ative method cannot be said to be much more complicated than the N/D.

As a final point, we would like to indicate that these results are substantially different from those obtained by Bransden *et al.*[4] in a similar calculation using almost the same input potential. Bransden *et al.* were unable to obtain trajectories rising up to $J=1$ in the $I=1$ partial wave for a pure ρ input, and were forced to include an elementary f^0 in the potential.[10]

The main differences between our calculation and theirs are improved accuracy and the different cutoff scheme. Their solution involves the smooth cutoff the potential $V^s(t,s)$ past a given s_1, and introduces no cutoff in $\rho^s(s,t)$. It can be easily checked that both these differences play an important role in the discrepancy between the two calculations. It is our impression that our cutoff procedure is the more natural one, as it does not interfere with the power blowup of the potential in the s direction, and also allows simple mathematical justification, as seen in Ref. 1.

To summarize, we can say that the above calculations seem to show that the Mandelstam iteration technique is indeed a feasible one from the computational point of view. It is quite able to produce reasonable output trajectories from a simple elementary-particle input potential, and it offers many advantages over the more usual N/D approach without an outrageous increase in the necessary computations.

At present attempts are being made at calculating fully Reggeized input potentials which will include Pomeranchuk repulsion effects. Also a more ambitious self-consistent scheme is being considered whereby the output $\rho(s,t)$ function obtained after the above iterations have been completed is used to compute a new potential $V^s(t,s)$ by means of crossing. This potential could then be used in a "macroiteration" to restart the whole calculation.

[10] This point is rather questionable, as has been pointed out by Chew [Progr. Theoret. Phys. (Kyoto) Suppl. Extra Number, 118 (1965)]: The inclusion of the f^0 as an elementary particle vastly exaggerates its effect.

ACKNOWLEDGMENTS

We would like to thank Professor G. F. Chew for innumerable discussions and suggestions throughout the course of this work. We also acknowledge valuable conversations with Shu-Yuan Chu and Dr. P. D. B. Collins.

The programming task of this work was greatly simplified by Dr. P. G. Burke, who generously made available to us the programs used in his calculations.

Finally, we thank Dr David Judd for the hospitality of the Theoretical Group of the Lawrence Radiation Laboratory.

PHYSICAL REVIEW VOLUME 150, NUMBER 4 28 OCTOBER 1966

Symmetry Predictions from Sum Rules without Saturation*

FREDERICK J. GILMAN† AND HOWARD J. SCHNITZER‡§
California Institute of Technology, Pasadena, California
(Received 23 May 1966)

The application of sum rules derived from current commutation relations to give $SU(6)$-like results is developed without the assumption of saturation. We first derive a consistency condition by comparing sum rules, and then study the sum rules for $1/g_A^2$ and the isovector charge radii, replacing the assumption of saturation with a simple dynamical model for the remaining continuuum contribution. This results in several $SU(6)$-like predictions which contain important correction factors to the results previously derived assuming saturation. We have also re-examined the experimental data for the sum rule for $\langle r_N^2\rangle^V/6$, and find that it converges below 1 GeV like the sum rule for $1/g_A^2$ when account is taken of the contribution to the sum rule of the large resonant M_{1+} and nonresonant E_{0+} multipoles in pion photoproduction, and of the $N^{**}(1520)$.

I. INTRODUCTION

RECENTLY, several exact sum rules have been derived[1,2] from the algebra of current commutators.[3] These have been used together with the available experimental data to check the agreement of the sum rules with experiment, as exemplified by the work of Adler and Weisberger.[2] Other applications have been the use of sum rules to obtain $SU(6)$-like predictions by assuming saturation of the sum rules by the low-lying 56-plet of baryons or 36-plet of mesons. However, the assumption of saturation is suspect since, for example, the value of $1/g_A^2$ derived under the assumption

* Work supported in part by the U. S. Atomic Energy Commission. Prepared under Contract No. AT(11-1)-68 for the San Francisco Operations Office, U. S. Atomic Energy Commission.
† National Science Foundation Post-Doctoral Fellow.
‡ Alfred P. Sloan Foundation Fellow.
§ On leave of absence from Brandeis University, Waltham, Massachusetts.

[1] S. Fubini and G. Furlan, Physics 1, 229 (1965); S. Fubini, G. Furlan, and C. Rossetti, Nuovo Cimento 40, 1171 (1965).

[2] S. L. Adler, Phys. Rev. Letters 14, 1051 (1965); Phys. Rev. 140, B736 (1965); W. I. Weisberger, Phys. Rev. Letters 14, 1047 (1965); Phys. Rev. 143, B1302 (1966).

[3] M. Gell-Mann, Physics 1, 63 (1964).

of saturation is a factor of 2 smaller than the experimental value obtained by Adler and Weisberger by evaluating the relevant sum rule using experimental pion-nucleon total cross sections.

In this paper we argue that saturation is not a particularly good assumption and we replace it with a simple dynamical model for those continuum contributions which are omitted in the usual saturation treatment. We show in particular that these contributions to the sum rules for $1/g_A^2$ and the isovector charge radii may give as much as 50% of the total contribution.

In Sec. II we derive a consistency relation,

$$\langle r_\pi^2\rangle/6 \approx \frac{1}{2M^2}\left(\frac{g_{\pi NN}}{G_{\rho\pi\pi}^2}\right)^2 \frac{1}{g_A^2} \approx \frac{1}{m_\rho^2},$$

by comparing two sum rules derived, assuming current commutation relations, partially conserved axial-vector currents (PCAC), and vector-meson dominance of vector currents. No assumptions are required for the continuum in the derivation of this relation, nor does the question of subtracted versus unsubtracted dispersion relations appear in Sec. II.

This relation is very useful when taken together with the results obtained in Secs. III and IV using a simple dynamical model for that part of the continuum contribution to the sum rules omitted under the saturation assumption. Among the $SU(6)$-like predictions obtained in Secs. III and IV, which contain sizable corrections from the additional continuum contribution, are

$$(\mu_p-\mu_n)_{\text{Tot}} \approx 2\sqrt{2} g_A M[\langle r_N^2\rangle^V/6]^{1/2} \approx \sqrt{2} g_A(2M/m_\rho),$$

$$(\mu_p/\mu_n)_{\text{Tot}} = -\tfrac{3}{2},$$

and

$$\langle r_\pi^2\rangle/6 \approx \frac{25}{9g_A^2}\left[\frac{3}{\alpha}\left(\frac{m_\omega}{m_\omega^2-m_\pi^2}\right)^3\right]\Gamma(\omega\to\pi^0+\gamma).$$

Finally, in Sec. V we use the available experimental information for a numerical evaluation of the sum rules for the isovector charge radii. In particular, it is shown that the sum rule for $\langle r_N^2\rangle^V/6$ converges below 1 GeV if one takes account of the contribution of both the large resonant M_{1^+} and nonresonant E_{0^+} multipoles in pion photoproduction at low energies and of the large contribution of the $N^{**}(1520)$ resonance to the sum rule. Thus, throughout this paper the emphasis is placed on avoiding the assumption of saturation by a few low-lying bound or resonant states.

II. A CONSISTENCY CONDITION FROM COMPARISON OF SUM RULES

In this section we obtain a consistency relation for the pion electromagnetic radius in terms of known coupling constants and masses from current commutation relations. The idea[4] is to relate the commutator of vector currents, whose matrix elements are taken between pseudoscalar-meson states, to the commutator of the divergence of axial-vector currents, whose matrix elements are taken between vector-meson states. The assumptions of chiral $SU(2)\times SU(2)$ equal-time commutation relations for the currents, PCAC, and vector-meson dominance of the vector form factors give two different representations for forward vector-meson–pseudoscalar-meson scattering with identical continuum contributions. When these are equated, we obtain the desired relation. The final result is not new,[4,5] but the derivation is quite different from the one given by Kawarabayashi and Suzuki, who consider the axial-vector commutator between vacuum and one-ρ-meson states and use PCAC.

First, consider the time-ordered amplitude

$$R_{\mu\nu} = \int d^4x\, e^{ip\cdot x}\langle q,a;\lambda|T(A_\mu^\alpha(x),A_\nu^\beta(0))|q,b;\sigma\rangle, \quad (1)$$

where $A_\mu^\alpha(x)$ denotes the axial-vector current density with isotopic-spin index α, and $|q,b;\sigma\rangle$ denotes a ρ-meson state with four-momentum q, isotopic-spin index b, and polarization σ. By taking $p_\mu p_\nu R_{\mu\nu}$ and using the techniques of Refs. 1 and 2 together with the algebra of chiral $SU(2)\times SU(2)$, we find the isotopic odd and even sum rules (averaged over ρ polarization $\lambda=\sigma$):

$$2(\epsilon_{\alpha\beta c}\epsilon_{abc})\left[1+\frac{c^2}{m_\pi^4}\frac{G_{\rho\pi\pi}^2K_{\pi\rho}^2(0)}{m_\rho^2}\right] = \lim_{p\to 0}\frac{\partial}{\partial\nu}\left[\left(\frac{c^2K_{\pi\rho}^2(0)}{m_\pi^4}\right)\left(\frac{i(2\pi)^3 2E_q}{K_{\pi\rho}^2(0)}\int_C d^4x \times e^{ip\cdot x}(\Box_x+m_\pi^2) \times\langle q,a;\lambda|T(\phi_\pi^\alpha(x),j_\pi^\beta(0))|q,b;\sigma\rangle\right)\right], \quad (2)$$

and

$$(-2\delta_{ab}\delta_{\alpha\beta}+\delta_{a\alpha}\delta_{b\beta}+\delta_{a\beta}\delta_{b\alpha})G_{\rho\pi\pi}^2 = \lim_{p\to 0}\left[\frac{-i(2\pi)^3 2E_q}{K_{\pi\rho}^2(0)}\int_C d^4x\, e^{ip\cdot x}(\Box_x+m_\pi^2) \times\langle q,a;\lambda|T(\phi_\pi^\alpha(x),j_\pi^\beta(0))|q,b;\sigma\rangle\right], \quad (3)$$

where $\nu=p\cdot q$, E_q is the ρ-meson energy, and the constant C arises from the use of PCAC, $\partial_\mu A_\mu^\alpha = C\phi_\pi^\alpha$, with $C=im_\pi^2 Mg_A/K_{\pi N}(0)g_{\pi NN}$.[6] The coupling constant $G_{\rho\pi\pi}$ is defined in terms of the interaction Lagrangian

$$\mathcal{L}_I = G_{\rho\pi\pi}(\phi_\pi^\alpha\partial_\mu\phi_\pi^\beta)\rho_\mu^\gamma\epsilon_{\alpha\beta\gamma},$$

[4] H. J. Schnitzer, Phys. Rev. 141, B1484 (1966).

[5] K. Kawarabayashi and M. Suzuki, Phys. Rev. Letters 16, 255 (1966); 16, 384E (1966).

[6] We write m_π, m_ρ, and M as the pion, ρ meson, and nucleon masses, respectively; g_A is the nucleon axial-vector coupling constant; $K_{\pi N}(p^2)$ is the pion-nucleon form factor with $K_{\pi N}(m_\pi^2)=1$; $g_{\pi NN}^2/4\pi=14.8$, and $G_{\rho\pi\pi}^2/4\pi\simeq 2.1$.

where ρ_μ is the ρ-meson field. $K_{\pi\rho}(0)$ is the ρ-meson–pion form factor evaluated at zero pion mass $(K_{\pi\rho}(m_\pi^2)=1)$. The subscript C in the integrals indicates that the Born terms have already been separated from the scattering amplitude.

We get another representation of the vector-meson–pion scattering amplitude if we consider Compton scattering of isovector photons from pions. By using the assumed $SU(2)$ commutation relations of vector currents, one can obtain a sum rule relating the pion electromagnetic radius to isovector photoproduction on pions.[7] Either by methods discussed by other authors,[1,2,7] or by an entirely equivalent method which considers

$$\lim_{q\to 0} q_\mu q_\nu \int d^4x\, e^{iq\cdot x}\langle p,\alpha|x_\lambda x_\sigma T(V_\mu{}^a(x),V_\nu{}^b(0))|p,\beta\rangle, \quad (4)$$

where $V_\mu{}^a(x)$ is a vector current with isotopic-spin index a, and $|p,\beta\rangle$ is a pion state with four-momentum p and isospin index β, we find, assuming the ρ meson is universally coupled to the isovector current, the isotopic odd and even sum rules (averaged over ρ polarization $\lambda=\sigma$):

$$2(\epsilon_{\alpha\beta c}\epsilon_{abc})F_\pi'(0)f_\rho^2 =\lim_{q\to 0}\frac{\partial}{\partial\nu}\left[\frac{-i(2\pi)^3 2\omega_p}{K_\rho(0)^2}\int_C d^4x\, e^{iq\cdot x}(\Box_x+m_\rho^2) \times\langle p,\alpha|T(\rho_\lambda{}^a(x),j_{\rho,\sigma}{}^b(0))|p,\beta\rangle\right], \quad (5)$$

and

$$[-2\delta_{ab}\delta_{\alpha\beta}+\delta_{\alpha a}\delta_{\beta b}+\delta_{\alpha b}\delta_{\beta a}]f_\rho^2 =\lim_{q\to 0}\left[\frac{-i(2\pi)^3 2\omega_p}{K_\rho^2(0)}\int_C d^4x\, e^{iq\cdot x}(\Box_x+m_\rho^2) \times\langle p,\alpha|T(\rho_\lambda{}^a(x),j_{\rho,\sigma}{}^b(0))|p,\beta\rangle\right]. \quad (6)$$

Here, ρ dominance of the vector current has been assumed in the form

$$\rho_\mu{}^a=\frac{f_\rho}{m_\rho^2}V_\mu{}^a. \quad (7)$$

The pion electromagnetic form factor $F_\pi(q^2)$ is normalized to $F_\pi(0)=1$, so that $F_\pi'(0)=\langle r_\pi^2\rangle/6$. The coupling constant f_ρ is the effective coupling of the ρ meson to the vector current, $j_{\rho,\sigma}{}^b$ the ρ current, and $K_\rho(0)$ is the form factor at the ρ-π-π vertex obtained by continuing the ρ meson to zero mass $[K_\rho(m_\rho^2)=1]$.

Comparing Eq. (2) with (5) and Eq. (3) with (6), it is easily seen that we now have two relations for the isotopic odd part and two relations for the isotopic even part of the forward ρ-meson–pion scattering amplitude with identical continua, since these equations just contain two equivalent forms of the reduction formula for the scattering amplitude. The continua are identical, neglecting threshold corrections, since we have continued all particles to their mass shells by means of the form factors standing in the denominators in front of the integrals. The threshold corrections are not expected to be important since there are no nearby resonances. Equating the continua in Eqs. (3) and (6), we find

$$f_\rho^2=G_{\rho\pi\pi}^2. \quad (8)$$

Equation (8) is the well-known result for the coupling of the ρ meson universally to the isospin current.[8] A more interesting result is obtained by comparing Eqs. (2) and (5), namely,

$$F_\pi'(0)=\frac{1}{M^2g_A^2}\left(\frac{g_{\pi NN}}{f_\rho}\right)^2\left(\frac{K_{\pi N}(0)}{K_{\pi\rho}(0)}\right)^2-\left(\frac{G_{\rho\pi\pi}}{f_\rho}\right)^2\frac{1}{m_\rho^2}, \quad (9)$$

and on using (8),

$$F_\pi'(0)=\frac{1}{M^2g_A^2}\left(\frac{g_{\pi NN}}{G_{\rho\pi\pi}}\right)^2\frac{K_{\pi N}^2(0)}{K_{\pi\rho}^2(0)}-\frac{1}{m_\rho^2}. \quad (10)$$

If we neglect mass-shell corrections, we have

$$F_\pi'(0)\simeq\frac{1}{M^2g_A^2}\left(\frac{g_{\pi NN}}{G_{\rho\pi\pi}}\right)^2-\frac{1}{m_\rho^2}. \quad (11)$$

Since ρ-meson dominance of the isospin current was already assumed in our derivation, we may further consistently approximate $F_\pi'(0)\approx 1/m_\rho^2$. Then from (11),

$$\frac{1}{m_\rho^2}\simeq\frac{1}{2M^2g_A^2}\left(\frac{g_{\pi NN}}{G_{\rho\pi\pi}}\right)^2\simeq F_\pi'(0), \quad (12)$$

which predicts $G_{\rho\pi\pi}^2/4\pi\simeq 3.3$ compared to the experimental result (from the ρ width) of 2.1. Our result was obtained in this form by Kawarabayashi and Suzuki.[5] We refer the reader to that paper, and in particular to their Footnotes 8 and 10, for a discussion of form factor effects.

It must be emphasized that Eqs. (7) and (9) are immediate consequences of assuming the commutation rules, PCAC, and the vector-meson pole dominance of the vector currents. There are no further assumptions such as saturation, and in particular the question of subtracted versus unsubtracted dispersion relations does not arise. If we generalize our discussion to chiral $U(3)\times U(3)$, then we can calculate the K-meson form factors in an analogous way and obtain equations similar to (12). One finds for the ratio of the K-meson

[7] S. L. Adler, unpublished, and Phys. Rev. **143**, B1144 (1966); J. D. Bjorken, unpublished, and Phys. Rev. **148**, 1467 (1966); N. Cabibbo and L. Radicati, Phys. Letters **19**, 697 (1966); R. Dashen and M. Gell-Mann in *Proceedings of the Third Coral Gables Conference on Symmetry Principles at High Energy* (W. H. Freeman and Company, San Francisco, California, 1966).

[8] M. Gell-Mann and F. Zachariasen, Phys. Rev. **124**, 953 (1961).

isovector and isoscalar radii

$$\frac{\langle r_K^2\rangle^V}{\langle r_K^2\rangle^S}=\frac{m_\omega^2}{3m_\rho^2}\frac{G_{\rho KK}^2}{G_{\omega KK}^2}\xrightarrow[SU(3)\text{ Exact}]{}1. \quad (13)$$

This of course is an immediate consequence of the assumption of vector-meson pole dominance of the vector currents alone, and does not depend on PCAC or the commutation relations, in contrast to the derivation of Eq. (9).

III. SYMMETRY PREDICTIONS FROM CHIRAL $SU(2)\times SU(2)$ SUM RULES

Let us now discuss sum rules which follow from chiral $SU(2)\times SU(2)$ so as to obtain $SU(6)$-like symmetry predictions without assuming saturation. Our treatment of the continuum contributions will be very similar in spirit to an earlier discussion of the axial-vector sum rules.[9] The integrals over cross sections will be divided into two parts, a low-energy continuum contribution dominated by a single resonance, and the part omitted under the saturation assumption for which we assume a simple dynamical model. These assumptions allow us to obtain relations involving only low-energy parameters and resonances, particularly when combined with the consistency condition derived in Sec. II.

In this section, we will only make use of sum rules which are odd under charge conjugation of the meson variables since: (1) the integrals over cross sections in the sum rules converge rapidly because of the Pomeranchuk theorem, (2) the dispersion relations do not seem to require subtractions, and (3) similar dynamical assumptions made for the continuum in even charge-conjugation relations are probably incorrect. All the sum rules discussed in this and the next section are derived using the methods discussed in Refs. 1, 2, and 7. Furthermore, we assume ρ-meson dominance of the isospin current so that when vector currents appear, they are related to strong-interaction currents and then to strong-interaction cross sections for zero-mass ρ mesons. We will neglect the mass shell and threshold corrections on the assumption that these will not introduce serious errors.[2]

We begin with the result derived by Cabibbo and Radicatti and others,[7] which when written for zero-mass ρ mesons is

$$\frac{\langle r_N^2\rangle^V}{6}=\frac{1}{2}\left(\frac{\mu_p-\mu_n}{2M}\right)^2+\frac{1}{\pi f_\rho^2}\int_{\nu_0}^{\infty}\frac{d\nu}{\nu}\times[\sigma(\rho^-p)-\sigma(\rho^+p)], \quad (14)$$

where

$$\frac{\langle r_N^2\rangle^V}{6}=\frac{\langle r_p^2\rangle-\langle r_n^2\rangle}{6}=G_{Ep}'(0)-G_{En}'(0)+\frac{1}{8M^2}, \quad (15)$$

with G_{Ep} (G_{En}) the proton (neutron) Sachs form factors and μ_p (μ_n) the proton (neutron) *total* magnetic moment. We also consider the Adler-Weisberger sum rule,[2]

$$\frac{1}{g_A^2}=1+\frac{2M^2}{g_{\pi NN}^2}\frac{1}{\pi}\int_{\nu_0}^{\infty}\frac{d\nu}{\nu}[\sigma(\pi^-p)-\sigma(\pi^+p)). \quad (16)$$

We explicitly separate the **33** resonance from the integrals in Eqs. (14) and (16), while for the residual (i.e., continuum) contribution we assume the validity of the simple quark model of scattering of Lipkin and Scheck.[10] An alternative model which gives the same results in the cases of interest to us here is a model which assumes that, in the odd amplitude for forward πN and ρN scattering, the t channel is dominated by the *same* octet coupled to the isospin current. In either model we have

$$\int_C\frac{d\nu}{\nu}[\sigma(\pi^-p)-\sigma(\pi^+p)]=\int_C\frac{d\nu}{\nu}[\sigma(\rho^-p)-\sigma(\rho^+p)], \quad (17)$$

which we assume for energies above the **33** resonance region.

The dynamical models for the continuum are not intended to give a "high-energy tail" to the low-energy resonance. Rather, their most important influence is in an *intermediate* region where there are still several distinct resonances [such as the $N^{**}(1520)$ and $N^{***}(1690)$]. Thus our model is to be thought of as giving a smooth estimate for the cross sections, taken as an *average* over the resonances occurring at intermediate energies. Of course, detailed information about the spins and parities of the intermediate energy states is lost in our treatment, but our goal is to study low-energy parameters rather than to predict new resonances, so that this is not an active concern to us. Also, since we make our dynamical assumption for forward scattering only, we need *not* assert that real vector-meson exchange is implied, which would give poor predictions for nonforward scattering, but rather we are assuming states which have certain transformation properties. For this reason, the Lipkin-Scheck model has greater appeal for us, particularly if one is ready to believe that the s-channel intermediate-baryon and meson resonances can be deduced from bound states of quarks.

If we now multiply Eq. (16) by $(1/2M^2)(g_{\pi NN}/f_\rho^2)$ and subtract it from Eq. (14), using Eq. (17) to cancel the continuum contributions, we have

$$\langle r_N^2\rangle^V/6-\frac{1}{2}\left(\frac{g_{\pi NN}}{g_Af_\rho M}\right)^2=\left[\frac{1}{2}\left(\frac{\mu_p-\mu_n}{2M}\right)^2-\frac{1}{2M^2}\left(\frac{g_{\pi NN}}{f_\rho}\right)^2\right]-\frac{2}{3}\frac{1}{f_\rho^2}\frac{1}{\pi}\int_R\frac{d\nu}{\nu}\times[\sigma(\rho p\to I=\tfrac{3}{2})-\sigma(\pi p\to I=\tfrac{3}{2})], \quad (18)$$

[9] H. J. Schnitzer, Phys. Letters **20**, 539 (1966).

[10] H. J. Lipkin and F. Scheck, Phys. Rev. Letters **16**, 71 (1966).

where the R indicates that the integrals extend from threshold over the low-energy resonance region (in this case, over the **33** resonance).[11] The $\sigma(\rho p)$-resonance contribution is entirely elastic and so for zero-mass ρ mesons we may use the photoproduction theory of Chew, Goldberger, Low, and Nambu.[12] Assuming that the photoproduction goes by a pure $M1$ transition,[13] we write

$$\sigma(\rho^0 p\to\rho^0 p)_{33}=\left(\frac{f_\rho}{g_{\pi NN}}\right)^2\left(\frac{\mu_p-\mu_n}{2}\right)^2\sigma(\pi^0 p\to\pi^0 p)_{33}. \quad (19)$$

Equation (18) then becomes

$$\frac{\langle r_N^2\rangle^V}{6}-\frac{g_{\pi NN}^2}{2M^2g_A^2f_\rho^2}=\left[\frac{1}{2}\left(\frac{\mu_p-\mu_n}{2M}\right)^2-\frac{1}{2M^2}\left(\frac{g_{\pi NN}}{f_\rho}\right)^2\right]\times\left[1-\frac{4M^2}{3g_{\pi N}^2}\frac{1}{\pi}\int_R\frac{d\nu}{\nu}\sigma(\pi p\to I=\tfrac{3}{2})\right]. \quad (20)$$

Direct numerical evaluation[13] of the second bracket on the right-hand side of Eq. (20) gives

$$\left[1-\frac{4M^2}{3g_{\pi N}^2}\frac{1}{\pi}\int_R\frac{d\nu}{\nu}\sigma(\pi p\to I=\tfrac{3}{2})\right]\simeq 0.5,$$

whereas the left-hand side is rather small experimentally when evaluated neglecting mass shell corrections to the coupling constants. In the ρ-dominance model, we have been assuming

$$\frac{1}{m_\rho^2}\simeq\frac{\langle r_N^2\rangle^V}{6}\simeq\frac{\langle r_\pi^2\rangle^V}{6}\simeq\frac{g_{\pi NN}^2}{2M^2g_A^2f_\rho^2}, \quad (21)$$

where the last approximate equality comes from the consistency condition of Sec. II. Thus the ρ-dominance model requires the left-hand side of Eq. (20) to be zero. To the same order of accuracy expected for Eq. (21), we have

$$\mu_p-\mu_n\simeq 2g_{\pi N}/f_\rho, \quad (22a)$$

$$\mu_p-\mu_n\simeq 2\sqrt{2}Mg_A/m_\rho, \quad (22b)$$

$$\mu_p-\mu_n\simeq 2\sqrt{2}Mg_A(\langle r_N^2\rangle^V/6)^{1/2}. \quad (22c)$$

The alternative forms of Eq. (22)[14] are obtained using the consistency relation of Sec. II and the ρ-dominance assumption. The resulting values for $\mu_p-\mu_n$ are 5.3, 4.1, and 5.9 nucleon magnetons, respectively, compared to the experimental value of 4.7. The range of values, 5.0 ± 0.9, gives a rough indication of the errors involved in the assumption of ρ dominance, omitting mass-shell corrections, and our continuum model. Also, we note the striking similarity of (22c) to the Dashen–Gell-Mann–Lee[15] sum rules, where our result differs in that $5/3\to\sqrt{2}g_A$, which is numerically very close. The factor $\sqrt{2}$ comes from the factor of 2 difference between Eq. (14) and the rest-frame sum rule, while g_A replaces 5/3, since saturation is not used.

Similar considerations can also be applied to the mesons. By methods identical to those already discussed, we find for the pion electromagnetic radius,[7]

$$F_\pi'(0)=\frac{1}{2f_\rho^2}\frac{1}{\pi}\int_{\nu_0}^\infty\frac{d\nu}{\nu}[\sigma(\rho^-\pi^+)-\sigma(\rho^+\pi^+)], \quad (23)$$

and for the isovector K-meson radius,

$$F_{K^+}'(0)-F_{K^0}'(0)=\frac{\langle r_K^2\rangle^V}{6}=\frac{1}{f_\rho^2}\frac{1}{\pi}\int_{\nu_0}^\infty\frac{d\nu}{\nu}\times[\sigma(\rho^-K^+)-\sigma(\rho^+K^+)]. \quad (24)$$

As before, we separate the low-energy resonance in both Eqs. (23) and (24) and use our continuum model which implies in this case

$$\int_C\frac{d\nu}{\nu}\tfrac{1}{2}[\sigma(\rho^-\pi^+)-\sigma(\rho^+\pi^+)]=\int_C\frac{d\nu}{\nu}[\sigma(\rho^-K^+)-\sigma(\rho^+K^+)].$$

Then subtracting Eq. (24) from (23), we have[16]

$$\frac{\langle r_\pi^2\rangle}{6}-\frac{\langle r_K^2\rangle^V}{6}=\frac{1}{6\pi f_\rho^2}\int_R\frac{d\nu}{\nu}\sigma(\rho\pi\to I=0)-\frac{2}{3f_\rho^2}\frac{1}{\pi}\int_R\frac{d\nu}{\nu}\sigma(\rho K\to I=\tfrac{1}{2}). \quad (25)$$

If $SU(3)$ is used to relate the ω and K^* contributions to the right-hand side of Eq. (25), we of course find that the right-hand side is zero (provided we take a degenerate pseudoscalar octet and degenerate vector nonet).

As an amusing exercise, we have computed the N^*, ω, and K^* low-energy resonance contributions to Eqs. (14), (23), and (24) using the simple quark model[17] and the narrow-resonance approximation. Inserting the

[11] We use the notation $\sigma(\rho p\to I=\frac{3}{2})$ to denote the contribution of $I=\frac{3}{2}$ states to the sum rule, where the relevant Clebsch-Gordan coefficients have already been separated out. Similarly, we later use $\sigma(\rho p\to 10)$ for the decuplet contribution to sum rules obtained from the chiral $SU(3)\times SU(3)$ current commutation relations.

[12] G. F. Chew *et al.*, Phys. Rev. **106**, 1345 (1957).

[13] See Sec. V for numerical details and corrections due to nonresonant multipoles.

[14] Equations (22c) and (37) have been derived by F. Buccella, G. Veneziano, and R. Gatto, Nuovo Cimento **42**, 1019 (1966), using the assumption of saturation where $g_A=5/3$, and also without the saturation assumption, but using an argument which seems to us somewhat weak (see the beginning of Sec. V). $SU(6)$-like results for magnetic moments have also been derived assuming saturation by S. Fubini, G. Segrè, and J. D. Walecka, Ann. Phys. (N.Y.) (to be published).

[15] R. F. Dashen and M. Gell-Mann, Phys. Letters **17**, 142 (1965); B. W. Lee, Phys. Rev. Letters **14**, 676 (1965).

[16] The low-energy resonances contributing to Eqs. (23) and (24) are the ω and K^*, respectively. The process $\phi\to\rho+\pi$ is forbidden in the quark model and very small in vector-mixing theories and experimentally (see Ref. 17).

[17] See the lectures of R. H. Dalitz, in *High Energy Physics*, edited by C. DeWitt and M. Jacob (Gordon and Breach, Science Publishers, Inc., New York, 1965), and references therein.

results of this computation into the sum rules, we obtain

$$\frac{\langle r_N^2\rangle^V}{6}=\frac{25}{18}\left(\frac{\mu_p}{2M}\right)^2-\frac{8}{9}\left(\frac{\mu_p}{2M}\right)^2+\frac{1}{f_\rho^2\pi}\int_C\frac{d\nu}{\nu}\times[\sigma(\rho^-p)-\sigma(\rho^+p)],\quad(26a)$$

$$\frac{\langle r_\pi^2\rangle}{6}=\frac{1}{2}\left(\frac{\mu_p}{2M}\right)^2+\frac{1}{2f_\rho^2\pi}\int_C\frac{d\nu}{\nu}\times[\sigma(\rho^-\pi^+)-\sigma(\rho^+\pi^+)],\quad(26b)$$

$$\frac{\langle r_K^2\rangle^V}{6}=\frac{1}{2}\left(\frac{\mu_p}{2M}\right)^2+\frac{1}{f_\rho^2\pi}\int_C\frac{d\nu}{\nu}\times[\sigma(\rho^-K^+)-\sigma(\rho^+K^+)],\quad(26c)$$

where C denotes the integral over the continuum part omitted under the usual saturation assumption. We see then that the contributions from the low-energy region to $\langle r_N^2\rangle^V$, $\langle r_\pi^2\rangle^V$, and $\langle r_K^2\rangle^V$ are all equal in the quark model. Furthermore, since within the simple quark model for scattering of Lipkin and Scheck

$$\int_C\frac{d\nu}{\nu}[\sigma(\rho^-p)-\sigma(\rho^+p)]=\int_C\frac{d\nu}{\nu}\tfrac{1}{2}[\sigma(\rho^-\pi^+)-\sigma(\rho^+\pi^+)]=\int_C\frac{d\nu}{\nu}[\sigma(\rho^-K^+)-\sigma(\rho^+K^+)]=\int_C\frac{d\nu}{\nu}[\sigma(\pi^-p)-\sigma(\pi^+p)],\quad(27)$$

the continua in Eq. (26) are also equal, so that $\langle r_N^2\rangle^V=\langle r_\pi^2\rangle=\langle r_K^2\rangle^V$. Moreover, since the contribution to $\langle r^2\rangle/6$ from the low-energy region is $\frac{1}{2}(\mu_p/2M)^2\approx1/2m_\rho^2$ and we expect from experiment that $\langle r^2\rangle/6\gtrsim1/m_\rho^2$, the contribution from the remaining part of the continuum must be about 50% of the sum rules (26a–c).[18]

Similarly, inserting quark-model predictions for the resonance contribution to Eq. (16), we find

$$\frac{1}{g_A^2}=1-\frac{16}{25}+\frac{2M^2}{g_{\pi NN}^2}\frac{1}{\pi}\int_C\frac{d\nu}{\nu}[\sigma(\pi^-p)-\sigma(\pi^+p)].\quad(28)$$

Since the experimental value of $1/g_A^2\simeq18/25$, again the continuum integral contributes $\approx50\%$ to the right-hand side of the sum rule, i.e.,

$$\frac{2M^2}{g_{\pi NN}^2}\frac{1}{\pi}\int_c\frac{d\nu}{\nu}[\sigma(\pi^-p)-\sigma(\pi^+p)]\simeq1/2g_A^2\quad(29)$$

in the quark model.

[18] The importance of the continuum contributions above the N^* region, particularly to the sum rule for $1/g_A^2$, has been repeatedly stressed by R. Dashen and M. Gell-Mann (see Ref. 7).

We use our continuum model, Eq. (27), to relate the integral in Eq. (29) to an integral over $\sigma(\rho^-\pi^+)-\sigma(\rho^+\pi^+)$, apply the consistency condition of Sec. II, and find that the continuum contribution to $\langle r_\pi^2\rangle/6$ is given by $\langle r_\pi^2\rangle/12$. We thus verify for this model that the continuum contribution to the sum rule for the pion charge radius gives 50% of the total contribution, as we had expected. The sum rule, Eq. (26a), now reads $\langle r_\pi^2\rangle/6=\frac{1}{2}(\mu_p/2M)^2+\langle r_\pi^2\rangle/12$, or

$$\frac{\langle r_N^2\rangle^V}{6}=\frac{\langle r_\pi^2\rangle^V}{6}=\frac{\langle r_K^2\rangle^V}{6}\simeq\left(\frac{\mu_p}{2M}\right)^2=\frac{1.25}{m_\rho^2}.\quad(30)$$

Thus, when the quark model is assumed for both the low-energy resonance and for the remaining continuum contribution, we can assert: (1) the equality of the isovector charge radii without ρ dominance, and (2) the high-energy continuum contributes 50% of the value of $\langle r^2\rangle^V/6$ and $1/g_A^2$, with the low-energy resonance contributions being $\frac{1}{2}(\mu_p/2M)^2$ and 9/25, respectively.

Still following the same line of thought, we are motivated by the quark model for scattering to equate continua of other sum rules and obtain more relations. For example, by comparing the continua in Eqs. (23) and (16), we find

$$\left[\frac{g_{\pi NN}^2}{2M^2g_A^2}-f_\rho^2\frac{\langle r_\pi^2\rangle}{6}\right]=\frac{g_{\pi NN}^2}{2M^2}-\frac{2}{3}\frac{1}{\pi}\int_R\frac{d\nu}{\nu}\langle(\pi p\to I=\tfrac{3}{2})-\frac{1}{6}\frac{1}{\pi}\int_R\frac{d\nu}{\nu}\sigma(\rho\pi\to I=0),\quad(31)$$

which is again an equation involving only low-energy parameters. On using the consistency relation of Eq. (11), this becomes

$$\frac{g_{\pi NN}^2}{2M^2}=\frac{2}{3}\frac{1}{\pi}\int_R\frac{d\nu}{\nu}\sigma(\pi p\to I=\tfrac{3}{2})+\frac{1}{6}\frac{1}{\pi}\int_R\frac{d\nu}{\nu}\sigma(\rho\pi\to I=0).\quad(32)$$

From direct integration over the N^* and ω resonances, we find (see Sec. V) that the right-hand side is about 80% of the left-hand side.

IV. PREDICTIONS FROM $SU(3)\times SU(3)$ SUM RULES

In this section we study the symmetry predictions that can be obtained from sum rules derived using the algebra of chiral $SU(3)\times SU(3)$. Our procedure will be identical to that of Sec. III in that we separate the continuum contribution to the sum rules into a low-energy part containing a single resonance and a residual part for which we use the continuum model discussed in detail in Sec. III.

We begin with the generalization of Eq. (14) to $SU(3)$ (involving zero-mass vector mesons),

$$\left[G_E'(0)_F+G_E'(0)_D+\frac{1}{8M^2}\right]=\frac{1}{2}\left[\frac{G_M(0)}{2M}\right]^2+\frac{1}{f_\rho^2}\frac{1}{\pi}\int_{\nu_0}^{\infty}\frac{d\nu}{\nu}[\sigma(\rho^-p)-\sigma(\rho^+p)],\quad(33a)$$

$$\left[G_E'(0)_F-G_E'(0)_D+\frac{1}{8M^2}\right]=\tfrac{1}{2}(1-2\beta)^2\left[\frac{G_M(0)}{2M}\right]^2+\frac{1}{f_\rho^2}\frac{1}{\pi}\int_{\nu_0}^{\infty}\frac{d\nu}{\nu}[\sigma(\bar{K}^{*0}p)-\sigma(K^{*0}p)],\quad(33b)$$

$$\left[G_E'(0)_F+\frac{1}{8M^2}\right]=\tfrac{1}{2}(1-2\beta+\tfrac{4}{3}\beta^2)\left[\frac{G_M(0)}{2M}\right]^2+\frac{1}{2f_\rho^2}\frac{1}{\pi}\int_{\nu_0}^{\infty}\frac{d\nu}{\nu}[\sigma(\bar{K}^{*-}p)-\sigma(K^{*-}p)],\quad(33c)$$

where $G(q^2)_F$ $[G(q^2)_D]$ is the F $[D]$-type Sachs form factor, and $G_{E,M}(q^2)=G_{E,M}(q^2)_F+G_{E,M}(q^2)_D$. Proceeding as in Sec. III and using our continuum model we write[19]

$$\int_C\frac{d\nu}{\nu}[\sigma(\rho^-p)-\sigma(\rho^+p)]=\int_C\frac{d\nu}{\nu}[\sigma(\bar{K}^{*0}p)-\sigma(K^{*0}p)]=\int_C\frac{d\nu}{\nu}\tfrac{1}{2}[\sigma(K^{*-}p)-\sigma(K^{*+}p)],\quad(34)$$

for energies above the low-lying decuplet of $\frac{3}{2}^+$ resonances, we find

$$\frac{1}{\pi}\int_R\frac{d\nu}{\nu}\sigma(\rho N\to 10)=\frac{8}{3}\beta^2f_\rho^2\left(\frac{G_M(0)}{2M}\right)^2,\quad(35a)$$

and

$$G_E'(0)_D=\beta\left(1-\frac{5}{3}\beta\right)\left(\frac{G_M(0)}{2M}\right)^2.\quad(35b)$$

Both sides of Eq. (35b) are very small, as the charge radius of the neutron is small and $\beta=-\frac{3}{2}\mu_n/(\mu_p-\mu_n)\simeq 3/5$; but putting in the experimental numbers we find the right-hand side of Eq. (35b) is an order of magnitude smaller than the left.

If we again make the assumption that the decuplet photoproduction is a pure $M1$ transition via the $SU(3)$ generalization of Eq. (19), and use the result[9]

$$\frac{1}{\pi}\int_R\frac{d\nu}{\nu}\sigma(\pi N\to 10)=\frac{24}{25}\left(\frac{g_{\pi NN}}{M}\right)^2,\quad(36)$$

obtained from the Adler-Weisberger sum rules by using the same assumptions about the continuum made in this paper, then we are able to conclude[14]

$$\beta=\tfrac{3}{5}\quad\text{or}\quad\frac{G_M(0)_D}{G_M(0)_F}=+\tfrac{3}{2},\quad\text{and}\quad G_E'(0)_D=0,\quad(37)$$

which are the standard $SU(6)$ results for the total baryon moments and the neutron charge radius.

It is also interesting to re-examine Eq. (32) using Eq. (36) for the decuplet contribution. One then finds

$$\frac{1}{\pi}\int_R\frac{d\nu}{\nu}\sigma(\rho\pi\to I=0)=\frac{27}{25}\frac{g_{\pi NN}^2}{M^2}.\quad(38)$$

In narrow-resonance approximation, the ω contribution to the integral on the left-hand side of Eq. (38) is

$$\frac{1}{\pi}\int_R\frac{d\nu}{\nu}\sigma(\rho\pi\to I=0)=\frac{18f_\rho^2}{\alpha}\left(\frac{m_\omega}{m_\omega^2-m_\pi^2}\right)^3\times\Gamma(\omega\to\pi\gamma).\quad(39)$$

Equating (38) and (39) and using Eq. (12), we find finally

$$\frac{\langle r_\pi^2\rangle}{6}=\frac{25}{9g_A^2}\left[\frac{3}{\alpha}\left(\frac{m_\omega}{m_\omega^2-m_\pi^2}\right)^3\right]\Gamma(\omega\to\pi^0+\gamma)\quad(40)$$

which, except for the factor of $25/9g_A^2\simeq 2$ contributed by the continuum as a correction, is identical to the result of Cabibbo and Radicati,[7] who assume saturation of Eq. (23) by the ω state. This is essentially the same factor of 2 which appeared in Sec. III, where the continuum accounted for 50% of the total contribution to $\langle r_\pi^2\rangle/6$ and $1/g_A^2$. From the experimental width,[20] $\Gamma(\omega\to\pi^0+\gamma)=1.3\pm0.3$ MeV, we predict

$$\frac{\langle r_\pi^2\rangle}{6}\simeq\frac{(0.045\pm0.010)}{m_\pi^2}=\frac{(1.4\pm0.3)}{m_\rho^2},$$

which is to be compared to the experimental value[21] of

$$\frac{\langle r_\pi^2\rangle}{6}=\frac{(0.041_{-0.020}^{+0.026})}{m_\pi^2}.$$

V. COMPARISON OF THE SUM RULES FOR CHARGE RADII WITH EXPERIMENT

We now turn to a detailed numerical evaluation of the sum rules for isovector charge radii which follow from

[19] Equations (34) are of the Johnson-Treiman type. See, for example, H. Harari and H. J. Lipkin, Phys. Rev. Letters **15**, 983 (1965) and references therein for a detailed discussion of this type of relation.

[20] A. H. Rosenfeld *et al.*, Rev. Mod. Phys. **37**, 633 (1965).

[21] C. W. Akerlof *et al.*, Phys. Rev. Letters **16**, 147 (1966).

assuming that the isospin-vector current densities form the algebra of $SU(2)$. With this assumption, we find the sum rules for isovector photon scattering[7] which were stated earlier under the assumtpion of ρ dominance (which is not assumed here):

$$\frac{\langle r_N^2\rangle^V}{6}=\frac{1}{2}\left(\frac{\mu_p-\mu_n}{2M}\right)^2+\frac{1}{\pi e^2}\int_{\nu_0}^{\infty}\frac{d\nu}{\nu}\times[\sigma_T(\gamma^-p)-\sigma_T(\gamma^+p)],\quad (41a)$$

$$\frac{\langle r_\pi^2\rangle}{6}=\frac{1}{2\pi e^2}\int_{\nu_0}^{\infty}\frac{d\nu}{\nu}[\sigma_T(\gamma^-\pi^+)-\sigma(\gamma^+\pi^+)],\quad (41b)$$

$$\frac{\langle r_K^2\rangle^V}{6}=\frac{1}{\pi e^2}\int_{\nu_0}^{\infty}\frac{d\nu}{\nu}[\sigma_T(\gamma^-K^+)-\sigma_T(\gamma^+K^+)],\quad (41c)$$

where, as before,

$$\langle r_N^2\rangle^V/6=G_{Ep}'(0)-G_{En}'(0)+1/8M^2,$$

$$\langle r_\pi^2\rangle/6=F_\pi'(0),$$

and

$$\langle r_K^2\rangle^V/6=F_{K^+}'(0)-F_{K^0}'(0),$$

but now $\nu=$ the photon lab energy, and $\gamma^+=(\gamma_1+i\gamma_2)/\sqrt{2}$ and $\gamma^-=(\gamma_1-i\gamma_2)/\sqrt{2}$ are isovector photons corresponding to $\gamma^0=\gamma_3$, the isovector part of the real photon. We note that there are several ways to derive these relations which differ by having the $\langle r^2\rangle$ and $(\mu_p-\mu_n)^2$ terms distributed in different ways between the contribution of the Born pole in the scattering amplitude and the contribution coming from the current commutator, but that all methods lead to the same result, Eq. (41).

Let us rewrite Eq. (41a), using an isospin rotation, as

$$\frac{\langle r_N^2\rangle^V}{6}=\frac{1}{2}\left(\frac{\mu_p-\mu_n}{2M}\right)^2+\frac{1}{4\pi^2\alpha}\int_{\nu_0}^{\infty}\frac{d\nu}{\nu}\times[2\sigma_T(\gamma^0+p\to I=\tfrac{1}{2})-\sigma_T(\gamma^0+p\to I=\tfrac{3}{2})],\quad (42)$$

where the notation is that of Cabibbo and Radicati.[7] Experimentally,[22] $\langle r_N^2\rangle^V/6=0.066/m_\pi^2$ and

$$\tfrac{1}{2}((\mu_p-\mu_n)/2M)^2=0.059/m_\pi^2.$$

The **33**-resonance [$N^*(1238)$] contribution to the integral is alone thus clearly of the wrong sign to satisfy the sum rule.

To obtain the magnitude of the contribution of the integral from threshold to $\nu=500$ MeV, which includes the region where the **33** resonance gives a large contribution, we have directly integrated over the photoproduction total cross sections using a theory which fits the low-energy photoproduction data fairly well. The theory consists in assuming that the dominant multipoles in this region are E_{0^+} and M_{1^+}. The total-cross-section data for $\gamma+p\to\pi^0+p$ are in fact very well fit in this region simply by assuming dominance by the **33** resonance in the M_{1^+} multipole.[23] For $\gamma+p\to\pi^++n$, however, there is in addition to the contribution of the **33** resonance in the M_{1^+} multipole a large s-wave (E_{0^+} multipole) contribution from the isovector photon Born terms, which roughly contributes equally with the **33** resonance at energies of a few hundred MeV.[23] On examining the isotopic-spin coefficients, we find that the E_{0^+} contribution to $\sigma_T(\gamma+p\to I=\frac{1}{2})$ is twice that to $\sigma_T(\gamma+p\to I=\frac{3}{2})$, so that it gives a positive contribution to the right-hand side of (42). Furthermore, due to its s-wave character, the E_{0^+} multipole will make a large contribution close to threshold in the integral in Eq. (42).

With the above theory, the contributions to $\langle r_N^2\rangle^V/6$ from the region from threshold to 500 MeV are, from $|M_{1^+}|^2$,

$$\frac{1}{4\pi^2\alpha}\int_{\nu_0}^{500}\frac{d\nu}{\nu}[-\tfrac{3}{2}\sigma_T(\gamma+p\to\pi^0+n)]=-\frac{0.028}{m_\pi^2},\quad (43)$$

and from $|E_{0^+}|^2$,

$$\frac{1}{4\pi^2\alpha}\int_{\nu_0}^{500}\frac{d\nu}{\nu}[\sigma_T(\gamma+p\to\pi^++n)-\tfrac{1}{2}\sigma_T(\gamma+p\to\pi^0+p)]=+\frac{0.016}{m_\pi^2},\quad (44)$$

where the numbers given are obtained from a direct integration over the total cross sections.[24,25] We can get a rough alternative estimate of the contribution of the **33** resonance to the sum rule by noting that in the CGLN theory of photoproduction,[12] the contribution of the **33** resonance to

$$\int\frac{d\nu}{\nu}[\sigma_T(\pi^-p)-\sigma(\pi^+p)]$$

should be the same as

$$\int\frac{d\nu}{\nu}\sigma(\gamma+p\to\pi^0+n)$$

except for a scale factor which we can set by the ratio of the peak in the total-cross-section curves. The relevant integral over pion-nucleon cross sections in the **33** region is contained in the work of Adler and Weisberger.[2]

[22] L. H. Chan *et al.*, Phys. Rev. **141**, B1298 (1966).

[23] M. Gell-Mann and K. M. Watson, Ann. Rev. Nucl. Sci. **4**, 219 (1954). Also see the detailed calculation of the multipoles and their fit to the experimental data in W. Schmidt, Z. Physik **182**, 76 (1964).

[24] W. S. McDonald *et al.*, Phys. Rev. **107**, 577 (1957); R. L. Walker *et al.*, *ibid.* **99**, 210 (1955). Also see the total cross-section data in Ref 23 and references given therein.

[25] R. L. Walker (private communication).

By such a scaling procedure we obtain $-0.030/m_\pi^2$ for

$$\frac{1}{4\pi^2\alpha}\int\frac{d\nu}{\nu}[-\tfrac{3}{2}\sigma_T(\gamma+p\to\pi^0+p)]$$

from Adler's calculation, where a correction for zero-mass pions has been made, and $-0.026/m_\pi^2$ from Weisberger's calculation, where such a correction is not made.

From the numbers given in Eqs. (43) and (44), we see that the **33** resonance makes a large contribution to the sum rule of the wrong sign, but that roughly 60% of it is canceled by the large, but nonresonant, s-wave contribution. If we were to now truncate the integral and neglect continuum contributions from above 500 MeV, the sum rule would read

$$\frac{0.066}{m_\pi^2}=\frac{\langle r_N^2\rangle^V}{6}\overset{?}{=}\frac{0.059}{m_\pi^2}+\frac{0.016}{m_\pi^2}-\frac{0.028}{m_\pi^2}=\frac{0.047}{m_\pi^2},\quad(45)$$

so that the right-hand side is roughly 30% too small when the integral is so truncated. It is amusing that although the numbers come from very different terms, this is fairly close to the Dashen–Gell-Mann–Lee result,[15]

$$\frac{0.066}{m_\pi^2}=\frac{\langle r_N^2\rangle^V}{6}\overset{?}{=}\left(\frac{\mu_p}{2M}\right)^2=\frac{0.041}{m_\pi^2},\quad(46)$$

derived from commutation relations between states at rest and assuming saturation by the N and N^* with the $SU(6)$ values for matrix elements.

To get an estimate of the contribution to the sum rule from above 500 MeV, we have again gone to the photoproduction total-cross-section curves. As the second resonance, $N^{**}(1520)$, is $I=\frac{1}{2}$ and is excited by isovector and not isoscalar photons,[25] it makes a positive contribution to the right-hand side of Eq. (52). Integrating numerically over the peak above the 50 μb background for $\sigma_T(\gamma+p\to\pi^++n)$, we find a contribution to $\langle r_N^2\rangle^V/6$ of

$$\frac{1}{4\pi^2\alpha}\int_{500\,\mathrm{MeV}}^{800\,\mathrm{MeV}}\frac{d\nu}{\nu}3\sigma_T(\gamma+p\to N^{**}\to\pi^++p)=+\frac{0.008}{m_\pi^2}.\quad(47)$$

As the second resonance is roughly 50% inelastic,[26] i.e., there is an equal contribution to $\sigma_T(\gamma+p\to N^{**})$ from $\sigma_T(\gamma+p\to N^{**}\to N+\pi+\pi)$, we estimate the total contribution to the sum rule from the region of the second resonance to be $+0.016/m_\pi^2$.

If we assume the third resonance, $N^{***}(1690)$, is also excited by isovector photons, which is not an established experimental fact, we find by a similar numerical integration over the resonant peak in $\gamma+p\to\pi^++n$ from 900 to 1100 MeV a contribution to $\langle r_N^2\rangle^V/6$ of $0.002/m_\pi^2$. It thus appears that contributions to the sum rule are rapidly falling off.

An alternative way to estimate the contribution from above $\nu=500$ MeV would be to use the ρ dominance and continuum model discussed above in Secs. III and IV. In this model we have

$$\frac{1}{4\pi^2\alpha}\int_c\frac{d\nu}{\nu}[\sigma_T(\gamma^-p)-\sigma_T(\gamma^+p)]=\frac{1}{\pi f_\rho^2}\int_c\frac{d\nu}{\nu}[\sigma(\rho^-p)-\sigma(\rho^+p)]=\frac{1}{\pi f_\rho^2}\int_c\frac{d\nu}{\nu}[\sigma(\pi^-p)-\sigma(\pi^+p)].\quad(48)$$

Taking the last integral from the calculation of Adler and Weisberger and using $f_\rho^2/4\pi\simeq2.1$ gives a contribution to $\langle r_N^2\rangle^V/6$ from above 500 MeV of $\simeq+0.015/m_\pi^2$, in fairly good agreement with the numbers calculated above by direct integration over the resonances, and also of the right magnitude to satisfy the sum rule (42).

Combining the various parts of our calculation, we find for the sum rule (42),

$$\frac{0.066}{m_\pi^2}=\frac{\langle r_N^2\rangle^V}{6}=\underset{N}{\frac{0.059}{m_\pi^2}}+\underset{|E_{0^+}|^2}{\frac{0.016}{m_\pi^2}}-\underset{N^*}{\frac{0.028}{m_\pi^2}}+\underset{N^{**}}{\frac{0.016}{m_\pi^2}}$$
$$+(\text{contributions from above 800 MeV}).\quad(49)$$

This should be contrasted with the Adler-Weisberger relation where we have roughly

$$0.7=\frac{1}{g_A^2}=\underset{N}{1}-\underset{N^*}{0.5}+0.2\quad(50)$$

(contribution of continuum above N^* region).

By comparison, it is interesting to note that: (1) both sum rules seem to converge below $\nu=1$ GeV; (2) when written as in Eqs. (49) and (50), the contribution from above the region of the **33** resonance is important in both cases and is 30 to 40% of the combined contribution from the nucleon and integral over the low-energy region; but (3) for the $\langle r_N^2\rangle^V/6$ sum rule there is an important nonresonant contribution to the sum rule at low energy which is neither in the Adler-Weisberger sum rule, nor in $SU(6)$ or representation mixing theories.

A similar numerical evaluation of the isotopically rotated sum rules coming from Eqs. (41b) and

[26] L. D. Roper *et al.*, Phys. Rev. **138**, B190 (1965); B. H. Bransden *et al.*, Phys. Letters **11**, 339 (1964); P. Auvil *et al.*, Phys. Letters **12**, 76 (1964). Also see L. D. Roper and R. M. Wright, Lawrence Radiation Laboratory (Livermore) Report UCRL-7846 (1964).

(41c),

$$\frac{\langle r_\pi^2\rangle^V}{6}=\frac{1}{8\pi^2\alpha}\int_{\nu_0}^{\infty}\frac{d\nu}{\nu}[\sigma_T(\gamma^0+\pi^0\to I=0)$$
$$+\sigma_T(\gamma^0+\pi^+\to I=1)-5/4\sigma_T(\gamma^0+\pi^0\to I=2)], \quad (51)$$

and

$$\frac{\langle r_K^2\rangle^V}{6}=\frac{1}{4\pi^2\alpha}\int_{\nu_0}^{\infty}\frac{d\nu}{\nu}[2\sigma_T(\gamma^0+K^+\to I=\tfrac{1}{2})$$
$$-\sigma_T(\gamma^0+K^+\to I=\tfrac{3}{2})], \quad (52)$$

is impossible owing to our lack of data for $\gamma+\pi$ and $\gamma+K$ cross sections, but we can proceed to give a rough evaluation from the known decay widths of meson resonances into $\gamma+\pi$ and $\gamma+K$, or into $\rho+\pi$ and $\rho+K$ together with a ρ-dominance model of the decay into $\gamma+\pi$ and $\gamma+K$. It is interesting that all the established meson resonances have isospins such that they contribute positively to the right-hand sides of Eqs. (51) and (52).

In the case of Eq. (51), the possible established resonances contributing to the sum rule are the ω, ϕ, and A_2. For (52), it is the $K^*(891)$ and $K^{**}(1405)$. From theories of $\phi-\omega$ mixing,[17] we know that $\phi\to\pi^0+\gamma^0$ is forbidden, or nearly so. Using Breit-Wigner resonance shapes for the total cross sections,

$$\sigma^I(s)=(2\pi)(2l+1)\frac{\Gamma_{R\to M+\gamma}{}^I\Gamma_{\rm tot}{}^I/k^2}{(s-s_R)^2+s_R\Gamma_{\rm tot}{}^2}, \quad (53)$$

where s is the total center-of-mass energy of the photon and meson, k is the photon momentum, and l is the orbital angular momentum, we find using the total widths of Rosenfeld *et al.*,[20] that the contributions of the ω and A_2 to $\langle r_\pi^2\rangle/6$ are $(0.018/m_\pi^2)\Gamma(\omega\to\pi+\gamma^0)$ and $(0.0077/m_\pi^2)\Gamma(A_2^+\to\pi^++\gamma^0)$ (widths in MeV). Similarly, the contributions to $\langle r_K^2\rangle^V/6$ from the K^* and K^{**} are $(0.15/m_\pi^2)\Gamma(K^{*+}\to K^++\gamma^0)$ and $(0.41/m_\pi^2)\times\Gamma(K^{**+}\to K^*+\gamma^0)$. As a check, we have calculated these same contributions in the narrow-resonance approximation, obtaining $(0.014/m_\pi^2)\Gamma(\omega\to\pi^0+\gamma)$, $(0.0071/m_\pi^2)\Gamma(A_2^+\to\pi^++\gamma^0)$,

$$(0.13/m_\pi^2)\Gamma(K^{*+}\to K^++\gamma^0),$$

and $(0.29/m_\pi^2)\Gamma(K^{**+}\to K^++\gamma^0)$, respectively.

For $\Gamma(\omega\to\pi+\gamma)$, we use the value 1.27 ± 0.30.[20] We calculate $\Gamma(A_2^+\to\pi^++\gamma^0)$ from $\Gamma(A_2^+\to\pi^++\rho^0)$ using the ρ-dominance model for the decay. As the decay is through D wave, we have

$$\Gamma(A_2^+\to\pi^++\gamma)=\left(\frac{e}{f_\rho}\right)^2\left(\frac{k_\gamma}{k_\rho}\right)^5$$
$$\times\Gamma(A_2^+\to\pi^++\rho^0)=1.2\text{ MeV}. \quad (54)$$

The matrix elements for $K^{*+}\to K^++\gamma^0$ and $K^{**+}\to K^*+\gamma^0$ are calculated from the matrix elements for $\omega\to\pi+\gamma$ and $A_2\to\pi+\gamma$ using $SU(3)$. We then find $\Gamma(K^{*+}\to K^*+\gamma^0)=0.17$ MeV and $\Gamma(K^{**+}\to K^++\gamma^0)=0.25$ MeV.

The resulting values for $\langle r_\pi^2\rangle^V/6$ and $\langle r_K^2\rangle^V/6$ are

$$\frac{\langle r_\pi^2\rangle}{6}=\underset{\omega}{\frac{0.023}{m_\pi^2}}+\underset{A_2}{\frac{0.009}{m_\pi^2}}=\frac{0.032}{m_\pi^2}, \quad (55)$$

and

$$\frac{\langle r_K^2\rangle^V}{6}=\underset{K^*}{\frac{0.026}{m_\pi^2}}+\underset{K^{**}}{\frac{0.010}{m_\pi^2}}=\frac{0.036}{m_\pi^2}. \quad (56)$$

The value obtained for $\langle r_\pi^2\rangle/6$ is very close to $1/m_\rho^2=0.033/m_\pi^2$ and agrees within errors with the value of $r_\pi=0.70\pm0.20$ F, or $\langle r_\pi^2\rangle/6=(0.041_{-0.020}^{+0.026})/m_\pi^2$ given by Akerlof *et al.*[21] It is, however, smaller by a factor of two than $\langle r_N^2\rangle^V/6$, which we conjectured in Sec. III should equal $\langle r_\pi^2\rangle/6$. It would be interesting to have a more accurate value for $\langle r_\pi^2\rangle/6$ to test this conjecture and to see whether either further resonant contributions to Eqs. (51) and (52), such as an octer of axial-vector mesons,[27] or nonresonant contributions as for $\langle r_N^2\rangle^V/6$, are required to bring the right-hand sides of Eqs. (51) and (52) into agreement with experiment for the left-hand sides.

ACKNOWLEDGMENTS

The authors would like to thank their colleagues at Caltech for interesting discussions during the course of this work. We particularly wish to thank Dr. R. L. Walker for discussions and for making available unpublished photoproduction total-cross-section data. One of us (HJS) would like to thank the California Institute of Technology for its hospitality during his visit.

[27] Assuming that the A_1 meson exists and that $\Gamma(A_1\to\rho+\pi)=125$ MeV, we calculate $\Gamma(A_1^+\to\pi^++\gamma)=2.5$ MeV using the ρ-dominance model for the decay, and find a contribution to $\langle r_\pi^2\rangle/6$ of $\simeq0.016/m_\pi^2$. There could, of course, be important contributions to the sum rule from $\gamma+\pi$ in the 1^+ state without an actual resonance in that state.