{K}}$ is[12,13]

$$\gamma_\varphi=\frac{2}{3}\frac{g_{\varphi K\bar{K}}{}^2}{4\pi}=\frac{2g_{\rho K\bar{K}}{}^2\cos^2\theta}{4\pi}. \tag{30}$$

The sum rule, with the inclusion of the φ meson, finally reads

$$2=\frac{[g_A{}^\Lambda(M_N+M_\Lambda)]^2}{G_{\Lambda KP}{}^2}\left[g_{\rho K\bar{K}}{}^2\left\{\frac{M_\rho{}^2-2M_K{}^2}{(M_\rho{}^2-M_K{}^2)}+3\sin^2\theta\frac{(M_\omega{}^2-2M_K{}^2)}{(M_\omega{}^2-M_K{}^2)^2}+F(M_\varphi)\tfrac{3}{2}\cos^2\theta\frac{(M_\varphi{}^2-4M_K{}^2)^{1/2}}{M_\varphi(M_\varphi{}^2-M_K{}^2)}\right\}+\frac{1}{\pi}\int_{4m_K{}^2}^\infty\frac{dW^2}{(W^2-M_K{}^2)}[\sigma_s{}^{I=0}(W)+\sigma_s{}^{I=1}(W)-\sigma_{K^+K^+}(W)]\right]. \tag{31}$$

[11] It is worth pointing out that some time ago the present author made an estimate of $g_{\rho K\bar{K}}$ from $\pi\pi\to K\bar{K}$ data [Nucl. Phys. **54**, 242 (1964)]. Using $g_{\rho\pi\pi}{}^2\approx 40$, somewhat larger than the presently known value, a value $g_{\rho K\bar{K}}{}^2\approx 5$ was obtained. If a more accurate value $g_{\rho\pi\pi}{}^2\approx 25$ is used $g_{\rho K\bar{K}}{}^2$ is raised to ≈ 8, in good agreement with $SU(3)$.

[12] J. J. Sakurai, Phys. Rev. Letters **9**, 472 (1962).

[13] R. F. Dashen and D. H. Sharp, Phys. Rev. **133**, B1585 (1964).

Numerical estimates can now be made. The value of $g_A{}^\Lambda$ is known reasonably well[14] to be ≈ 0.83, and $SU(3)$ gives

$$\frac{G_{\Lambda KP}{}^2(0)}{4\pi} \approx 14 K_{\Lambda KP}{}^2(0), \qquad (32)$$

with a F/D mixing parameter $f \approx 0.35$. The off-mass-shell factor $K_{\Lambda KP}(0)$ drops out of all our numerical estimates because of our off-mass-shell correction. We shall simply ignore it. Using the $\varphi-\omega$ mixing angle[12] such that $\sin^2\theta \approx 0.38$ and $\cos^2 \approx 0.62$, we get

$$2 = 0.45 + \frac{[g_A{}^\Lambda(M_N+M_\Lambda)]^2}{G_{\Lambda KP}{}^2} \frac{1}{\pi} \int_{4M_K{}^2}^{\infty} \frac{dW^2}{W-M_K{}^2} \times [\sigma_s{}^{I=0}(W) + \sigma_s{}^{I=1}(W) - \sigma_{K^+K^+}(W)]. \qquad (33)$$

One might be tempted to conclude that there is a large S-wave $K\bar{K}$ scattering. We shall see, in the next section, that this is not necessarily so.

A work of caution must also be spoken. Because of the relatively large mass of the K meson, the off-mass-shell corrections, in the absence of a better way of handling them, become somewhat unreliable. The question we now wish to investigate is: Does the assumption of the existence of a scalar unitary singlet, the σ meson of Brown and Singer,[15] help us in any way with the $\pi\pi$ and $K\bar{K}$ sum rules?

IV. A SCALAR UNITARY SINGLET AND THE SUM RULES

First consider the $\pi\pi$ sum rule. It appears beyond doubt that the S-wave scattering length in $I=2$ state is small.[16] It may be reasonable to assume that the contribution of the $I=2$ state to the sum rule is small. We shall neglect it completely. Adler[1] has made an estimate of the ρ and the f mesons to the $\pi\pi$ sum rule, giving, with our assumed neglect of $I=2$ scattering [the reader is referred to Adler's equation (77)],

$$\frac{4M_N{}^2}{g_{\pi NN}{}^2} \frac{1}{2\pi} \int \frac{dW^2}{W^2 - M_\pi{}^2} \tfrac{2}{3}\sigma_s{}^{I=0} \approx 0.9. \qquad (34)$$

We now assume the existence of a scalar unitary singlet, the σ meson, which we shall identify with the particle postulated by Brown and Singer.[15] We can work out the left-hand side of Eq. (34) with the assumption

$$\sigma_s{}^{I=0}(W) \approx \frac{4\pi\gamma_\sigma{}^2/(q^2+M_\pi{}^2)}{(M_\sigma{}^2-W^2)^2 + \gamma_\sigma{}^2 q^2/(q^2+M_\pi{}^2)}. \qquad (35)$$

What evidence there is for the existence of the σ meson suggests that its mass is quite close to the two-pion threshold and the σ resonance is not very narrow; thus, a narrow-resonance approximation would not be a good one. We carried out the numerical integration of the left-hand side of Eq. (34) with

$$M_\sigma \approx 3M_\pi, \quad \Gamma_\sigma \approx 70 \text{ MeV}, \qquad (36)$$

implying

$$\gamma_\sigma \approx 2. \qquad (37)$$

The estimate of the left-hand side of Eq. (34) was ≈ 0.6, not in very satisfactory agreement with the right-hand side of Eq. (34). Evidently either a much broader resonance must be postulated or the position of the resonance must be pulled closer to the two-pion threshold. The numerical estimate turns out to be very sensitive to the variation in the position of σ meson. We evaluated the left-hand side of Eq. (34) with

$$M_\sigma \approx 2.67 M_\pi, \quad \Gamma_\sigma \approx 80 \text{ MeV}, \qquad (38)$$

implying

$$\gamma_\sigma \approx 2. \qquad (39)$$

The estimate for the left-hand side of Eq. (34) was ≈ 1.4. This is not in disagreement with the sum rule considering that we neglected all $I=2$ scattering.

The σ meson will also couple to the $K\bar{K}$ system. If we assume the interaction

$$\mathcal{L} = \tfrac{1}{2} i g_{\sigma\pi\pi} \sigma \varphi^\dagger \varphi + i g_{\sigma K\bar{K}} \sigma \varphi_{\bar{K}} \varphi_K, \qquad (40)$$

then by $SU(3)$ symmetry

$$g_{\sigma K\bar{K}} = g_\sigma \pi\pi. \qquad (41)$$

The additional contribution to the $K\bar{K}$ sum rule is then

$$\frac{[g_A{}^\Lambda(M_N+M_\Lambda)]^2}{G_{\Lambda KP}{}^2} \frac{g_{\sigma K\bar{K}}{}^2}{(M_K{}^2 - M_\sigma{}^2)^2} \approx 0.4 \frac{g_{\sigma K\bar{K}}{}^2}{4\pi}, \qquad (42)$$

with

$$G_{\Lambda KP}{}^2/4\pi \approx 14, \quad M_\sigma \approx 2.67. \qquad (43)$$

If we now assume the σ meson width to be ≈ 70 MeV we get[17]

$$g_{\sigma\pi\pi}{}^2/4\pi \approx 5. \qquad (44)$$

The σ contribution to the sum rule is hence ≈ 2.0.

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[14] R. P. Feynman, in *Symmetries in Elementary Particle Physics*, edited by A. Zichichi (Academic Press Inc., New York, 1965), 156–157.

[15] L. M. Brown and P. Singer, Phys. Rev. Letters **8**, 460 (1962); Phys. Rev. **133**, B812 (1964).

[16] H. J. Schnitzer, Phys. Rev. **125**, 1059 (1962); J. Kirz, J. Schwartz, and R. D. Tripp, *ibid*, **126**, 763 (1962); N. Schmitz, Nuovo Cimento **31**, 255 (1964); A. N. Kamal, Ph.D. thesis, University of Liverpool, 1962 (unpublished).

[17] L. M. Brown and H. Faier, Phys. Rev. Letters **13**, 73 (1964).

One might well ask if the assumption of a nonresonant S-wave scattering in the $\pi\pi$ and $K\bar{K}$ systems will not make large contribution to the sum rules just as σ meson does in $\pi\pi$ case and in $K\bar{K}$ case. We look at the $\pi\pi$ case first. We make the crude but (we hope) adequate zero-range approximation

$$\frac{q}{(q^2+M_\pi^2)^{1/2}}\cot\delta^{(0)}\approx-\frac{1}{a^{(0)}M_\pi}. \tag{45}$$

The fact that the integrals are expected to be dominated by the low-energy regions makes this assumption a fairly good one. With this approximation we can write Eq. (34) as

$$\frac{4M_N^2}{g_{\pi NN}^2}\frac{16}{3}\frac{a^{(0)2}}{M_\pi^2(1-3a^{(0)2})}\ln\frac{4}{3(1+a^{(0)2})}\approx 0.9. \tag{46}$$

A value of $|a^{(0)}|\approx 0.9M_\pi^{-1}$ satisfies Eq. (46). Hence as far as the $\pi\pi$ sum rule is concerned a large $I=0$ scattering length will serve to saturate the sum rule just as well as the σ meson.

If we try to do a similar calculation for the $K\bar{K}$ scattering, we do not have the same degree of success. Again let us assume a zero-range formula

$$\frac{q}{(q^2+M_K^2)^{1/2}}\cot\delta^{(I)}\approx-\frac{1}{a^{(I)}M_\pi}, \tag{47}$$

so that

$$\delta^I(q\to 0)=-a^I q\frac{M_\pi}{M_K}; \tag{48}$$

then, with the S-wave dominance,

$$\frac{1}{\pi}\int_{4M_K^2}^{\infty}\frac{dW^2}{W^2-M_K^2}[\sigma^{I=0}(W)+\sigma^{I=1}(W)]\approx\frac{4}{M_K^2}\left(\frac{a^{(0)2}}{1-3a(0)^2}\ln\frac{3}{3(1+a^{(0)2})}+(a^{(0)}\to a^{(1)}\right). \tag{49}$$

The contribution to the sum rule with

$$|a^{(0)}|\approx|a^{(1)}|\approx 3M_\pi^{-1} \tag{50}$$

is only ≈ 0.38. If we raise $|a^{(0)}|$ to $\approx 6M_\pi^{-1}$, this contribution increases to ≈ 0.55. It looks extremely unlikely that a reasonable choice of $|a^{(0)}|$ and $|a^{(1)}|$ will contribute significantly to the sum rule without any aid from the σ meson.

Our calculations, hence, show that the assumption of the existence of the σ meson with the position and width suggested by the work of Brown and Singer[15] is not in disagreement with the $\pi\pi$ and $K\bar{K}$ sum rules. In the $\pi\pi$ case, there is no way of distinguishing the effect of the σ meson from a nonresonant scattering with a large scattering length. The $K\bar{K}$ sum rule, however, cannot be saturated unless we do postulate the σ meson. The evaluation of the $K\bar{K}$ sum rule is more vulnerable to criticism owing to the largeness of K-meson mass and the uncertainties of extrapolation from zero mass to the physical mass.

As for the σ meson itself, no $\pi\pi$ calculation that does not assume its presence from the start has succeeded in producing a low-energy S-wave resonance. In fact the dynamical calculations of $\pi\pi$ scattering are weighted in favor of repulsive S-wave ($I=0$) phase shifts, even though the semiphenomenological calculations on $\pi\pi$ scattering using experimental πN data seem to imply strongly attractive S-wave ($I=0$) phase shifts.

V. THE πN SUM RULE

The sum rule relating g_A to the total cross section for π^+p and π^-p can also be obtained using the commutation rule of Eq. (7). Here we use the phase convention of Bethe[18] for the Clebsch-Gordan coefficients to be consistent with Eq. (11). Taking the matrix element of Eq. (7) between $\langle p|$ and $|n\rangle$ states one gets, after some manipulations of the kind already indicated in Sec. II,

$$\begin{aligned}2q^0(2\pi)^3\delta(\mathbf{q}'-\mathbf{q})=&\sqrt{2}\left[\frac{M_Ng_A}{gK(0)}\right]^2\\&\times(2\pi)^3\delta(\mathbf{q}'-\mathbf{q})\sum_{\substack{\text{int}\\ \mathbf{q}=\mathbf{q}_j}}\frac{1}{2q_j^0}\frac{1}{(q^0-q^0_j)^2}\\&\times[\langle p(q')|J^3(0)|j\rangle\langle j|J^+(0)|n(q)\rangle\\&-\langle p(q')|J^+(0)|j\rangle\langle j|J^3(0)|n(q)\rangle],\end{aligned} \tag{51}$$

the sum extending over both discrete states and the continuum. The discrete states contributing are a single proton state and a single neutron state. With

$$\frac{1}{(4q^0q'^0)^{1/2}}\langle p(q')|\begin{pmatrix}J^3(0)\\J^+(0)\end{pmatrix}|n(q)\rangle=\begin{pmatrix}1\\\sqrt{2}\end{pmatrix}igK[(q'-q)^2]\frac{\boldsymbol{\sigma}\cdot(\mathbf{q}'-\mathbf{q})}{2M}, \tag{52}$$

we get (note that for the π^3nn vertex $-g$ should be used) for the discrete part of the summation

$$\sqrt{2}2q^0(2\pi)^3\delta^3(\mathbf{q}'-\mathbf{q})2\sqrt{2}\left[\frac{M_Ng_A}{gK}\right]^2\left(\frac{gK}{2M}\right)^2\frac{(\mathbf{q}_j-\mathbf{q})^2}{(q_j^0-q^0)^2}, \tag{53}$$

which in the limit $q_0\to\infty$ reduces to

$$2q^0(2\pi)^3\delta^3(\mathbf{q}'-\mathbf{q})g_A^2. \tag{54}$$

This, apart from the different normalization of the states, is the same quantity as that appearing in Adler's and Weisberger's sum rules.[1,2]

[18] H. A. Bethe and F. DeHoffmann, *Mesons and Fields* (Row, Peterson, Inc., Evanston, Illinois), Vol. II, Sec. 31(g).

The contribution of the continuum can be written as follows:

$$\sqrt{2}(2\pi)^3 2q^0\delta^3(\mathbf{q}'-\mathbf{q}) \left[\frac{M_N g_A}{gK}\right]^2 \int_{M_N+M_\pi}^{\infty} dW \sum_{\text{int}} \frac{\delta(W-M_j)}{(M_j^2-M_N^2)^2} \times\{\langle p(q')|J^3(0)|j\rangle\langle j|J^+(0)|n(q)\rangle - \langle p(q')|J^+(0)|j\rangle\langle j|J^3(0)|n(q)\rangle\}. \quad (55)$$

One notices at this stage that

$$\sum_{\mathbf{q}=\mathbf{q}_j} \langle p(q')|J^3(0)|j\rangle\langle j|J^+(0)|n(q)\rangle \quad (56)$$

is related to the imaginary part of the forward amplitude (zero momentum transfer across the nucleons) for $\pi^+n\rightarrow\pi^3p$. The precise relation in the center-of-mass frame is

$$\frac{2W}{\pi}\mathrm{Im}T_{n\pi^+\rightarrow p\pi^3}(\theta=0,W)=\sum_{\text{int}} \langle p(q')|J^3(0)|j\rangle \times\langle j|J^+(0)|n(q)\rangle\delta(W-M_j). \quad (57)$$

Using charge independence and the phase convention of Bethe, the $|n\pi^+\rangle$ and $|p\pi^3\rangle$ states are written as (states $|I,I_3\rangle$ are eigenstates of I^2 and I_3):

$$|n\pi^+\rangle=\frac{1}{\sqrt{3}}[-|\tfrac{3}{2},\tfrac{1}{2}\rangle+\sqrt{2}|\tfrac{1}{2},\tfrac{1}{2}\rangle] \quad (58)$$

$$|p\pi^3\rangle=\frac{1}{\sqrt{3}}[\sqrt{2}|\tfrac{3}{2},\tfrac{1}{2}\rangle+|\tfrac{1}{2},\tfrac{1}{2}\rangle]. \quad (59)$$

Equations (58) and (59) imply that

$$\mathrm{Im}T_{n\pi^+\rightarrow p\pi^3}(\theta=0,W)=-\frac{\sqrt{2}}{3}[\mathrm{Im}T^{3/2}(\theta=0) - \mathrm{Im}T^{1/2}(\theta=0)]. \quad (60)$$

In the same manner

$$\mathrm{Im}T_{n\pi^3\rightarrow p\pi^-}(\theta=0,W)=\frac{\sqrt{2}}{3}[\mathrm{Im}T^{3/2}(\theta=0) - \mathrm{Im}T^{1/2}(\theta=0)]. \quad (61)$$

The continuum contribution finally reads

$$\sqrt{2}(2\pi)^3\delta^3(\mathbf{q}'-\mathbf{q})2q^0\left[\frac{M_N g_A}{gK}\right]^2\frac{2\sqrt{2}}{3\pi} \times\int\frac{dW^2}{(W^2-M_N^2)^2}[\mathrm{Im}T^{1/2}(\theta=0)-\mathrm{Im}T^{3/2}(\theta=0)]. \quad (62)$$

To make identification with π^+p and π^-p cross sections, one notices that charge independence implies

$$\mathrm{Im}T_{\pi^+p\rightarrow\pi^+p}(\theta=0,W)=\mathrm{Im}T^{3/2}(\theta=0,W), \quad (63)$$

$$\mathrm{Im}T_{\pi^-p\rightarrow\pi^-p}(\theta=0,W)=\tfrac{1}{3}[\mathrm{Im}T^{3/2}(\theta=0,W) +2\,\mathrm{Im}T^{1/2}(\theta=0,W)], \quad (64)$$

so that

$$\mathrm{Im}T^{1/2}(\theta=0,W)-\mathrm{Im}T^{3/2}(\theta=0,W) =\tfrac{3}{2}[\mathrm{Im}T_{\pi^-p\rightarrow\pi^-p}(\theta=0,W) -\mathrm{Im}T_{\pi^+p\rightarrow\pi^+p}(\theta=0,W)]. \quad (65)$$

The use of optical theorem (we are treating the π meson off its mass-shell) now gives

$$\mathrm{Im}T_{\pi^-p\rightarrow\pi^-p}(\theta=0,W)=(W^2-M_N^2)\sigma_{\pi^-p}(W). \quad (66)$$

Making use of Eqs. (65) and (66) in Eq. (62), we finally get the continuum contribution

$$(2q^0)(2\pi)^3\delta^3(\mathbf{q}'-\mathbf{q})\left[\frac{M_N g_A}{gK}\right]^2\frac{2}{\pi}\int_{(M_N+M_\pi)^2}^{\infty}\frac{dW^2}{W^2-M_N^2} \times[\sigma_{\pi^-p}(W)-\sigma_{\pi^+p}(W)]. \quad (67)$$

The sum rule is hence

$$1=g_A^2+\left[\frac{M_N g_A}{gK}\right]^2\frac{2}{\pi}\int_{(M_N+M_\pi)^2}^{\infty}\frac{dW^2}{W^2-M_N^2} \times[\sigma_{\pi^-p}(W)-\sigma_{\pi^+p}(W)-\sigma_{\pi^+p}(W)]. \quad (68)$$

This is the same equation as that of Adler[1] and Weisberger[2].

Note added in proof. If σ meson was treated as an isosinglet member of a scalar octet then $g_{\sigma K\bar{K}}=-\tfrac{1}{2}g_{\sigma\pi\pi}$ and the σ-meson contribution to the $K\bar{K}$ sum rule would be cut down to 0.5. Some errors in the unpublished version of this paper have been corrected.