Single-Channel Calculation of $\pi\pi$ Scattering Using the Mandelstam Iteration*

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The Mandelstam iteration with appropriate cutoff is shown to be a practical technique for the study of strong-interaction dynamics in the framework of the strip approximation. Comparison with known potential (nonrelativistic) scattering problems shows that the method is accurate enough to allow workable numerical calculations. Calculations of single-channel relativistic $\pi\pi$ scattering with an elementary ρ potential are reported.

I. INTRODUCTION

IN a previous paper[1] the Mandelstam iteration technique was analyzed in the context of the strip approximation and shown to be, in principle, a workable method for the calculation of strongly interacting amplitudes. In this paper we describe some preliminary calculations which use this technique to study a single-channel zero-spin case (π-π), and which assures us that this approach is numerically feasible.

In Sec. II we describe the calculation method in detail. Section III is devoted to a comparison of solutions of potential problems obtained by the iteration method and by integration of Schrödinger's equation. Section IV describes some preliminary calculations in the fully relativistic π-π problem. These solutions are compared with solutions of the equivalent problem obtained by the N/D technique of the "new strip approximation."[2]

II. CALCULATION METHOD

The iteration technique involves the integration of the pair of coupled equations,[3]

$$\rho^s(s,t)=\frac{g(s)}{2\pi q_s^2(s)}\int\int dt'dt''\frac{M_t^*(t',s)M_t(t'',s)}{K^{1/2}[q_s^2(s);t,t',t'']}, \quad (2.1)$$

$$M_t(t,s)=V_t^s(t,s)+\frac{1}{\pi}\int_{s_0}^{\infty}\frac{ds'}{s'-s}\rho^s(s',t), \quad (2.2)$$

with

$$K(q^2;y,y',y'')=y^2+y'^2+y''^2-2(yy'+yy''+y'y'')-(yy'y'')/q^2) \quad (2.3)$$

and

$$g(s)=q_s(s) \quad (2.4)$$

for potential scattering, and

$$g(s)=(2q_s(s)/(s)^{1/2})h(s,s_1) \quad (2.5)$$

for relativistic scattering, where

$$q_s^2(s)=\tfrac{1}{4}s-1. \quad (2.6)$$

Here $h(s,s_1)$ is a cutoff function equal to unity below s_1 and rapidly going to zero above s_1. As usual t' and t'' integrals in Eq. (2.1) are carried over the region where K is positive.

A detailed study of the solution of a similar set of equations has been made by Bransden *et al.*,[4] and it is continued here.

A computer program designed to solve these equations has been written, and it operates as follows: Given an initial potential discontinuity function $V_t^s(t,s)$ for all s and t, it can, using (2.1), compute $\rho^s(s,t)$ for a limited range of t. Equation (2.2) then allows one to compute $M_t(t,s)$ for this same range of t, which upon return to Eq. (2.1), can be further extended. The trick is that, because of the nature of the region of integration in Eq. (2.1), to compute $\rho^s(s,t)$ for t, say, equal to t_1, only values of $M_t(t,s)$ for t less than t_1 are required. This "iteration" process can, in principle, be repeated indefinitely. However, after a sufficient number of iterations, we can expect that the power behavior of the discontinuity function $M_t^s(t,s)$ will emerge, dominated by the leading Regge pole in s,

$$M_t^s(t,s)\approx\beta(s)t^{\alpha(s)} \quad (2.7)$$

and it is unnecessary to proceed any further. The trajectory function $\alpha(s)$ and residue function $\beta(s)$ can then be obtained from the relations

$$\begin{aligned}\ln|M_t^s(t,s)|&=\ln|\beta(s)|+\mathrm{Re}\alpha(s)\ln t,\\ \arg(M_t^s(t,s))&=\arg(\beta(s))+\mathrm{Im}\alpha(s)\ln t,\end{aligned} \quad (2.8)$$

by simple least-squares straight-line fit to $\ln|M_t^s(t,s)|$ and $\arg(M_t^s(t,s))$ over a sufficiently large range of $\ln t$. The functions $\alpha(s)$ and $\beta(s)$ can now be used to define by analytic continuation the scattering amplitude

$$M^s(s,t)=\frac{1}{\pi}\int\frac{M_t(t,s)}{t'-t}dt' \quad (2.9)$$

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[1] Naren F. Bali, Geoffrey F. Chew, and Shu-Yuan Chu, preceding paper, Phys. Rev. **150**, 1352 (1966).

[2] G. F. Chew, Phys. Rev. **129**, 2363 (1963); G. F. Chew and C. E. Jones, *ibid.* **135**, B208 (1964).

[3] We use the notation of Ref. 1.

[4] B. M. Bransden, P. G. Burke, J. W. Moffat, R. G. Moorhouse, and D. Morgan, Nuovo Cimento **30**, 207 (1963).

even if the above integral does not converge, as will be the case if a resonance is present.

The program to carry out this calculation is reasonably straightforwward, if a bit complicated. Accuracy in the integrals is of great importance if stable solutions are to be obtained. Particular care has to be exercised in Eq. (2.1) close to the boundaries of the t' and t'' integrals, as the denominator vanishes there like an inverse square root.

As is evident from this description of the calculation, only the leading trajectory is detected. In principle, once this trajectory were known, its effect could be subtracted, and lower trajectories could then be calculated. At present it seems unlikely that the over-all accuracy of the method is enough to allow this subtraction to be carried out successfully.

III. POTENTIAL PROBLEM

The potential-scattering problem involving the exchange of spin-zero particles can be solved by this method without a cutoff. This allows us to check the accuracy and reliability of the iteration solution by comparing it with solutions obtained by direct integration of Schrödinger's equation. For this purpose, an attractive potential with the discontinuity

$$\begin{aligned} V_t^s(t,s) &= \frac{A\epsilon}{(t-t_R)^2+\epsilon^2} \quad \text{for} \quad t>4m_\pi^2, \\ &= 0 \qquad\qquad\qquad \text{for} \quad t\leqslant 4m_\pi^2 \end{aligned} \tag{3.1}$$

was chosen. It corresponds to a superposition of Yukawa potentials of range close to $1/(t_R)^{1/2}$ and it attempts to model the exchange of a spin-zero particle of width ϵ. Although a single Yukawa potential would perhaps have been preferable, its corresponding discontinuity is a δ function which makes its numerical treatment awkward.

The same potential can then be used to integrate Schrödinger's equation. Since we are mainly interested in comparing trajectory and residue functions, this integration can be best performed numerically, using a modified version of Burke and Tate's TREGGE program.[5]

In Figs. 1–6 we exhibit $\alpha(s)$ and $\beta(s)$ for the iterative and the Schrödinger solution of this problem for different values of the width ϵ, and strength A. It is seen that the agreement is in general quite good throughout the ranges of s explored. In particular, the iterative calculation seems to give reasonable residue functions $\beta(s)$, which are usually more difficult to calculate than the trajectory functions $\alpha(s)$.

As can be expected the agreement is poorer for narrower or stronger potentials, the errors arising mainly from inaccuracies in the (2.1) integration. Also

[5] Philip G. Burke and Cecil Tate, University of California Radiation Laboratory Report No. UCRL-10384, 1962 (unpublished).

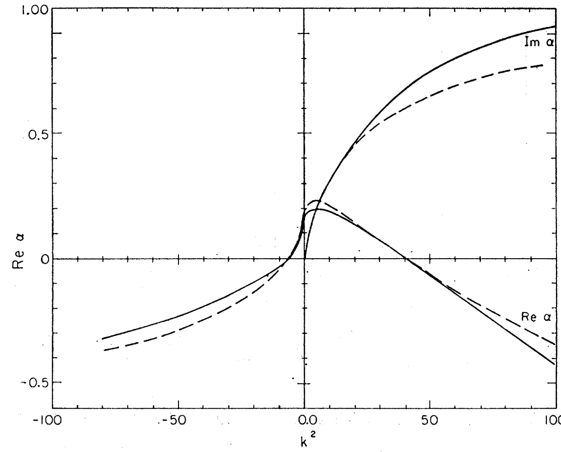


FIG. 1. Real and imaginary parts of α for $A=15.0$, $\epsilon=3.0$, $t_R=6.0$, – – – Iterative —— Schrödinger.

it was noted that the residue functions of potentials whose trajectories did not rise much above zero were rather poorly determined. This is probably due to error buildup in $\rho(s,t)$, which in these case does not increase much as a function of t.

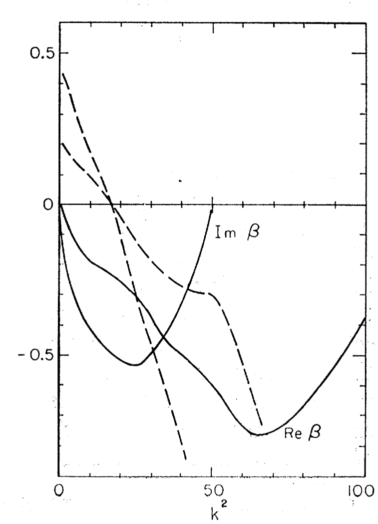
All trajectories shown were obtained at $t\approx 19600m_\pi^2$. It is necessary to go that far in t to eliminate oscillations which appear in $M_t(t,s)$ from interference with lower trajectories.

All these calculations were performed in a CDC-6600 computer, and required about 7 min per set.

IV. RELATIVISTIC π-π SCATTERING

Having ascertained the accuracy of the iteration procedure in nonrelativistic problems, we turn to the interesting case, relativistic $\pi\pi$ scattering. As pointed out in Ref. 1, the major difference between the potential and relativistic problems is the necessity of introducing a cutoff, as otherwise the integral in Eq. (2.2) cannot be performed. The cutoff procedure adopted is the one

FIG. 2. Real and imaginary parts of β for $A=15.0$, $\epsilon=3.0$, $t_R=6.0$, – – – Iterative —— Schrödinger.



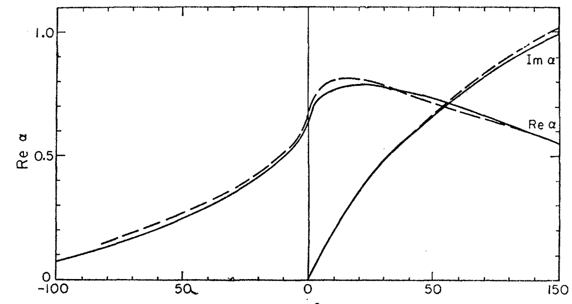


FIG. 3. Real and imaginary parts of α for $A=35.0$, $\epsilon=3.0$, $t_R=6.0$, - - - Iterative —— Schrödinger.

suggested in Ref. 1, which has the advantage of both being mathematically tractable and at the same time modeling closely the "strip" structure assumed for the amplitude. To this end, the function $h(s,s_1)$ in Eq. (2.5) was set to

$$h(s,s_1)=\frac{1}{1+\exp[(s-s_1)/\Delta]}. \qquad (4.1)$$

The solutions of the relativistic problem can be expected to depend rather critically on s_1, as it presumably represents the extremely complicated higher s structure of $\rho^s(s,t)$ arising from the increasing number

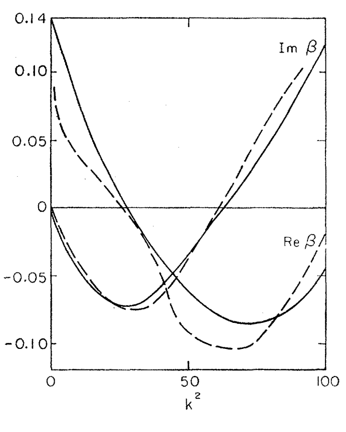


FIG. 4. Real and imaginary parts of β for $A=35.0$, $\epsilon=3.0$, $t_R=6.0$, - - - Iterative —— Schrödinger.

of inelastic channels open to the reaction. However, if the strip approximation is a sensible one, the dependence on Δ should not be too severe.

In these preliminary calculations, the iteration technique was used to calculate the trajectory and residues of the ρ and Pomeranchuk ($I=1$ and $I=0$) trajectories in π-π scattering with an "elementary" ρ exchanged in the u and t channel as potential. No attempt to obtain self-consistent or "bootstrap" solutions was made, as the presumably important Pomeranchuk repulsion[6] was entirely neglected. These calculations are not expected to reproduce too closely the physical values of the position and widths of the resonances involved.

[6] G. F. Chew, Phys. Rev. 140, B1427 (1965).

The input potential was taken to be

$$V_{t^s}(t,s)^I=2\beta^{I1}3P_1\left(1+\frac{s}{12}\right)\left(\frac{t_R}{t_R-4}\right)^{1/2}\left[\frac{\Gamma^2 t_R}{(t-t_R)^2+\Gamma^2 t_R}\right],$$

where t_R and Γ are the mass and the width of the input ρ particle, and β^{I1} is the familiar π-π crossing matrix. The leading factor of 2 is introduced to take into account the effect of the potential in both the t and u channels. The parameters Γ, s_1, and Δ were then adjusted to obtain reasonable output trajectories, consistent with the physical situation. It was found, however, that Δ has little effect over the lower part of the trajectory, which is mainly controlled by Γ and s_1.

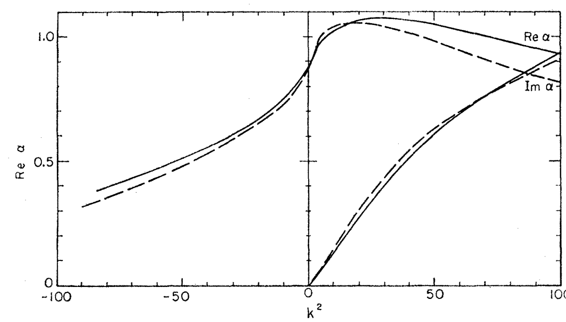


FIG. 5. Real and imaginary parts of α for $A=50.0$, $\epsilon=3.0$, $t_R=6.0$, - - - Iterative —— Schrödinger.

The cutoff point s_1 is expected to be anywhere from about (200 to 600)m_π^2, the width of the resonance region which characterizes the strip approximation,[2] and Γ is known experimentally to be $0.9m_\pi$.[7]

For parameters in this region, it is possible to obtain a continuum of solutions which yield a trajectory with $\mathrm{Re}\alpha(28)=1$ in the $I=1$ partial wave, including one for Γ slightly higher than the physical width of the ρ meson. The real and imaginary parts of α for two such examples are shown in Figs. 7–10 corresponding to (a) $\Gamma=1.1m_\pi$ and $s_1=400m_\pi^2$, and (b) $\Gamma=1.6m_\pi$ and $s_1=256m_\pi^2$. For case (b) we also show the effect of changing the parameter Δ from $30.0m_\pi^2$ to $100.0m_\pi^2$.

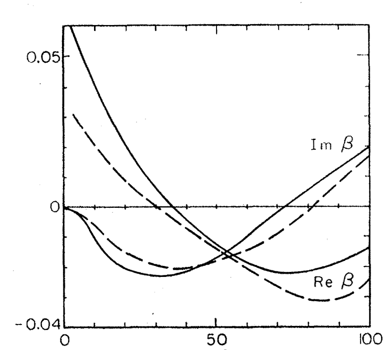


FIG. 6. Real and imaginary parts of β for $A=50.0$, $\epsilon=3.0$, $t_R=6.0$, - - - Iterative —— Schrödinger.

[7] A. M. Rosenfeld, A. Barbaro-Galtieri, W. M. Barkas, P. L. Bastien, J. Kirz, and M. Roos, Lawrence Radiation Laboratory Report UCRL-8030, Part I (unpublished).

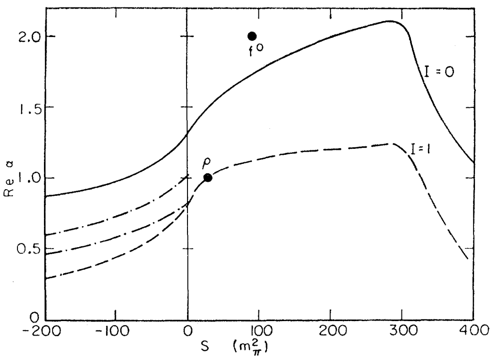
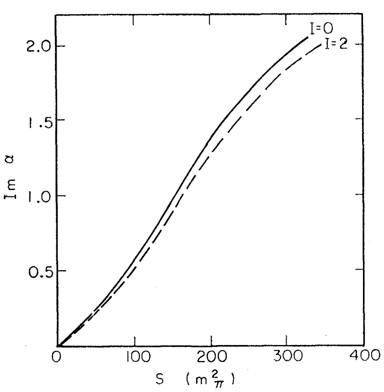


FIG. 7. Real part of α, $\pi\pi$ problem, case (a), --- $I=1$; —— $I=0$; —·— N/D solution.

As can be readily seen, this has little effect for low positive energies, but becomes more important towards the high end of the strip and at large negative energies, where this calculation is not expected to be accurate anyway. Although neither set of trajectories is very close to the physical one, the second one seems to be the better, as it is initially steeper, more in accordance with the experimentally determined trajectories.[8] In case (a) the $I=0$ trajectory rises up to $J=2$ to produce the f^0 resonance while in case (b) the trajectory does not reach $J=2$ for a real value of s but probably does for a slightly complex one. Also, as expected, no trajectory above $J=0$ is observed for $I=2$, as in this case the ρ potential is repulsive.

FIG. 8. Imaginary part of α, $\pi\pi$ problem, case (a), --- $I=1$; —— $I=0$.



The width of the output ρ meson is in both cases too large: $3.7m_\pi$ in case (a) and $2.7m_\pi$ in case (b). This is not surprising, as we have not yet included the effect of the Pomeranchuk trajectory, which can be expected to narrow this resonance.

As we approach the high end of the strip all trajectories bend downwards, as they can be expected to do if they satisfy a dispersion relation of the form

$$\alpha(s)=\alpha(\infty)+\frac{1}{\pi}\int_{s0}\frac{\mathrm{Im}\alpha(s')}{s'-s}ds', \tag{4.2}$$

since the imaginary part of $\alpha(s)$ should go to zero outside the strip.

We also show for comparison the $0<s$ part of the trajectories calculated using the new form of the strip approximation and the N/D method.[9] It is seen that for this particular potential, it is not very different from the iterative solution. However, this is quite reasonable, as we are dealing with a purely attractive elementary potential, where the N/D solution can be expected to perform rather well. Even so, the trajectories are flatter than the corresponding iterative ones, indicating that it will probably be easier to obtain steeper trajectories required by experiment with the new technique, once a better input potential is used.

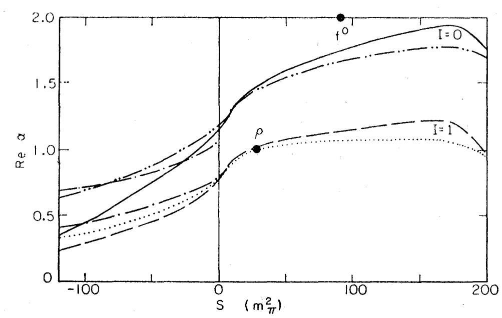
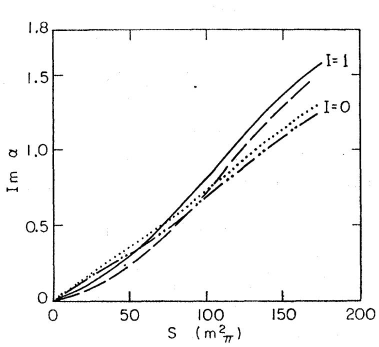


FIG. 9. Real part of α, $\pi\pi$ problem, case (b), --- $I=1$, $\Delta=30.0$; —— $I=0$, $\Delta=30.0$; ···· $I=1$, $\Delta=100.0$; —··— $I=0$, $\Delta=100.0$; —·— N/D solution.

The flatness of the N/D trajectories persists for $s>0$ where their effect can be seen by calculating cross sections. This lack of slope in turn implies that the N/D method will require considerably stronger potentials than the iterative method to give the correct mass to the ρ and f^0 resonances. Thus, for $s>0$ the N/D calculation yields less binding than the iterative one.

All trajectories shown were calculated at $t=10\,000m_\pi^2$. The time required to perform these calculations in a CDC 6600 was about 4 min per value of the isotopic spin. The time needed to solve the N/D equations for an equivalent range of J is about 1.7 min, so the iter-

FIG. 10 Imaginary part of α, $\pi\pi$ problem, case (b), --- $I=1$, $\Delta=30.0$; —— $I=0$, $\Delta=30.0$; ···· $I=1$, $\Delta=100.0$; —·— $I=0$, $\Delta=100.0$.



[8] C. Chiu, Lawrence Radiation Laboratory (private communication).

[9] P. D. B. Collins and V. L. Teplitz, Phys. Rev. **140**, B663 (1965).

ative method cannot be said to be much more complicated than the N/D.

As a final point, we would like to indicate that these results are substantially different from those obtained by Bransden *et al.*[4] in a similar calculation using almost the same input potential. Bransden *et al.* were unable to obtain trajectories rising up to $J=1$ in the $I=1$ partial wave for a pure ρ input, and were forced to include an elementary f^0 in the potential.[10]

The main differences between our calculation and theirs are improved accuracy and the different cutoff scheme. Their solution involves the smooth cutoff the potential $V^s(t,s)$ past a given s_1, and introduces no cutoff in $\rho^s(s,t)$. It can be easily checked that both these differences play an important role in the discrepancy between the two calculations. It is our impression that our cutoff procedure is the more natural one, as it does not interfere with the power blowup of the potential in the s direction, and also allows simple mathematical justification, as seen in Ref. 1.

To summarize, we can say that the above calculations seem to show that the Mandelstam iteration technique is indeed a feasible one from the computational point of view. It is quite able to produce reasonable output trajectories from a simple elementary-particle input potential, and it offers many advantages over the more usual N/D approach without an outrageous increase in the necessary computations.

At present attempts are being made at calculating fully Reggeized input potentials which will include Pomeranchuk repulsion effects. Also a more ambitious self-consistent scheme is being considered whereby the output $\rho(s,t)$ function obtained after the above iterations have been completed is used to compute a new potential $V^s(t,s)$ by means of crossing. This potential could then be used in a "macroiteration" to restart the whole calculation.

[10] This point is rather questionable, as has been pointed out by Chew [Progr. Theoret. Phys. (Kyoto) Suppl. Extra Number, 118 (1965)]: The inclusion of the f^0 as an elementary particle vastly exaggerates its effect.

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Symmetry Predictions from Sum Rules without Saturation*

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The application of sum rules derived from current commutation relations to give $SU(6)$-like results is developed without the assumption of saturation. We first derive a consistency condition by comparing sum rules, and then study the sum rules for $1/g_A^2$ and the isovector charge radii, replacing the assumption of saturation with a simple dynamical model for the remaining continuuum contribution. This results in several $SU(6)$-like predictions which contain important correction factors to the results previously derived assuming saturation. We have also re-examined the experimental data for the sum rule for $\langle r_N^2\rangle^V/6$, and find that it converges below 1 GeV like the sum rule for $1/g_A^2$ when account is taken of the contribution to the sum rule of the large resonant M_{1+} and nonresonant E_{0+} multipoles in pion photoproduction, and of the $N^{**}(1520)$.

I. INTRODUCTION

RECENTLY, several exact sum rules have been derived[1,2] from the algebra of current commutators.[3] These have been used together with the available experimental data to check the agreement of the sum rules with experiment, as exemplified by the work of Adler and Weisberger.[2] Other applications have been the use of sum rules to obtain $SU(6)$-like predictions by assuming saturation of the sum rules by the low-lying 56-plet of baryons or 36-plet of mesons. However, the assumption of saturation is suspect since, for example, the value of $1/g_A^2$ derived under the assumption

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‡ Alfred P. Sloan Foundation Fellow.
§ On leave of absence from Brandeis University, Waltham, Massachusetts.

[1] S. Fubini and G. Furlan, Physics 1, 229 (1965); S. Fubini, G. Furlan, and C. Rossetti, Nuovo Cimento 40, 1171 (1965).

[2] S. L. Adler, Phys. Rev. Letters 14, 1051 (1965); Phys. Rev. 140, B736 (1965); W. I. Weisberger, Phys. Rev. Letters 14, 1047 (1965); Phys. Rev. 143, B1302 (1966).

[3] M. Gell-Mann, Physics 1, 63 (1964).