Regge-Pole Phenomenology and Forward Dispersion Relations

M. RESTIGNOLI, L. SERTORIO,* AND M. TOLLER
Istituto Nazionale di Fisica Nucleare, Sezione di Roma, Istituto di Fisica dell'Università di Roma, Rome, Italy
(Received 10 May 1966)

Six sum rules derived from forward dispersion relations are used to improve the analysis of the forward π-N, K-N, and $\bar{K}$-N scattering amplitudes in terms of Regge poles. A numerical analysis shows that these sum rules are not in disagreement with the available data regarding total cross sections and forward charge-exchange differential cross sections. It is also shown that they are rather effective in improving the accuracy of the determination of the Regge-pole parameters. The position of the second Pomeranchuk pole and the possible existence of a third vacuum singularity are discussed.

RECENTLY, Phillips and Rarita[1] have shown that the available data on π-N, K-N, and $\bar{K}$-N scattering at high energy and low momentum transfer can be fitted using a five-Regge-pole model. These authors have determined all the Regge-pole parameters from the analysis of the high-energy experimental data, taking into account the theoretical constraints.

Our aim is to exploit some additional independent information, deriving from another firm source of constraints, that is provided by the forward dispersion relations. Thus, confining ourselves to vanishing momentum transfer, we are able to write some relations between the Regge-pole parameters and the low-energy scattering data. We shall show that a considerable amount of information can be extracted from these relations.

The method we use is a modification of the method used by Igi[2,3] to prove the necessity of introducing a second vacuum pole. We start from the assumption that above a certain energy both the imaginary and the real part of the forward amplitude can be written as a sum of Regge-pole contributions[4]. The expression that we use is the following:

$$F_\pm(E)=\sum_{i=1}^{n} a_i[i-\cot(\tfrac{1}{2}\pi\alpha_i)]\left(\frac{k}{k_0}\right)^{\alpha_i} \pm\frac{E}{k}\sum_{i=n+1}^{n+m} a_i[i+\tan(\tfrac{1}{2}\pi\alpha_i)]\left(\frac{k}{k_0}\right)^{\alpha_i}, \quad (1)$$

for

$$E>\bar{E},\quad (k_0=1\ \mathrm{GeV}/c),$$

TABLE I. Evaluation of the right-hand sides of Eqs. (2), (3), and (5).

Calculated quantity	$\bar{k}$ (GeV/c)	Contribution of the integrals (mb) Physical region	Unphysical continuum	Poles	Contribution of $D(\mu)$ (mb)	Final result (mb)
$\frac{4\pi}{k_0}U(\pi^\pm p)$	5	166.3±0.19	0	−0.65	−0.08±0.23	165.6±0.3
$\frac{4\pi}{k_0}V(\pi^\pm p)$	5	−19.8±0.2	0	1.24	0	−18.6±0.2
$\frac{4\pi}{k_0}U'(K^\pm p)$	6	125.7±0.55	6.5	−1.4 ±3.2	17.2 ±0.9	148.0±3.3
$\frac{4\pi}{k_0}V(K^\pm p)$	6	−84.4±1.95	−5.7	0.9 ±2.5	0	−89.2±3.2
$\frac{4\pi}{k_0}U'(K^\pm n)$	6	113.7±1.12	1.9	−1.4 ±3.2	7.4 ±1.3	121.6±3.6
$\frac{4\pi}{k_0}V(K^\pm n)$	6	−46.1±3.1	−1.7	0.9 ±2.5	0	−46.9±4.0

* Present address: Department of Physics, Iowa State Unversity, Ames, Iowa.
[1] R. J. N. Phillips and W. Rarita, Phys. Rev. **139**, B1336 (1965).
[2] K. Igi, Phys. Rev. **130**, 820 (1963).
[3] K. Igi, Phys. Rev. Letters **9**, 76 (1962).
[4] Note, however, that W. R. Frazer [Phys. Rev. **131**, 491 (1963)] has given some arguments in favor of the assumption that the real part approaches the value given by the Regge-pole expansion less rapidly than the imaginary part. Our point of view is a somehow stronger interpretation of the Regge-pole model.

where k and E are the momentum and the total energy of the meson in the laboratory system, and $F_\pm(E) = D_\pm(E)+iI_\pm(E)$ are the forward amplitudes for the scattering of positive and negative mesons on nucleons.

In the model that we consider, α_i for $i=1, \cdots, n$ means α_P, $\alpha_{P'}$, or α_R, and for $i=n+1, \cdots, n+m$ means α_ρ or α_ω. A similar convention has been used for the residues a_i. Equations similar to Eq. (1) have to be written for $\pi^\pm$-p, $K^\pm$-p, and $K^\pm$-n scattering. The parameters α_i are the same for the three cases, but the residues a_i are in general different. They satisfy some relations due to isotopic-spin conservation, which we take into account in our calculations, but do not write here explicitly. In particular, for $\pi^\pm$-p scattering we have $a_R=0$ and $a_\omega=0$.

As shown previously,[5] the consistency of Eq. (1) with dispersion relations requires that the following three sum rules be satisfied:

$$\sum_{i=1}^{n} \frac{a_i}{\alpha_i}\left(\frac{\bar{k}}{k_0}\right)^{\alpha_i} = U = -\tfrac{1}{4}\pi[D_+(\mu)+D_-(\mu)]+\frac{1}{2}\int_0^{\bar{E}} [I_+(E)+I_-(E)]\frac{EdE}{E^2-\mu^2}, \quad (2)$$

$$\sum_{i=n+1}^{n+m} \frac{a_i}{\alpha_i+1}\left(\frac{\bar{k}}{k_0}\right)^{\alpha_i+1} = V = \frac{1}{2}\int_0^{\bar{E}} [I_+(E)-I_-(E)]\frac{dE}{k_0}, \quad (3)$$

$$\sum_{i=n+1}^{n+m} \frac{a_i}{\alpha_i-1}\left(\frac{\bar{k}}{k_0}\right)^{\alpha_i-1} = W = -\frac{\pi k_0}{4\mu}[D_+(\mu)-D_-(\mu)]+\frac{k_0}{2}\int_0^{\bar{E}} [I_+(E)-I_-(E)]\frac{dE}{E^2-\mu^2}, \quad (4)$$

where μ is the meson mass and $\bar{k}=(\bar{E}^2-\mu^2)^{1/2}$.

Note that the functions $I_\pm(E)$ contain delta functions which give rise to pole terms which have not been written explicitly. In the case of K-N scattering, the integrals written above contain also a contribution from a nonphysical continuum. However, as can be seen in Table I, the largely dominant contribution comes from the physical zone, where the absorptive parts are given by the optical theorem.

It turns out that the sum rule (4) is not useful for our purposes because the right-hand side is a small difference between large quantities and therefore is subject to a large error. For π-N scattering we use only the conditions (2) and (3). For K-N scattering, instead of condition (2), we use a linear combination of Eqs. (2) and (4), which offers the advantage of being less dependent on the properties of the absorptive part in the unphysical region. The formula obtained in this way is

$$\sum_{i=1}^{n} \frac{a_i}{\alpha_i}\left(\frac{\bar{k}}{k_0}\right)^{\alpha_i} + \frac{\mu}{k_0}\sum_{i=n+1}^{n+m} \frac{a_i}{\alpha_i-1}\left(\frac{\bar{k}}{k_0}\right)^{\alpha_i-1} = U' = -\tfrac{1}{2}\pi D_+(\mu)+\tfrac{1}{2} \times\int_0^{\bar{E}}\left[\frac{I_+(E)}{E-\mu}+\frac{I_-(E)}{E+\mu}\right]dE. \quad (5)$$

The contributions to the expressions U, U', and V for the processes we are considering are given in Table I. The integrals over the total cross sections have been calculated numerically using the best data available. The low-energy parameters for π-N scattering have been taken from Hamilton and Woolcock[6] and the scattering lengths for K^+-N interaction have been taken from Stenger *et al.*[7] The contribution of the hyperon poles to the K-p amplitude has been taken into account using the effective pole residue given by Cook *et al.*[8] The nonphysical continuum has been treated in the same way as in Ref. 8, i.e., by extrapolation of the amplitude below the threshold using the S-wave zero-effective-range analysis of Humphrey and Ross.[9] The errors introduced by this procedure should be compensated by the effective pole residue.

The pole contributions to the K-n amplitude have to be evaluated by means of some hypothesis. They vanish for isoscalar poles, and for isovector poles they are twice the corresponding contributions to the K-p amplitude. We have arbitrarily used the same effective pole residue used for the K-p case. A more detailed analysis is not required for our purposes.

Besides the quantities U, U', and V, we have used in our analysis the total-cross-section data of Galbraith *et al.*,[10] the charge exchange π-p differential cross sections of Stirling *et al.*,[11] and Mannelli *et al.*[12] extrapolated at zero scattering angle, and the forward differential cross section for the reaction $K^-+p\rightarrow\bar{K}^0+n$ taken from Astbury *et al.*[13]

[5] L. Sertorio and M. Toller, Phys. Letters **18**, 191 (1965).
[6] J. Hamilton and W. S. Woolcock, Rev. Mod. Phys. **35**, 737 (1963).
[7] V. J. Stenger, W. E. Slater, D. H. Stork, H. K. Ticho, G. Goldhaber, and S. Goldhaber, Phys. Rev. **134**, B1111 (1964).
[8] V. Cook, D. Keefe, L. T. Kerth, P. G. Murphy, W. A. Wentzel, and T. F. Zipf, Phys. Rev. **129**, 2743 (1963).
[9] W. E. Humphrey and R. R. Ross, Phys. Rev. **127**, 1305 (1962).
[10] W. Galbraith, E. W. Jenkins, T. F. Kycia, B. A. Leontic, R. H. Phillips, A. L. Read, and R. Rubinstein, Phys. Rev. **138**, B913 (1965).
[11] A. V. Stirling, P. Sonderegger, J. Kirz, P. Falk-Variant, O. Guisan, C. Bruneton, P. Borgeaud, M. Yvert, J. P. Guillaud, C. Caverzasio, and B. Amblard, Phys. Rev. Letters **14**, 763 (1965).
[12] I. Mannelli, A. Bigi, R. Carrara, M. Wahlig, and L. Sodickson, Phys. Rev. Letters **14**, 408 (1965).
[13] P. Astbury, G. Finocchiaro, A. Michelini, C. Verkerk, D. Websdale, C. West, W. Beutsch, B. Gobbi, M. Pepin, M. Ponchon, and E. Polgar, Phys. Letters **16**, 328 (1965).

TABLE II. Results of least-squares fits.

Input	Sum rules	No.	$\alpha_{P'}$	α_ρ	α_ω	α_R	$\alpha_{P''}$	Degrees of freedom	χ^2
$\pi^\pm N$ only	No	1	0.10 ±0.36	0.559±0.032	...	...	...	22	10.25
	Yes	2	0.663±0.035	0.588±0.006	...	...	...	24	14.18
	Yes	3	0.41 ±0.15	0.588±0.006	...	...	0.11	22	12.22
$K^\pm N$ only	No	4	0.62 ±0.24	0.39 ±0.29	0.410±0.063	0.38 ±0.16	...	23	45.62
	Yes	5	0.64 ±0.10	0.38 ±0.12	0.425±0.030	0.297±0.064	...	27	47.04
	Yes	6	0.53 ±0.30	0.38 ±0.12	0.425±0.031	0.297±0.064	0.33±0.50	25	47.03
$\pi^\pm N$ and $K^\pm N$	No	7	0.45 ±0.14	0.556±0.031	0.388±0.045	0.34 ±0.18	...	47	57.67
	Yes	8	0.658±0.033	0.587±0.006	0.402±0.027	0.286±0.054	...	53	64.60
	Yes	9	0.40 ±0.13	0.587±0.006	0.406±0.029	0.289±0.054	0.07	50	62.58

The Regge-pole parameters a_i, α_i have been determined from these quantities by means of the least-squares method.

In order to test the consistency of the model and to understand which data are more effective in determining the parameters, we have performed several least-squares fits with different input data, as shown in Table II. Moreover, in order to investigate the usefulness of introducing a third vacuum pole, we have performed fits Nos. 3, 6, and 9 with the same input data used for the preceding ones, but with one more vacuum pole. From the last column, we see that the decrease of χ^2 due to the introduction of the new parameters is never sufficiently large to justify this complication of the model.

For every fit we give the resulting values of α_i. If we call the adjustable parameters x_j and take

$$A_{jk}=\frac{\partial^2\chi^2}{\partial x_j \partial x_k},$$

the errors quoted in the Table are an estimate of $[2(A^{-1})_{jj}]^{1/2}$.

The value of χ^2 should not be very different from the number of degrees of freedom. However for the first three fits it is significantly smaller, probably because the errors of the experimental points are not statistically independent. For the other fits the value of χ^2 is too large. This is due to the fact that the experimental total K-N cross sections present fluctuations which cannot be fitted by a smooth curve (see Ref. 1).

Comparing the results of fits which differ in that the sum rules are present as input data, we note a strong decrease of the errors. This is a proof of the usefulness of the sum rules.

The consistency of the sum rules with the other data is proved by the fact that χ^2 does not increase much more than the number of degrees of freedom when the sum rules are introduced. The worst case from this point of view is given by fit No. 2. Here the increase of χ^2 is significant and is related to the strong change of the value of $\alpha_{P'}$. Note that the new value of $\alpha_{P'}$ is in accord with the value obtained independently from the K-meson scattering data (fits Nos. 4–5). The consistency with the results of Ref. 1 is acceptable for α_ω and α_R, but our values of $\alpha_{P'}$ and α_ρ are greater than those given by Phillips and Rarita, and the difference is large compared with the errors. However, the errors given in Table II must not be taken seriously. This difference is not unexpected, because the high accuracy in the determination of these parameters is obtained in two completely different and independent ways in the two approaches. We increase the accuracy by using dispersion relations and a stronger interpretation of the Regge-pole hypothesis, whereas in Ref. 1 the smallness of the errors is due to the fact that they use the data at non-vanishing momentum transfer and a somewhat arbitrary expression for the dependence of the Regge-pole parameters on t.

Another kind of data regarding the high-energy pion-nucleon scattering is given by the real part of the forward scattering amplitude measured by Foley *et al.*[14] We have not used these data in the fits of Table II. The real part calculated from the parameters obtained from any of the fits Nos. 2, 3, 8, 9 does not differ significantly from the one calculated in Ref. 5, and is substantially in accord with the calculations of Barashenkov[15] for energies between 5 and 10 GeV. The accord of all these calculations with the experimental data of Ref. 14 is rather poor (see also Höhler and Baacke[16]).

We have also tried to insert these data into the least-

TABLE III. Values of the parameters $(4\pi/k_0)a_i$ (expressed in millibarns) obtained from the best fit No. 8. The values given refer to the π^+p and K^+p elastic scattering. The values for the other reactions follow from isotopic spin conservation.

Reaction	$(4\pi/k_0)\alpha_P$	$(4\pi/k_0)\alpha_{P'}$	$(4\pi/k_0)\alpha_\rho$	$(4\pi/k_0)\alpha_\omega$	$(4\pi/k_0)\alpha_R$
π^+p	17.48	17.85	−2.30	0	0
K^+p	16.35	6.81	−1.58	−7.97	2.06

[14] K. J. Foley, R. S. Gilmore, R. S. Jones, S. J. Lindenbaum, W. A. Love, S. Ozaki, E. H. Willen, R. Yamada, and L. C. L Yuan, Phys. Rev. Letters 14, 862 (1965).
[15] V. S. Barashenkov, Phys. Letters 19, 699 (1966).
[16] G. Höhler and J. Baacke, Phys. Letters 18, 181 (1965).

squares fits (together with the sum rules). The resulting parameters are not appreciably affected by the presence of the new data. This seems to mean that the parameters cannot be modified in such a way as to fit the data on the real part without destroying the accord with the sum rules and with the data on the total and charge-exchange cross sections. The introduction of a third Pomeranchuk pole does not improve the situation. More accurate experimental data on the real part of the amplitude are needed for a conclusive analysis.

Concluding, we think that our best fit, No. 8, gives the most reliable values of the parameters which can be obtained from the Regge-pole hypothesis and the dispersion relations without using the cross sections at nonvanishing momentum transfer. The values obtained for the parameters a_i are given in Table III.

Note added in proof. Note, however, that J. Scanio [University of California Radiation Laboratory Report No. UCRL 16766 (unpublished)] obtains a large value for $\alpha_{P'}(0)$, namely, 0.69.

PHYSICAL REVIEW VOLUME 150, NUMBER 4 28 OCTOBER 1966

Sum Rules and a Scalar Unitary Singlet

A. N. KAMAL
Department of Physics, University of Alberta, Edmonton, Alberta, Canada
(Received 25 April 1966)

The sum rules relating the axial-vector coupling constant to $\pi\pi$ and πN total cross sections are derived using the commutation relations between the chiral currents $\chi^+(t)$ and $\chi^3(t)$. The sum rules obtained are the same as those obtained by other authors using the commutation relation between $\chi^+(t)$ and $\chi^-(t)$. It is also shown that the assumption of the existence of the σ meson as a scalar unitary singlet helps in saturating the $\pi\pi$ and $K\bar{K}$ sum rules. A definite conclusion cannot be arrived at about the existence of σ meson from the $\pi\pi$ sum rule, since the sum rule can equally well be saturated by postulating a large low-energy $\pi\pi$ scattering in $I=0$ state, but the $K\bar{K}$ sum rule does seem to require the presence of a particle with properties similar to those of the σ meson. $SU(3)$ symmetry is assumed in the latter case.

I. INTRODUCTION

In the recent past, some interesting sum rules for hadrons have been derived by various workers[1–4] on the assumption of the partially conserved-axial-vector-current hypothesis (PCAC).[5,6] These sum rules, as in the case of Adler[1] and Weisberger,[2] have had spectacular success in the calculation of g_A, in remarkable agreement with experiments. Here the information on the total π^+p and π^-p scattering cross section is used as input. In other cases, like $\pi\pi$ scattering, reliable information on total cross section does not exist, and one can use the sum rules with the experimental value of g_A as input to deduce the size of the $\pi\pi$ scattering cross section. Such sum rules have been investigated by authors in Refs. 1–4. The purpose of the present paper is twofold: First, to show that the same sum rules as those of Adler[1] can be derived using current commutation relations which have not been exploited so far (the reasons will be evident later); second, to see if a scalar unitary singlet makes sense when put in the sum rules for the $\pi\pi$ and $K\bar{K}$ scattering.

[1] S. L. Adler, Phys. Rev. Letters 14, 1051 (1965); Phys. Rev. 140, B736 (1965).
[2] W. I. Weisberger, Phys. Rev. Letters 14, 1047 (1965); Phys. Rev. 143, 1302 (1966).
[3] L. K. Pandit and J. Schechter, Phys. Letters 19, 56 (1965).
[4] V. S. Mathur and L. K. Pandit, Phys. Rev. 143, 1216 (1966).
[5] M. Gell-Mann and M. Lévy, Nuovo Cimento 16, 705 (1960).
[6] Y. Nambu, Phys. Rev. Letters 4, 380 (1960).

Sections II and III are devoted to the derivation of $\pi\pi$ and $K\bar{K}$ sum rules, respectively. In Sec. IV we have looked into the implications of the existence of a scalar unitary singlet (the σ meson) with regard to the sum rules. In Sec. V the πN sum rule of Adler[1] and Weisberger[2] has been rederived using a different current commutation relation.

II. $\pi\pi$ SUM RULE

The notations for the currents and "charges" will be

$$A_j{}^i(t)=i\int d^3x[V_j{}^i(\mathbf{x},t)]_0\,, \tag{1}$$

$$B_1{}^k(t)=i\int d^3x[P_1{}^k(\mathbf{x},t)]_0\,, \tag{2}$$

where $V_j{}^i$ and $P_1{}^k$ are the vector and the pseudo-scalar octet of currents. The commutation relations assumed are,[3,7]

$$[A_j{}^i(t),A_1{}^k(t)]=\delta_j{}^kA_1{}^i(t)-\delta_1{}^iA_j{}^k(t)\,, \tag{3}$$

$$[B_j{}^i(t),B_1{}^k(t)]=\delta_j{}^kA_1{}^i(t)-\delta_1{}^iA_j{}^k(t)\,, \tag{4}$$

with

$$\begin{aligned} A_2{}^1(t)=I^+(t)\,,\quad B_2{}^1(t)=\chi^+(t)\,,\\ A_1{}^2(t)=I^-(t)\,,\quad B_1{}^2(t)=\chi^-(t)\,,\\ A_1{}^1(t)-A_2{}^2(t)=2I^3(t)\,,\quad B_1{}^1(t)-B_2{}^2(t)=2\chi^3(t)\,,\end{aligned} \tag{5}$$

[7] M. Gell-Mann, Physics 1, 63 (1964).