Mandelstam Iteration in a Realistic Bootstrap Model of the Strong-Interaction S Matrix*

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The Mandelstam iteration is analyzed for a strip model of the four-line connected part that conforms to most known strong-interaction experimental requirements at both high and low energies. It is shown that, with the Froissart limit as a supplementary condition, asymptotic behavior is controlled by Regge poles, the amplitude being meromorphic in the right-half angular-momentum complex plane. The results support the practicality of the Mandelstam iteration as a numerical technique for realistic bootstrap computations.

I. INTRODUCTION

IT has been realized for a number of years that the Mandelstam iteration procedure is appropriate for dynamical calculations which concentrate on the strip regions of a Mandelstam diagram.[1] The motivation for M^s manifests itself in two ways: (a) In the s physical region there may be strong peaks in low-energy cross sections; these peaks are associated with s poles of definite J_s whose residues have a corresponding polynomial dependence on $z_s=\cos\theta_s$ and thus on t (or u). The inevitable dying out of such peaks above about 2 GeV in center-of-mass energy indicates that even if resonances continue at high s the partial widths for individual two-particle channels are small. (b) When there exist low-s poles on or near the physical sheet, emphasis of these regions is experimental. It has been observed that four-line connected parts are large within three narrow strips, as shown in Fig. 1. The strip labeled

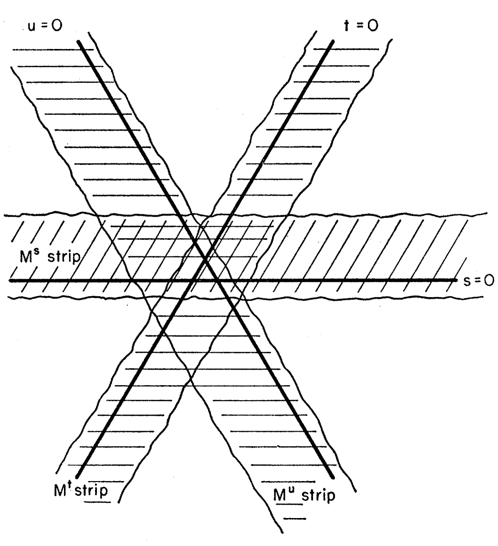

FIG. 1. The strip regions of the Mandelstam diagram.

peaks are systematically observed in the t and u reactions near the forward (or backward) direction where $|s|$ is small. These peaks have widths in $|s|$ of the order $\frac{1}{3}$ GeV^2 and persist to indefinitely high values of t (or u), as one might expect from the s pole position's independence of crossed-channel invariants. Outside such forward and backward peaks, high-energy four-line connected parts are very small. By suitably permuting the variables s, t, u, equivalent statements can be made about the strips labeled M^t and M^u. To summarize: In physical regions, four-line connected parts are experimentally observed to be small unless the absolute value of at least one of the three-channel invariants is no larger than a few GeV.[2]

Since the existence of the three strips seems to have a connection with poles, it is tempting to construct a model

$$M(s,t,u)=M^s(s,t,u)+M^t(s,t,u)+M^u(s,t,u), \quad (1.1)$$

where M^s contains all the s poles, M^t contains all the t poles, and M^u all the u poles. If we, in addition, assume that M^s is large only when $|s|$ is small, with corresponding properties for M^t and M^u, the required strip structure is immediately achieved. (It is, of course, not certain that the strip structure holds in unphysical regions. If it does not, the model will fail.) To further define the model, another experimental fact may be invoked. Within the s strip, where s poles are prominent, the most important s normal thresholds are for two-particle channels—provided we include unstable particles. Only at large values of s, above the strip, do multiparticle channels become dominant. Since poles and nearby important branch points inevitably interact, it is natural to concentrate all two-particle s thresholds inside the s strip into M^s, along with the s poles. In a similar fashion, M^t absorbs the low-t two-particle thresholds and M^u the low-u two-particle thresholds. Conversely, since M^t and M^u dominate at high s, it is natural to assign all multiparticle s thresholds to these two components, leaving such singularities out of M^s. Similarly M^t will contain no multiparticle t thresholds and M^u no multiparticle u thresholds.

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[1] G. F. Chew and S. C. Frautschi, Phys. Rev. 123, 1478 (1961).
[2] G. F. Chew, S. C. Frautschi, and S. Mandelstam, Phys. Rev. 126, 1202 (1962).

At this point the relevance of the Mandelstam iteration becomes apparent. This iteration allows us to construct that part of the amplitude containing s poles and two-particle s thresholds if we are given the t and u singularities that are independent of two-particle s thresholds. In other words, if we know $V^s=M^t+M^u$ (which may be called the generalized "potential" for the s reaction), Mandelstam tells us how to compute M^s. Evidently, we are also told how to compute M^t from a knowledge of M^s+M^u, as well as M^u from a knowledge of M^s+M^t. This reciprocal dynamics constitutes the bootstrap.

The model outlined here does not ignore multiparticle thresholds, but at the same time it fails to take direct account of the associated discontinuity formulas. Only two-particle discontinuities are individually and accurately treated. Multiparticle cuts in the strip model arise indirectly, via the Mandelstam iteration. Provided that individual multiparticle discontinuities are small and only the sum is large—from the accumulation of many different channels—there is a chance for such an approach to make sense.

What obstructed the original strip model proposal of Chew and Frautschi?[1] Fundamentally it was a matter of asymptotic behavior. In realistic situations, where the potential may contain poles corresponding to spin 1 or greater, the Mandelstam iteration becomes unstable and in the absence of delicate cancellations leads to nonsensical results.[2] In particular, there is no tendency for M^s to become small at large $|s|$. It thus seems probable that the ultimate dynamical origin of the experimentally observed strip structure involves a more intimate interaction between two-particle and many-particle channels than is included in the model. In other words, the strip structure must be *imposed*; it will not automatically emerge. Nevertheless, it remains plausible that, if strip structure can be built in, the model may have great utility.

An attempt to impose strip structure via a cutoff prescription was made by Bransden *et al.*,[3] with results that were generally encouraging. The underlying mathematical character of their prescription was never established, however, so it was not sure that the iteration was converging to the limit assumed. The purpose of our paper is to formulate a cutoff procedure that is susceptible to analysis and that guarantees manageable asymptotic behavior in the iteration. The following paper describes certain numerical tests of the procedure that confirm its practicability.

Before we proceed to detailed analysis a word is in order about the "new form of strip approximation"[4] that attempted to bypass the Mandelstam iteration, going directly to N/D equations. The physical motivation of both "old" and "new" forms is identical, the advantage of the "new" being the reduction of the problem to functions of a single variable—the Regge trajectory and residue functions. A great deal has been learned by studying the new form,[5] but its inadequacies have proved serious. Like other N/D methods it does not permit a proper treatment of the left-hand cut in the partial-wave amplitude, and the joining of elastic to inelastic regions on the right is awkward. A crucial aspect of the former deficiency is the inability to calculate reliably when the (direct reaction) angular momentum is larger than 1. The "old" form with a suitable cutoff has no trouble with such questions and, as described in the following paper, does not require appreciably more computing time. The disadvantage is that functions of two variables are unavoidable.

From a concrete practical standpoint we hope by, returning to the Mandelstam iteration, to remedy the following specific inadequacies of alternative procedures.

(a) It has not so far been possible to confidently follow Regge trajectories to angular momenta greater than 1.

(b) It has been impossible to include the generalized potential the repulsive effect of trajectories like the Pomeranchuk, where "ghosts" occur.[6]

(c) Multiparticle production at intermediate energies (i.e., the upper half of the strip) has never been included in the dynamics. We shall see that the Mandelstam iteration handles all three of these points in a natural fashion.

II. THE MANDELSTAM EQUATIONS WITH A CUTOFF

A preliminary study by Drummond[7] of the complex normal thresholds associated with unstable two-particle channels shows their qualitative similarity to ordinary stable-channel thresholds. There are of course complications, but Drummond finds it possible to preserve the key elements of the strip model. The discussion here will proceed, therefore, as if one had to deal only with stable channels. The inclusion, furthermore, of any finite number of two-particle channels in the dynamics creates no difficulties beyond those already present with a single channel. The calculations merely become more lengthy. For pedagogical reasons, then, we write down only those equations appropriate to a single two-particle channel, with zero spins for both channel particles.

The first step is to introduce the standard "one-sided" functions corresponding to $M(s,z_s)$, $V^s(s,z_s)$, and $M^s(s,z_s)$. Each of the new functions has only a right-hand cut in z_s, a superscript $(+)$ conventionally

[3] R. H. Bransden, P. G. Burke, J. W. Moffat, R. G. Moorhouse, and D. Morgan, Nuovo Cimento **30**, 207 (1963).

[4] G. F. Chew, Phys. Rev. **129**, 2363 (1963); G. F. Chew and C. E. Jones, *ibid.* **135**, B208 (1964).

[5] V. L. Teplitz, Phys. Rev. **137**, B136 (1965); P. D. B. Collins and V. L. Teplitz, *ibid.* **140**, B663 (1965); P. D. B. Collins, *ibid.* **142**, 1163 (1966).

[6] G. F. Chew, Phys. Rev. **140**, B1427 (1965); Phys. Rev. Letters **16**, 60 (1966).

[7] I. Drummond, Phys. Rev. **140**, B482 (1965).

designating a function of this type whose even part in z_s coincides with the even part in an original "two-sided" function, and a superscript $(-)$ designating a function whose odd part so coincides. Thus, for example,

$$M(s,z_s)=\tfrac{1}{2}[M^+(s,z_s)+M^+(s,-z_s)]+\tfrac{1}{2}[M^-(s,z_s)-M^-(s,-z_s)]. \quad (2.1)$$

The relation

$$M^{\pm}(s,z_s)=V^{s\pm}(s,z_s)+M^{s\pm}(s,z_s) \quad (2.2)$$

will hold, just as for the original functions. In fact, all the equations to follow have a form independent of the $(\pm)$ signature, so we shall suppress these clumsy superscripts.

By definition the functions M^s are supposed to contain all the s poles as well as the normal threshold branch point at $s_0=(m_a+m_b)^2$. The potential V^s lacks this branch point although it contains multiparticle s thresholds. The potential may be expressed as a Cauchy integral over its cut (and poles, if any) in z_s or equivalently in t:

$$V^s(s,t)=\frac{1}{\pi}\int_{t_0}^{\infty}\frac{dt'}{t'-t}V_t^s(t',s), \quad (2.3)$$

with similar expansions of M and M^s in terms of their t discontinuities M_t and M_t^s, respectively, so that from Eq. (2.2) we have

$$M_t(t,s)=V_t^s(t,s)+M_t^s(t,s). \quad (2.4)$$

The lower limit t_0 of the t spectrum in the potential necessarily lies below the beginning of the t spectrum in M^s because all t and u poles and two-particle thresholds have been assigned to V^s.

The double spectral function $\rho^s(s,t)$ is defined to be the s discontinuity of M_t^s, that is,

$$M_t^s(t,s)=\frac{1}{\pi}\int_{s_0(t)}^{\infty}\frac{ds'}{s'-s}\rho^s(s',t), \quad (2.5)$$

where the lower limit $s_0(t)$ is related to t_0 by

$$q_s^2(s_0(t))=\frac{t_0^2}{t-4t_0}, \quad (2.6)$$

if $q_s(s)$ is the barycentric system momentum. (The reader is assumed to be familiar with the Mandelstam representation.) The fundamental equation derived by Mandelstam then is

$$\rho^s(s,t)=\frac{g(s)}{2\pi q_s^2(s)}\int\int dt'dt''\frac{M_t^*(t',s)M_t(t'',s)}{K^{1/2}(q_s^2(s);t,t',t'')}, \quad (2.7)$$

where

$$g(s)=2q_s(s)/\sqrt{s} \quad (2.8)$$

and

$$K(q^2;y,y',y'')=y^2+y'^2+y''^2-2(yy'+yy''+y'y'')-(yy'y''/q^2), \quad (2.9)$$

the range of integration in (2.7) being confined to the region where K is positive. From Eqs. (2.4) and (2.5) we have

$$M_t(t,s)=V_t^s(t,s)+\frac{1}{\pi}\int_{s_0(t)}^{\infty}\frac{ds'}{s'-s}\rho^s(s',t), \quad (2.10)$$

giving a pair of equations in (2.7) and (2.10) on which a Mandelstam iteration may be based, starting from knowledge of V^s, or equivalently of V_t^s.

The flaw in the above equations is the absence of any guarantee that the function

$$M^s(s,t)=\frac{1}{\pi^2}\int\int ds'dt'\frac{\rho^s(s',t')}{(s'-s)(t'-t)} \quad (2.11)$$

should become small for large $|s|$. Such a requirement is essential to the consistency of the strip model, but a study of Eqs. (2.7) and (2.10) shows on the contrary a tendency for each successive iteration to increase more strongly at large s than its predecessor, if for any real positive t the potential itself grows faster than the first power of s.[2,8] Since particles certainly exist with spin greater than one, there will at least sometimes be poles in t whose dependence on s goes with a corresponding large power; and delicate cancellations will be required to prevent the Mandelstam iteration from "running away."

A crude but simple prescription that might circumvent the dilemma is to replace the two-particle phase-space factor $g(s)$ in Eq. (2.7) by a modified factor $g_1(s)$, equal to $g(s)$ for $s<s_1$ but dropping rapidly to zero for $s>s_1$. Although the parameter s_1 (naturally dubbed the "strip width") is not an arbitrary parameter in a complete bootstrap calculation, where only energy ratios are significant, the introduction of such a crude cutoff cannot be regarded as satisfactory and inevitably will serve as a focus for efforts to improve the model. At the same time it is of importance to know whether this prescription, crude or not, suffices to ensure that the Mandelstam iteration approaches a sensible limit that can be given a physical interpretation. In the following section we attack this question.

One immediate consequence of the cutoff is worthy of notice. We are requiring that at high energies the entire s discontinuity should approach that of the potential. Thus, to the extent that experimental data at high energies has been successfully fitted by Regge-pole expansions,[9] the model is in good shape if it can be demonstrated that the high-s behavior of $V^s=M^t+M^u$ is dominated by Regge poles (in J_t and J_u). Such domination will emerge by crossing considerations from the analysis to follow. In other words, we shall demonstrate that M^s has only pole singularities in the right-

[8] S. Mandelstam, Ann. Phys. (N. Y.) **21**, 302 (1963).
[9] See, for example, R. J. N. Phillips and W. Rarita, Phys. Rev. **139**, 1336 (1965).

half J_s plane and that the large-t behavior of M^s is controlled by these poles. Crossing then implies that M^t is controlled at large s by poles in J_t, and M^u by poles in J_u.

III. THE POWER BOUND IN t

There are three elements in the demonstration that iteration of Eqs. (2.7) and (2.10), with a cutoff phase-space factor, leads to simple Regge behavior at large t:

(1) We first employ Mandelstam's direct analysis of the iteration[8] to establish that $M_t(t,s)$ is bounded by a finite power of t, say t^L. The Froissart-Gribov formula then defines the analytic continuation in J_s for $\mathrm{Re}J_s>L$.

(2) N/D equations of the kind proposed by Frye and Warnock,[10] but including the cutoff phase-space factor, are next shown to be Fredholm in character and to allow continuation through the right-half J_s complex plane. The only singularities allowed by this continuation are poles.

(3) The Sommerfeld-Watson transform then leads to simple Regge asymptotic behavior at large t, if the partial-wave amplitude vanishes sufficiently rapidly for $|J_s|\to\infty$, $\mathrm{Re}J_s>0$.

What are the differences between our problem and that already analyzed by Mandelstam?[8] They are relatively minor:

(a) We make no requirement that the potential should vanish at large s. In fact, by crossing we expect the potential to exhibit Regge behavior, sometimes increasing with a large positive power of s.

(b) Because his potential did not increase with energy, Mandelstam was able to show that he needed no cutoff for his two-particle phase-space factor if the generalized potential was sufficiently weak. As noted above, we do need a cutoff, but there is then no limitation on potential strength.

(c) Mandelstam did not concern himself with the imaginary part of the potential, which corresponds to multiple production (inelastic scattering). The model considered here is more realistic than that of Mandelstam, but we shall see that the essentials of his analysis manage to survive.

The first step, the establishment of a finite power bound in t, can be taken over almost without change. The point is that because of the cutoff factor in Eq. (2.7), the behavior of the potential for $s\gg s_1$ is irrelevant to the development of $M_t^s(t,s)$. Thus, for the investigation of the latter function we can introduce a modified potential equal to the original throughout the strip but ultimately cutoff at *very* large s so as to satisfy Mandelstam's requirement. The fact that our double spectral function $\rho(s,t)$ is also cutoff does not hinder any of his arguments; it only makes then simpler.

[10] G. Frye and R. L. Warnock, Phys. Rev. **130**, 478 (1962).

The analysis to establish the power bound on $M_t(t,s)$ does not require the potential to be real. This property Mandelstam invoked only for step (2), which we take up in the following section. It is essential that the potential itself have a power bound in t, but such a property is an intrinsic feature of the strip model. In fact, we see by crossing that, since M^s falls off at large s at least as fast as $1/s$, the potential V^s will fall off at large t at least as fast as $1/t$.

Being assured the existence of a maximum power behavior $<t^L$, we may define the Froissart-Gribov partial-wave amplitude for $\mathrm{Re}J>L$ by the contour integral[11]

$$A_J(s)=\frac{1}{2\pi i}\int_c dz_s Q_J(z_s)M(s,z_s)\,, \tag{3.1}$$

the contour c passing around the cut in z_s of $M(s,z_s)$. An equivalent form is

$$A_J(s)=\frac{1}{\pi}\int_{z_s(s,t_0)}^{\infty} dz_s Q_J(z_s)M_t(t(z_s,s),s)\,. \tag{3.2}$$

The object of the final section is to continue the partial-wave amplitude so defined as a meromorphic function throughout the entire right-half J_s complex plane.

IV. MEROMORPHY IN THE RIGHT-HALF COMPLEX ANGULAR-MOMENTUM PLANE

Let us define the "reduced" partial-wave amplitude

$$B_J(s)=(q_s^2)^{-J}A_j(s) \tag{4.1}$$

and break this function into two components corresponding to Eqs. (2.2) and (2.4):

$$B_J(s)=V_J^s+B_J^s. \tag{4.2}$$

The "potential" component,

$$V_J^s(s)=\frac{(q_s^2)^{-J}}{\pi}\int_{z_s(s,t_0)}^{\infty} dz_s Q_J(z_s)V_t^s(t(z_s,s),s)\,, \tag{4.3}$$

is immediately defined and analytic for $\mathrm{Re}J>-1$, i.e., throughout the holomorphy domain of $Q_J(z_s)$. In fact, if $V_t^s(t,s)$ decreases for large t faster than any power, as will follow from most cutoff prescriptions, then J singularities of V_J^s can at most be fixed poles at the negative integers—these poles arising from the corresponding poles of Q_J. The difficulty in continuation throughout the right-half J plane lies in the function

$$B_J^s(s)=\frac{(q_s^2)^{-J}}{\pi}\int_{z_s(s,4t_0)}^{\infty} dz_s Q_J(z_s)M_t^s(t(z_s,s),s)\,, \tag{4.4}$$

[11] M. Froissart, La Jolla Conference on Strong and Weak Interactions, 1961 (unpublished); V. N. Gribov, Zh. Eksperim. i Teor. Fiz. **41**, 1962 (1962) [English transl.: Soviet Phys.—JETP **14**, 1395 (1962)].

which is immediately defined only for Re$J>L$. For future reference we note that, since $M_t^s(t,s)$ is of order $1/s$ for large s, $B_J^s(s)$ is of order $(1/s)^{J+1}$, and for Re$J>L$

$$B_J(s) \xrightarrow[J\text{ fixed}]{s\to\infty} V_J^s(s)+\text{order}\left(\frac{1}{s}\right)^{J+1}. \quad (4.5)$$

Thus, we do not require our partial-wave amplitude to vanish at large s (which would be contrary to experimental indications), but we do require it to approach the potential plus a remainder that vanishes.

In order to extend the region of J analyticity of B_J^s and thus of B_J we consider the quotient

$$B_J(s)=N_J(s)/D_J(s)\,, \quad (4.6)$$

and attempt to define the numerator and deominator so that each is analytic in J. The denominator $D_J(s)$ is to have only a right-hand cut from the elastic threshold s_0 to $+\infty$, while the numerator $N_J(s)$ carries not only the left-hand cuts but is also cut from the inelastic threshold s_{in} on the right to $+\infty$. Following the procedure of Frye and Warnock,[10] we shall be able to derive Fredholm equations for N_J and D_J in which both the inhomogeneous terms and the kernels are determined by V_t^s and M_t^s.

To begin the derivation observe that in the physical region along the upper side of the right cut it follows from Eqs. (2.7) and (2.10) that

$$\text{Im}B_J^s(s)=\rho_J^1(s)\,|B_J(s)|^2\,, \quad (4.7)$$

where the cutoff phase-space factor

$$\rho_J^1(s)=(q_s^2)^J g_1(s) \quad (4.8)$$

goes rapidly to zero for $s>s_1$. The discontinuity of B_J^s on the left arises, according to Eq. (4.4), from the discontinuity of $Q_J(z_s)$. The integral over this discontinuity is given by a straightforward calculation to be

$$B_J^{s,L}(s)=\frac{1}{\pi}\int_{s_L}^{-\infty}\frac{ds'}{s'-s}(-q_{s'}^2)^{-J} \times\frac{1}{2}\int_{z_s(s',4t_0)}^{1} dz_s P_J(-z_s)M_t^s(t(z_s,s'),s')\,, \quad (4.9)$$

where s_L is given by $z_s(s_L,4t_0)=1$. Inspection of the integral here reveals that, if $M_t^s(t,s)$ is bounded by t^N for $|s|$ sufficiently large, the integral is defined and analytic in J for Re$J>N-1$. The Froissart limit assures us that $N<1$, so if we manage to satisfy this limit the function $B_J^{s,L}(s)$ will be analytic throughout the right-half angular-momentum plane.

Although no explicit use is to be made of the following nonlinear equation, it may help the reader to keep track of what has been said so far:

$$B_J(s)=V_J(s)+B_J^{s,L}(s)+\frac{1}{\pi}\int_{s_0}^{\infty}\frac{ds'}{s'-s}\rho_J^1(s') \times|B_J(s')|^2. \quad (4.2')$$

The first two terms on the right side of Eq. (4.2′) are analytic throughout the right-half J plane. Together, these terms will determine the inhomogeneity and the kernel of the linear Fredholm equation that is being sought.

To proceed with the Frye-Warnock method we define a function S_J^1 by the equation

$$S_J^1=1+2i\rho_J^1(s)B_J(s)\,. \quad (4.10)$$

This artificial function is equal to the elastic S-matrix element for $s<s_1$ but goes rapidly to 1 for $s>s_1$. Let us then require that in the physical region the denominator function should have a phase equal to the negative of half the phase of S_J^1. That is,

$$S_J^1=\eta_J^1(D_J^*/D_J)\,, \quad (4.11)$$

where η_J^1 is the absolute value of S_J^1. Comparing Eqs. (4.11), (4.10), and (4.6) it follows that in the physical region

$$\eta_J^1 D_J^*-D_J=2i\rho_J^1 N_J\,, \quad (4.12)$$

or

$$\text{Im}D_J=-\left(\frac{2}{1+\eta_J^1}\right)\rho_J^1\,\text{Re}N_J\,,\quad s>s_0\,, \quad (4.13)$$

$$\text{Im}N_J=\left(\frac{1-\eta_J^1}{2\rho J^1}\right)\text{Re}D_J\,,\qquad s>s_{\text{in}}\,. \quad (4.14)$$

(It should be remembered that N_J also has left-hand cuts.) The parameter η_J^1 may be evaluated from the equation

$$\text{Im}B_J=\rho_J^1|B_J|^2+\text{Im}V_J^s\,,$$

leading to

$$\tfrac{1}{4}[1-(\eta_J^1)^2]=\rho_J^1\,\text{Im}V_J^s\,. \quad (4.15)$$

It follows that η_J^1 equals 1 for $s<s_{\text{in}}$ and rapidly approaches 1 for $s>s_1$. Evidently, there is a requirement on the potential that

$$0\leqslant\rho_J^1\,\text{Im}V_J^s\leqslant\tfrac{1}{4}\,, \quad (4.16)$$

a constraint always to be checked before the dynamical calculation is begun.

We are now in a position to write down equations for N_J and D_J, using as a guide the requirement that for Re$J>L$ the partial wave must coincide with formula (3.2). Inspection of the latter shows that there should be no physical sheet poles in s and that $A_J(s)$ must decrease exponentially as $\mathrm{Re}J \to +\infty$. It follows from the definition (4.10) that the phase of S_J^1, for sufficiently large J, must be the same at $s=s_0$ as at $s=\infty$. The absence of physical sheet poles, together with the phase requirement, eliminates CDD ambiguities[12] at least for large ReJ, and if we normalize the denominator function to unity at $s=\infty$, its equation immediately follows from formula (4.13) to be

$$D_J(s)=1-\frac{1}{\pi}\int_{s_0}^{\infty}\frac{ds'}{s'-s}\bar{\rho}_J(s')\bar{N}_J(s'), \quad (4.17)$$

where

$$\bar{\rho}_J=\rho_J{}^1/\eta_J{}^1 \quad (4.18)$$

and

$$N_J=[2\eta_J{}^1/(1+\eta_J{}^1)]\,\mathrm{Re}N_J. \quad (4.19)$$

The equation for N_J (or $\bar{N}_J$) takes longer to derive, but our problem is formally equivalent to that considered by Frye and Warnock[10] except that we have the asymptotic condition (4.5) in place of a simple vanishing requirement. The result is the same (and is unique):

$$\bar{N}_J(s)=\bar{B}_J{}^L(s)+\frac{1}{\pi}\int_{s_0}^{\infty}\frac{ds'}{s'-s}[\bar{B}_J{}^L(s')-\bar{B}_J{}^L(s)]\times\bar{\rho}_J(s')\bar{N}_N(s'), \quad (4.20)$$

with

$$\bar{B}_J{}^L(s)=\mathrm{Re}\left\{V_J{}^s(s)+B_J{}^{s,L}(s)+\frac{1}{\pi}\int_{s_{\mathrm{in}}}^{\infty}\frac{ds'}{s'-s}\frac{[1-\eta_J{}^1(s')]^2}{4\rho_J{}^1(s')}\right\}. \quad (4.21)$$

In order to see that (4.21) is the formula derived by Frye and Warnock, it is only necessary to recall Eq. (4.15) together with the identity

$$\frac{1-\eta^2}{4}+\frac{(1-\eta)^2}{4}=\frac{1-\eta}{2}.$$

[12] L. Castillejo, R. H. Dalitz, and F. J. Dyson, Phys. Rev. **101**, 453 (1956).

The linear integral equation (4.20) for $\bar{N}_J(s)$ is nonsingular if

$$\int\int dsds'\left|\frac{\bar{B}_J{}^L(s')-\bar{B}_J{}^L(s)}{s'-s}\right|^2\bar{\rho}_J(s')\bar{\rho}_J(s)<\infty$$

and

$$\int ds\,\bar{\rho}_J(s)\,|\bar{B}_J{}^L(s)|^2<\infty\,.$$

With a cutoff at s_1 there is no trouble in these conditions from the upper range of integration and also none at the lower limit s_0 if $\mathrm{Re}J>-\frac{3}{2}$. Since $\bar{B}_J{}^L$ is analytic in J for $\mathrm{Re}J>0$, if we manage to satisfy the Froissart limit, it follows that the Fredholm equation (4.20) defines a unique function $\bar{N}_J(s)$ analytic at least throughout the right-half angular-momentum complex plane except for possible fixed poles in J arising from zeros of the Fredholm determinant. According to Eq. (4.17) such fixed poles would also occur in $D_J(s)$ and thus cancel in the quotient yielding $B_J(s)$. Thus, the only J singularities of $B_J(s)$ for $\mathrm{Re}J>0$ would be Regge poles, arising from the zeros of $D_J(s)$.

The final demonstration of Regge asymptotic behavior requires, beyond meromorphy in J, an investigation of the limiting behavior of $A_J(s)$ as $J\to i\infty$; but Mandelstam's[8] analysis of this latter question can, fortunately, be taken over directly. Provided that the Froissart limit is satisfied, one can justify the neglect of those portions of the distorted Sommerfeld-Watson contour at $|J|=\infty$, $\mathrm{Re}J>0$. The usual Regge formula for $M(s,t)$ as $t\to\infty$ with s fixed, in terms of pole trajectories and residues, then follows.

The following paper[13] shows in detail how the understanding of asymptotic behavior at large t allows a practical numerical calculation of $M(s,t)$ to be based on Eqs. (2.7) and (2.10).

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[13] N. F. Bali, following paper, Phys. Rev. **150**, 1358 (1966).