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Dispersion Calculation of the Lamb Shift*

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It is shown that the Lamb shift can be calculated by dispersion theory. In this context no infrared divergence appears. This approach leads to a calculation which is different from the conventional one, of which it can be an independent check (hopefully at least as accurate). Only the leading contributions are discussed here.

IT is generally acknowledged[1] that many difficulties of quantum electrodynamics arise from: (a) a too detailed description of space-time, and (b) the use of perturbation theory. As Feynman[2] suggested, the use of dispersion techniques might remove these difficulties. In the present paper, we apply these techniques to the calculation of the Lamb shift. Infrared divergences appears in this approach. We give here a preliminary nonrelativistic calculation. A more complete analysis will be given in forthcoming papers. It is completely independent from the field-theoretical method[3] and, we may hope, at least as accurate.

The amplitude for electron-proton scattering has poles at the energies of the hydrogen levels. We shall also take into account the following singularities:

(1) the electron-proton elastic cut,

(2) the "left-hand cut" due to the exchange of any number of photons in the t channel,

(3) inelastic cuts arising from intermediate states such as $H^*+\gamma$, and $e^-+p+\gamma$ (bremsstrahlung), and

(4) A contribution to the left-hand cut due to intermediate states (e^-e^+) in the t channel.

The Dirac amplitude $a_0=N_0/D_0$ has singularities (1) and that part of (2) which comes from the γ_μ part of the photon pole residue. Nonrelativistically, one has[4]

$$D_0(k^2)=\Gamma^{-1}(l+1-i/k)\,, \tag{1}$$

where l is the orbital angular momentum, k the momentum in Coulomb units, and Γ the Euler gamma function.

We now write the true amplitude as

$$a=N/D=a_0+a_1\,.$$

Apart from the singularities of a_0 one must take into account, for a, those of type (3) and (4), and the remaining part of (2), which is nothing but the effect of the anomalous magnetic moment. Also the positions of the poles of a are not exactly the same as those of a_0, and to calculate the shifts of these poles, we propose to study the function

$$B(k^2)=e^{-\pi/k}(ND_0-N_0D)\,. \tag{2}$$

This function[5] has no poles. It does not have the singularities of a_0 but only the other cuts that we have mentioned. Furthermore its discontinuities can be easily evaluated at first order. Therefore it can be calculated by a Cauchy formula. The shift δ_n of the nth pole is given by

$$\delta_n=-[N_0(k_n^2)D_0'(k_n^2)]^{-1}e^{\pi/k_n}B(k_n^2)\,. \tag{3}$$

All the contributions are small and we can treat them separately.

The main contribution comes from the inelastic right-hand cuts. It can be compared to the "self-energy" term of the field-theoretical calculation. Let us evaluate this term. The discontinuity of B due to inelastic processes is given by

$$\mathrm{Im}B(k^2)=(\pi^2m/k^2)\,|D_0^2|\,e^{-\pi/k}\sum_m\langle k|T|m\rangle \times\langle m|T^\dagger|k\rangle\delta[E_m-E(k)]\,, \tag{4}$$

where the quantities $\langle k|T|m\rangle$ are the transition amplitudes $e^-+p\to H^*+n\gamma$ and $e^-+p\to e^-+p+n\gamma$. They contain the normalization factor of the wave function of the state $|k\rangle$ which is $e^{\pi/2k}D_0^{-1}$ [hence the exponential factor in Eq. (2)].

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[1] J. Schwinger, *Quantum Electrodynamics* (Dover Publications, Inc., New York, 1958), Preface; see also Ref. 2.

[2] R. P. Feynman, Report of the Solvay Conference, 1961, pp. 61–91 (unpublished).

[3] H. A. Bethe, Phys. Rev. **72**, 339 (1947); N. M. Kroll and W. E. Lamb, *ibid.* **75**, 388 (1949); M. Baranger, H. A. Bethe, and R. P. Feynman, *ibid.* **92**, 482 (1953); R. Karplus, A. Klein, and J. Schwinger, *ibid.* **86**, 288 (1952).

[4] See, for instance, A. Messiah, *Mécanique Quantique* (Dunod Cie., Paris, 1959), p. 412.

[5] Our function resembles that introduced by Dashen and Frautschi. [R. Dashen and S. Frautschi, Phys. Rev. **135**, B1190 (1964)]. In fact, when D_0 is not zero we have $B\sim a_1D_0^2e^{-\pi/k}$. The exponential factor is necessary to damp out the effect of the essential singularity at threshold, in order to make the integrals exist.

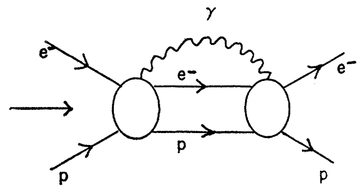

FIG. 1. Dispersion diagram representing the leading contribution due to inelasticity. This corresponds to the conventional "self-energy" term.

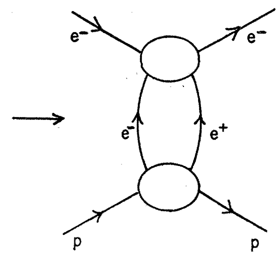

FIG. 2. Contribution of the e^-e^+ state in the t channel.

A first estimate of the order of magnitude of this effect can be obtained along the following lines:

(a) Consider only intermediate states containing one photon (Fig. 1).

(b) Use the electric dipole approximation for the transition amplitudes. The dispersion integral for B then appears as

$$B(k_n^2)=\frac{1}{\pi}\int_{s_0}^{\infty}\frac{dk^2}{k^2-k_n^2}e^{-\pi/k}|D_0^2|\frac{m}{4k^2}$$
$$\times\tfrac{2}{3}e\sum_{E_m<E(k)}[E(k)-E_m]\langle k|\mathbf{v}|m\rangle\cdot\langle m|\mathbf{v}|k\rangle; \qquad (5)$$

s_0 is the position of the lowest inelastic branch point. Note that there is no infrared divergence.

(c) Extend the summation to all states and neglect the integration between s_0 and threshold.

(d) Take into account the decrease of bremsstrahlung with energy. It acts essentially as a cutoff around the electron mass.

The result is then for the shift in energy $\Delta E_{n,l}$,

$$\Delta E_{n,l}=\delta_{l,0}(8/3\pi)(1/n^3)mZ^4\alpha^5\ln(n/Z\alpha). \qquad (6)$$

This is in striking agreement with the usual formulas.[3]

The corrections to the left-hand cut are twofold:

(a) The effect of the anomalous magnetic moment can be treated in the same spirit. It is identical to the usual result.

(b) The e^-e^+ intermediate state in the t channel (Fig. 2) will play the same role as the vacuum polarization in field theory. It can be evaluated easily by writing unitarity in the t channel and we will not have a different result from field theory.[6]

The simplicity of this calculation can be taken as an encouragement to a further study of the S matrix (as far as it exists) in quantum electrodynamics.[7,8]

One of us (J. L. B.) wishes to thank Professor S. Gorodetzky for his support.

[6] *Note added in proof.* In (b) it is necessary to subtract an infinite term proportional to the discontinuity of the one-photon pole. This is the well-known effect of charge renormalization.

There is no infrared divergence in (a) at $k=0$, due to the oscillating $e^{-i\pi/|k|}$ factor.

[7] The calculation of the anomalous magnetic moment of the electron done by Drell and Pagels [S. Drell and H. Pagels, Phys. Rev. **140**, B397 (1965)] sustains this point.

[8] *Note added in proof.* After the completion of this work, we learned of a paper on the same object by H. Abarbanel to appear in Ann. Phys. (N. Y.). We thank Dr. Abarbanel for correspondence.