Relativistic Three-Particle Equations. I*

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Linear integral equations for three-particle scattering amplitudes in any Lorentz-invariant local field theory are written; they are three-particle analogs of the Bethe-Salpeter equation. The kernels of these equations are off-mass-shell relativistic generalizations of two-particle and three-particle potentials. We transform the equations by the method of Faddeev so that the two-particle potential no longer appears, but only the two-particle scattering amplitude. Particular cases of these equations are presented. We then show that two- and three-particle unitarity is satisfied provided the relativistic potentials are real in the relevant energy region.

I. INTRODUCTION

THERE is a considerable and increasing amount of experimental data on strongly interacting three-particle systems. In particular the three-pion system has four possible resonances—the ω, $H(959)$[1], $A_1(1080)$[1], and A_2 (1300)[1]. The $KK\pi$ system has two possible resonances—the $D(1285)$ and $E(1415)$, the $K\pi\pi$ system has $C(1220)$. There are also numerous resonances[1] with nucleon number 1, whose understanding may very likely require three- or more-particle intermediate states. Thus we need a theory of three-particle scattering to explain this data. Such a theory does not not yet exist, since at the same time such a theory would require some knowledge of four-, five-, six- $\cdots$ particle scattering.

An approximation scheme for three-particle scattering which breaks this infinite chain is to neglect contributions from connected processes with not less than four particles in intermediate states. Such an approximation scheme might be expected to be reasonable for energies below the production threshold of an extra particle, which would usually be a single meson. Such an approximation scheme may be regarded as analogous to setting the three-particle potential equal to zero in potential scattering theory. Even when this is done there are considerable difficulties left in the nonrelativistic potential scattering equations. These difficulties are generated by the presence of the two-particle potentials in the kernel of the three-particle Lippmann-Schwinger equation; upon iteration of this three-particle kernel, resonances or bound states of the two-particle channels may cause the divergence of the iteration. Further practical difficulties are that the two-particle potentials are not as well known as the the two-particle scattering amplitudes and their presence in the equations does not allow the use of the bound states and resonances in the two-particle amplitudes to approximate the two-particle contributions.

These difficulties were all removed by the work of Faddeev,[2] who showed how it was possible to introduce partial three-particle amplitudes which satisfied integral equations—the Faddeev equations—whose kernels were the complete two-particle scattering amplitudes. He then gave a careful analysis of these kernels,[3] and showed that suitable iterates of them were completely continuous, and could be approximated by separable kernels. This essentially means that the two-particle amplitudes entering the kernels could be expressed as sums of bound-state and resonance contributions.

The Faddeev equations have been applied in nonrelativistic scattering theory to the 3-nucleon system,[4] for 3α particles,[5] for the $\pi\pi N$ system,[6] and for the Kd system.[7]

When we wish to extend these discussions to relativistic three-particle scattering we meet the difficulty that we no longer have a unique equivalent of the Lippmann-Schwinger equation. The lack of uniqueness arises from the possibility of considering three-particle intermediate states either completely on the mass shell for each particle and on the energy shell for the total three-particle energy, or off the mass shell for each particle but on the total energy shell, or on the mass shell for each particle but off the total energy shell. Similar possibilities exist for relativistic two-particle scattering; the first possibility described above is two-particle unitarity as practiced by S-matrix theorists, the second corresponds to using the Bethe-Salpeter (B.S.) equation, the third the reduction of the B.S. equation by means of the Blankenbecler-Sugar kernel,[8] which puts to zero the relative energies of the two intermediate particles in the B.S. equation; it may be derived from the B.S. equation, but has the same structure and number of variables as the Lippmann-Schwinger nonrelativistic equation.

* Supported in part by the National Science Foundation.

[1] A. H. Rosenfeld *et al.*, Rev. Mod. Phys. **36**, 977 (1964). S. Ya. Nikiten, in *Proceedings of the International Conference on High-Energy Physics, Dubna, 1964* (Atomizdat, Moscow, 1965).

[2] L. D. Faddeev, Zh. Eksperim i Teor. Fiz. **39**, 1459 (1960) [English transl.: Soviet Phys.—JETP **12**, 1014 (1961)]; Dokl. Akad. Nauk SSSR **138**, 565 (1961); **145**, 301 (1962) [English transls.: Soviet Phys.—Doklady **6**, 384 (1961); **7**, 600 (1963)].

[3] L. D. Faddeev, Atomic Energy Research Establishment Translation, Harwell, England, 1964 (unpublished).

[4] R. Amado, Phys. Rev. **132**, 485 (1963); R. Aaron, R. Amado and Y. Yam, *ibid.* **136**, B651 (1964) and Phys. Rev. Letters **13**, 574 (1964).

[5] D. Harrington, Bull. Am. Phys. Soc. **11**, 28 (1966).

[6] C. Lovelace, Phys. Rev. **135**, B1225 (1964).

[7] H. Hetherington and R. Schick, Phys. Rev. **137**, B935. (1965).

[8] R. Blankenbecler and R. Sugar, Phys. Rev. **142**, 1051 (1966).

In the three-particle case the first and the last possibilities have been pursued. As for the first possibility, unitarity contributions have been analysed[9] and three-particle N/D equations have been written down.[10] There is a considerable difficulty in this approach arising from the unknown analytic properties of the amplitudes, which has not yet been satisfactorily solved.[11] Also it has not been possible to apply this approach to actual three-particle systems due to nonseparability. The third possibility has been very actively pursued[12] and applied specifically to the problem of generating the ω resonance from the three-pion system.[13] The numerical results of these calculations give resonances of too high an energy to the ω, though they may be the A mesons. It is possible that the further approximations made in these calculations, in particular that of just taking $\pi\pi$ scattering through the ρ resonance, may be the reason for this failure to obtain the ω; this is not yet known.

From this brief survey of recent work on the three-particle problem it would appear useful to see if the second possibility discussed above, that of off-mass-shell methods, will give a more satisfactory theory of relativistic three-particle scattering. That is the purpose of this and following papers.

In the present paper we wish to set up the general equations describing the exposure of off-mass-shell three-particle intermediate states in scattering amplitudes; we also wish to show that we may rewrite these equations in a form only involving the two-particle scattering amplitudes. Thus we will have our equations in a form ready to approximate by neglect of higher than three-particle intermediate states. We then write down such approximate equations and consider their unitarity properties briefly here.

One might ask if the crucial difficulties encountered in off-mass-shell theories of (*i*) too many variables and (*ii*) occurrence of conditionally convergent integrals will prevent us from ever using the equations described in this paper. We show in the next paper[14] how both these difficulties may be faced and surmounted, at least under the reasonable approximation of taking only resonance and bound-state contributions in two-particle scattering amplitudes.

It is very pertinent to remark in this introduction that the equations obtained in Secs. 3 and 4 may be used as the starting point for deriving equations in either the on-mass-shell unitary S-matrix approach or the off-energy-shell approach using the Blankenbecler and Sugar kernel. In other words, what we are doing in this paper is solving the combinatorial problem of exposing three-particle states in scattering amplitudes. Once we have solved this problem so that we have the correct order of the various scatterings, then the resulting integral equations may be obtained for the cases mentioned above by suitable choice of propagators and vertices.

The solution of this combinatorial problem was given in an earlier work of the author.[15] Since the details for three-particle scattering were not spelled out in that paper, there still seems to be considerable confusion as to the correct solution to this problem. We feel it necessary to go into this solution in some detail for the three-particle case here. In particular it appears very necessary to discuss the manner in which elementary and composite particles enter, and the role of renormalization. Both these topics are very scantily discussed in the references to the three-particle problem made earlier in this paper, and in some places errors have even crept in due to confusion over the solution of the combinatorial problem given in Ref. 14. Further, as we remarked earlier in this section, our off-mass-shell equations may be regarded as a basis for on-mass-shell approaches. Thus a complete and careful discussion of the off-mass-shell equations may be regarded as laying the foundations for these other approaches as well.

In the next section we describe the method of exposing three-particle intermediate states in a general scattering process. In Sec. 3 we show how we may transform these equations so that they do not involve the two-particle "potential" explicitly; the next section describes the resulting equations in detail for various physical processes, when the three-particle "potential" is taken as zero. The final section gives a formal derivation of two- and three-particle unitarity, emphasizing the properties of the "potentials" on which this unitarity depends.

II. EXPOSING THREE PARTICLES

We discuss here how we may expose three-particle intermediate-state contributions to scattering processes.[15] This is done for off-mass-shell as well as on-mass-shell particles, and the exposure is performed in S-matrix elements which are in general off mass shell in all the particle momenta. We base our discussion on the existence of fields $\phi_\alpha(x)$, where α is a variable describing the various types of particle we are concerned with. We assume that the usual properties of general field theory[16] are valid, that is we have Lorentz invariance, positive energies, local commutativity, and the existence of a

[9] M. O. Taha, Department of Applied Mathematics and Theoretical Physics Report, Cambridge, 1965 (unpublished).

[10] S. Mandelstam, Phys. Rev. **140**, B375 (1965).

[11] The corresponding nonrelativistic problem appears to have been solved in recent work of M. Rubin, R. Sugar, and G. Tiktopoulos, Phys. Rev. **146**, 1130 (1966).

[12] D. Freeman, C. Lovelace, and J. Namyslowski, CERN Report, 1965 (unpublished); R. Alessandrini and R. Omnes, Phys. Rev. **139**, B167 (1965).

[13] J. L. Basdevant and R. Kreps, Phys. Rev. **141**, 1398; **141**, 1404 (1966); **141**, 1409 (1966).

[14] H. Cohen, A. Pagnamenta, and J. G. Taylor (to be published).

[15] J. G. Taylor, Nuovo Cimento Suppl. **1**, 857 (1964), in particular Paper III, pp. 945 ff.

[16] See, for example, R. Jost, in *Applied Mathematics* (American Mathematical Society, Providence, Rhode Island, 1965), Vol. IV; or R. Streater and A. Wightman, *PCT, Spin and Statistics, and All That* (W. Benjamin and Company, New York, 1964).

unique invariant vacuum. We also require the asymptotic condition to be valid,[17] so that off-mass-shell S-matrix elements $G(p_1,\cdots,p_n)$ are defined by

$$G(p_1,\cdots,p_n)=\int \exp(-i\sum_{j=1}^{n} p_j x_j)\prod_{j=1}^{n} d^4x_j(\Box_{x_j}{}^2-m_j{}^2)$$
$$\times\langle 0|T(\phi_1(x_1)\cdots\phi_n(x_n)|0\rangle.$$

We denote the connected part of $G(p_1,\cdots,p_n)$ by

$$\delta^4(\sum_{j=1}^{n} p_j)M'(p_1\cdots p_n);$$

$M'(p_1,\cdots,p_n)$ is denoted graphically by the bubble $\bigcirc'n$, and the propagators $iD_F(p)$, $iD_F'(p)$ is denoted by $(-)$ and $(+)$, respectively. We wish to expose the three-particle intermediate states between the sets of

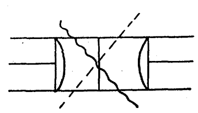
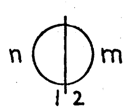


FIG. 1. A counterexample to the use of the "last cut" lemma without removal of suitable intermediate states. The wiggly and dashed lines cut two overlapping three-particle states, and there are no other three-particle intermediate states.

particles n and m in an amplitude $n\bigcirc'm$. The method for doing this was based in Ref. 15 on the "last cut" lemma. This lemma proves that it is possible to expose a three-particle intermediate state in a unique fashion which is closest to the set of particles n or the set of particles m (this intermediate state being the "last cut"). This is only possible provided that intermediate states involving fewer particles have been removed. If this prior removal is not effected, then the "last cut" need not be unique. This is evident from a study of perturbation diagrams, of which a possible counterexample is given in Fig. 1. In that figure the dashed and wiggly lines intersect. The propagators in three-particle

FIG. 2. The contribution to a process involving $n+m$ external particles without any one- or two-particle intermediate states between n of the particles and the remaining m.



states, but neither of these lines is nearer the left- or right-hand sides. Such a diagram is not allowed if all one- and two-particle intermediate states have first been removed. It may also be forbidden by selection rules, such as occur in $\pi\pi\pi$ or $\pi\pi N$ scattering, though not in πNN scattering. We will return later, in Sec. 4, to the case when such selection rules occur, but initially such selection rules will not be assumed. The contribution without one- or two-particle intermediate states is termed one- and two-particle irreducible, and denoted as in Fig. 2. We may now expose the three-particle intermediate

[17] H. Lehmann, K. Symanzik, and W. Zimmermann, Nuovo Cimento 1, 205 (1955). This condition has been partly derived from the earlier conditions by the work of Haag, Ruelle, and Hepp. This is carefully discussed, for example, in Jost's book in Ref. 16.

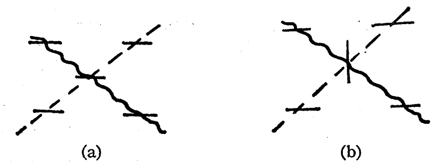


FIG. 3. The general form of overlapping three-particle intermediate states. Case (a) involves a common "horizontal" line, case (b) a common "vertical" line.

states in between n and m and choose a unique such state nearest to n or m. For if we cannot do this, then the two three-particle states "overlap" in the manner of Fig. 3. There are two possible overlaps. In case (a) we may choose a three-particle state nearer to the left-hand side; in case (b) a two-particle state may be exposed between left and right, and this is not allowed—hence our result.

A similar result on the exposure of one- and two-particle intermediate states is immediate. We also need to remove the possible self-energy contributions on external lines. This may be achieved by amputating each momentum with respect to the complete propagator: We multiply the function $M'(p_1\cdots p_n)$ by

$$\prod_{j=1}^{n} D_F(p_j)/D_F'(p_j),$$

which does not change the on-mass-shell observable S matrix element; $D_F(p)=(p^2-m^2)^{-1}$, and $D_F'(p)$ is the complete propagator

$$D_F'(p)=\int d^4(x-y)e^{-ip(x-y)}(\Box_x{}^2-m^2)(\Box_y{}^2-m^2)$$
$$\times\langle 0|T(\phi(x)\phi(y))|0\rangle.$$

Thus we define

$$M(p_1,\cdots,p_n)=[\prod_{j=1}^{n} D_F(p_j)/D_F'(p_j)]M'(p_1\cdots p_n).$$

If we now use the M functions throughout in place of the M' functions, we will at the same time have to use complete propagators for internal lines. The process of removing self-energies on both internal and external lines has now been achieved if we only use M functions, and not M' functions. We note here that if we try to expose a two-particle intermediate state, for example, in the complete amplitude, and not in the one-particle irreducible amplitude, then renormalization effects may explicitly arise. Thus if we consider exposing the two-particle intermediate state in $n\bigcirc'$-p nearest to the particle of momentum p we have the equation of Fig. 4.

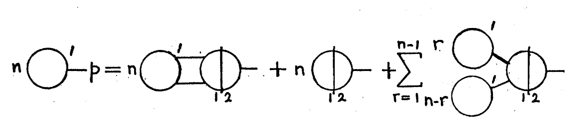


FIG. 4. The graphical equation exposing two-particle intermediate states in the complete amplitude between a single particle of momentum p and n other external particles.

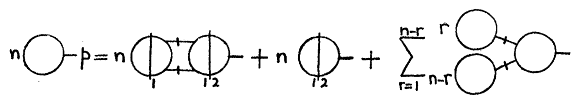


FIG. 5. The graphical equation exposing two-particle intermediate states, in the amputated amplitude, between a single particle of momentum p and n other external particles.

We take $\langle 0|\phi(0)|0\rangle=0$ in Fig. 4, so that the sum over r in the last term does not include the values $r=0$ or n. The right-hand side of Fig. 4 evidently contains self-energy additions to the propagator of momentum p; these are automatically taken care of if we expose the corresponding two-particle intermediate state in $n\bigcirc\text{-}p$, as in Fig. 5. Only for $n=1$ will self-energy effects arise in Fig. 5, and these can be removed by differentiation and integration on the external momentum, as carefully discussed elsewhere.[15,18] Before that can be done other

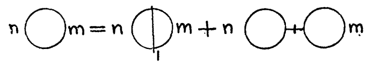


FIG. 6. The exposure of single-particle intermediate states in the amputated amplitude between n and m particles.

renormalizations have to be performed on any possible vertex functions. Vertex functions enter equations such as Fig. 5 in the last term on the right of Fig. 5; renormalization of these terms can again be achieved by differentiation and integration on the external momenta.[15,18]

The renormalization effects can only be taken account of explicitly in the case that we have a particular set of local-field equations, arising, say, from some Lagrangian. It is not yet known whether or not such

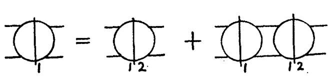


FIG. 7. The exposure of two-particle intermediate states in the amputated amplitude between two-particle states.

equations have solutions, though at least for quantum electrodynamics approximate solutions seem to make a great deal of experimental sense. Even if there do exist solutions it would be best if we could derive equations relating off-mass-shell S-matrix elements, which involve as little as possible of the details of the local interactions specifying how we are to go off-mass-shell; in such equations we expect there to be minimal effects from renormalization. These two properties, for our equations,

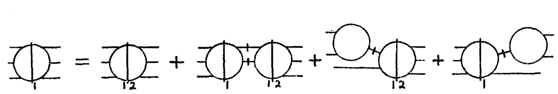


FIG. 8. The exposure of two-particle intermediate states in the amputated amplitude between three-particle states.

[18] K. Symanzik, in *Lectures on High Energy Physics*, edited by B. Jaksic (Gordon and Breach, Science Publishers, New York, 1966).

that is (*i*) of being independent of the particular local interactions present and (*ii*) of being independent of renormalization effects, are evidently desirable. We attempt to derive equations describing three-particle scattering which possess these properties to the highest degree.

From the remarks of the previous section we use the amputated M functions, and expose n-particle intermediare states after exposing $(n-1)$-particle intermedi-

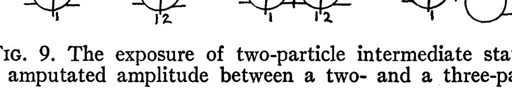


FIG. 9. The exposure of two-particle intermediate states in the amputated amplitude between a two- and a three-particle state.

ate states. The exposure of single-particle intermediate states is given in Fig. 6. We use M functions here, so the last term on the right-hand side of Fig. 6 automatically contains no further one-particle contributions than that exposed in the complete propagator. We remark that the exposure will always be by complete propagators, which is just a manifestation of what we elsewhere termed complete unitarity.[15]

It is necessary to underline here the fact that there must be consistency in our process of exposure; if we

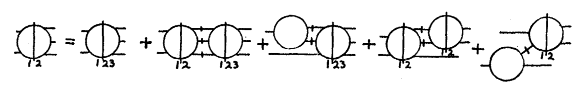


FIG. 10. The exposure of three-particle intermediate states in the amputated amplitude between a two- and a three-particle intermediate state.

have exposed a certain particle in the last term of the right-hand side of Fig. 6, then the same particle is absent in the first term on the right-hand side of Fig. 6. To obtain a set of equations with properties (*i*) and (*ii*) mentioned above we want to avoid distinguishing which particles are elementary and which are composite. By elementary particles we mean those related to fields with dynamical terms in a local Lagrangian; composite

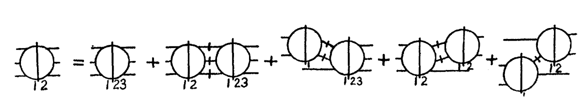


FIG. 11. The exposure of three-particle intermediate states in the amputated amplitude between three-particle states.

particles are "composed" of the elementary particles and are related to fields which are local functions of the elementary-particle fields.[19]

The process of exposing particles does not distinguish between elementary or composite particles; it only

[19] The definition of, and distinction between, elementary and composite particles is more fully discussed by M. M. Broido and J. G. Taylor, Phys. Rev. **147**, 993 (1966). See also references given therein to related works.

depends on the concept of "particle," as made specific in terms of a particle propagator which describes the way simple poles occur in various S-matrix amplitudes. Thus we may expose either composite or elementary particles or both by means of the last-cut lemma; if we expose a particle at any step, then we cannot, at a later stage, require it to arise as a composite-particle pole in the same channel from which it was originally removed.[20]

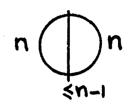


FIG. 12. The irreducible amplitude analogous to a nonrelativistic amplitude in an n-particle scattering process.

We now proceed to expose two-particle intermediate states. We will not explicitly distinguish between different particles, but assume that all our particles are distinct; we will discuss the equal-particle case later. We have, for two-particle scattering, the equation of Fig. 7. The three-particle scattering case is shown in Fig. 8. We also discuss the two → three particle amplitude in Fig. 9. The equations expressed by Figs. 7, 8, and 9 are seen to follow from the last-cut lemma by recognizing that each involves exposure of a two-particle state furthest to the right, and that disconnected scattering either before or after that two-particle state may occur.

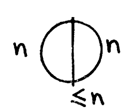


FIG. 13. The irreducible amplitude analogous to a potential in an n-particle scattering process.

We now expose the three-particle intermediate state in the one- and two-particle irreducible amplitudes as in Figs. 10 and 11. Let us now try to relate the equations expressed by Figs. 7, 10, and 11 to potential scattering equations. A simple approximation for the two-particle irreducible term in Fig. 7 is just that given by single-particle exchange; this term thus plays a role analogous to that played by a potential in potential-scattering theory. This is, indeed, well known from analyses of the nonrelativistic reduction of the B.S. equation expressed by Fig. 7.[21] We see that the same two-particle potential enters the last two terms of Figs. 10 and 11; we may thus regard the three-particle irreducible amplitude as the relativistic analogue of a three-particle potential, and Figs. 10 and 11 as the relativistic analogues of three-particle Lippmann-Schwinger potential-scattering equations. We note that we expect the analogy of relativistic theory to potential theory to be good for the irreducible diagrams of Fig. 12; this is due to the fact that in potential scattering, particles cannot be created or destroyed. This situation is just that which occurs in the equation for the term in Fig. 12, if that in Fig. 13 is considered as a given potential; in Figs. 7 or 11 only two- or three-particle intermediate states occur.

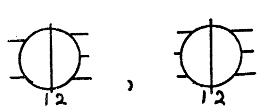


FIG. 14. The amplitude solutions of the equations expressed by Figs. 8 and 9.

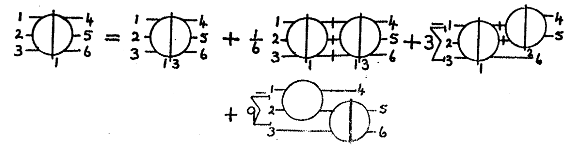


FIG. 15. The symmetrized form of three-particle exposure in three-particle scattering for identical particles.

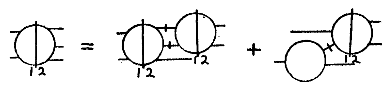


FIG. 16. The three-particle exposure in two → three particle scattering when the three-body potential is neglected.

We do not expect the equations expressed by Figs. 8 and 9 to have a potential-scattering analog, and their structure is not discussed in any great detail. However, the one important fact about them is that neither of them involve the two-body potential of Fig. 13 for $n=2$. In fact if the equations expressed by Figs. 10 and 11 could be solved for the amplitudes of Fig. 14, then knowledge of the complete two-particle scattering amplitude, the complete vertex functions, and the complete propagators allow the complete two → three and three → three scattering amplitudes to be obtained from Figs. 8 and 9 by simple integration. Thus if we assume that we are given the complete propagator, vertex function, and two-particle scattering amplitude (in some suitable approximation), then the main difficulty in obtaining the two → three and three → three particle amplitudes is that of solving the equations expressed by Figs. 10 and 11. In order to be able to use approximations to the two-particle scattering amplitude which allow the two-particle

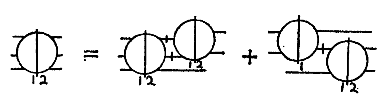


FIG. 17. The three-particle exposure in three-particle scattering when the three-body potential is neglected.

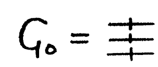


FIG. 18. The free Green's function G_0 for three-particle scattering.

[20] We can, if we wish, impose the condition of being composite on a particle; this will correspond to taking the wave function renormalization constant to be zero, as was more fully considered in Ref. 19.

[21] E. E. Salpeter and H. A. Bethe, Phys. Rev. 84, 1232 (1951); G. C. Wick, *ibid.* 96, 1124 (1954).

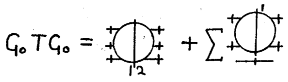


FIG. 19. The three-particle T-matrix T.

bound state and resonance structure to be exploited, we have thus to transform only Figs. 10 and 11 so as to replace the two-particle potentials by complete scattering amplitudes. This is exactly the problem solved by Faddeev[2]; we will attempt to use the same method here. Before we do this let us first discuss the change in Figs. 7 to 11 when identical particles occur. We consider first the case when only a single type of particle is present. We assume the particle to be a neutral scalar, so no selection rules occur to put to zero the amplitudes with an odd number of external particles. Then Figs. 7–11 have exactly the same graphical structure, though

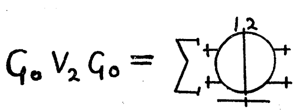


FIG. 20. The three-particle potential V_2 arising from two-body scattering.

now they must lead to scattering amplitudes with the correct symmetry. The symmetry in irreducible amplitudes is in all the momentum variables which enter on one side or another of the vartical line denoting absence of up to so many particles in an intermediate state; no symmetry is expected with respect to variables on either side of such a line. This symmetry is preserved on the right-hand sides of Figs. 8–11 by symmetrization over the external momenta on the left and on the right of the vertical line; further, each intermediate state of n particles occurring on the right of 8 to 11 must have a factor $(n!)^{-1}$ to account for the symmetry. As an example of this, Fig. 11, for pseudoscalar mesons, becomes

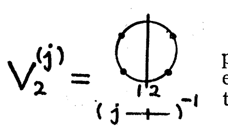


FIG. 21. The contribution to the three-particle potential V_2 arising from the process in which the jth particle is not scattered by the other two particles.

Fig. 15, where the operator $\bar{\Sigma}$ denotes symmetrization over the variables 1, 2, 3 and over 4, 5, 6:

$$\bar{\Sigma}=(3!)^{-2}\sum_{p(123)}\sum_{p(456)},$$

where $\sum_{p(123)}$ denotes summation over all permutations of 1, 2, 3, of the function on which it acts.

Similar symmetrization and factors $(n!)^{-1}$ arise when considering the scattering, say, of two identical particles by a different one, such as in $\pi\pi N$ scattering. We need not spell out these details further here.

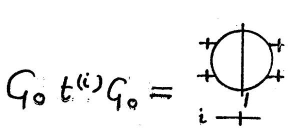


FIG. 22. The three-body partial t-matrix $t^{(i)}$.

Finally we will write down the equations resulting from Figs. 10 and 11 on neglect of the three-body potentials of Fig. 13 with $n=3$ in Figs. 16 and 17. It is these equations which we would attempt to solve below the production threshold for an extra particle; in particular for three-pion scattering we hope to use Fig. 17 later to see if we can obtain an ω meson.[14] We

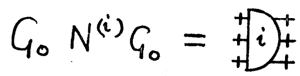


FIG. 23. The three-body partial amplitude $N^{(i)}$.

note that the equations expressed by Figs. 16 and 17 are linear integral equations with a kernel depending only on the two-particle potential. We turn in the next section to replacement of this kernel by one depending only on the two-particle amplitude.

III. REMOVAL OF THE POTENTIAL

In order to remove the two-particle potential of Fig. 13 with $n=2$ from Figs. 10 and 11 it will be useful to rewrite Fig. 11 in a form similar to the Lippmann-Schwinger integral equation. We do this by introducing the following notation:

(a) The free Green's function G_0 for three-particle scattering, as in Fig. 18. We do not insert the variables,

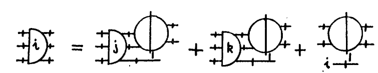


FIG. 24. The graphical expression of Fig. 6 for the three-body partial amplitudes.

or consider distinct or identical particles, since these are trivial.

(b) The three-particle T matrix, as in Fig. 19, where the summation is over all possible choices of pairs of particles, the third particle being unscattered.

(c) The three-particle potential V_2 arising from two-body scattering, as in Fig. 20, with the summation being the same as in the definition of T.

(d) The three-particle potential V_3 arising from three-particle scattering, as in Fig. 14 with $n=3$.

(e) The three-particle scattering amplitude G.

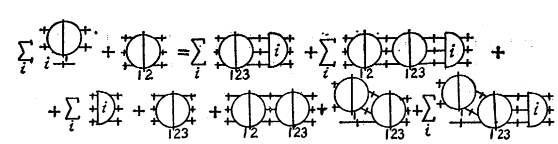


FIG. 25. The graphical expression of Fig. 7 for the three-body two-particle irreducible amplitude.

Then Fig. 11 is, in the above notation:

$$T=V_2+V_3+TG_0(V_2+V_3), \qquad (1)$$

where we are using the obvious notation of integration over internal variables in the last term in (1). In order to remove V_2 we use the fact that V_2 in Fig. 20 is a sum of terms

$$V_2=\sum_i V_2^{(1)};$$

$V_2^{(i)}$ is given in Fig. 21 where $i=1$, 2, or 3, and denotes the particle type we are considering (or the momentum

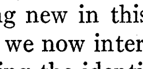


FIG. 26. The two-particle propagators with no interaction.

variable, if the particles are identical). We may rewrite (1) formally as

$$T=[V_2+V_3+TG_0V_3](1-G_0V_2)^{-1}. \tag{2}$$

It is not evident that the kernel $(1-G_0V_2)$ is invertible; we certainly are not able to discuss whether it is or not at present, due to the presence of conditionally convergent integrals. We return to this elsewhere,[14,22] and only remark here that our arguments can be put into a form independent of the invertibility of $(1-G_0V_2)$; we do not do that here since some of the simplicity of the argument is lost in so doing.

We consider $N=(1-G_0V_2)^{-1}$, so $N=NG_0V_2+I$. Define $N^{(i)}$ so that $G_0N^{(i)}=NG_0V_2^{(i)}$, so

$$N=\sum_i G_0N^{(i)}+I.$$

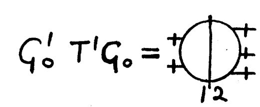


FIG. 27. The two → three particle irreducible amplitude T^1.

Then

$$N^{(i)}=[N^{(j)}+N^{(k)}]G_0V_2^{(i)}+N^{(i)}G_0V_2^{(i)}+V_2^{(i)},$$

so if $t^{(i)}=V_2^{(i)}[1-G_0V_2^{(i)}]^{-1}$ is the scattering amplitude when the ith particle does not interact, we have

$$N^{(i)}=[N^{(j)}+N^{(k)}]G_0t^{(i)}+t^{(i)}. \tag{3}$$

Also if $V_2N=M=V_2(1-G_0V_2)^{-1}$, then

$$M=\sum_i N^{(i)}.$$

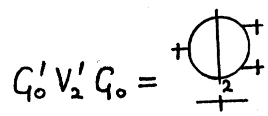


FIG. 28. The two → three-particle connected potential V_3.

Equation (3) may be solved from knowledge of $t^{(i)}$ alone, and we may now write (2) as

$$T=(V_3+TG_0V_3)(\sum_i G_0N^{(i)}+I)+\sum_i N^{(i)}. \tag{4}$$

[22] J. G. Taylor (to be published).

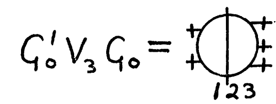


FIG. 29. The two → three particle disconnected potential V_2^1.

We have thus obtained a linear integral equation for T, in which the two-particle potential V_2 has disappeared and been replaced by the "partial" scattering amplitudes $N^{(i)}$. There is nothing new in this; it is exactly Faddeev's trick.[2] However, we now interpret the terms entering (3) and (4) following the identification set up at the beginning of this section. In graphical form we may write (3), for example, as follows. We have $G_0t^{(i)}G_0$ as in Fig. 22, and if we denote $G_0N^{(i)}G_0$ as in Fig. 23 (where the curved side makes explicit the side we are considering the interaction on), then (3) becomes Fig. 24. This is exactly the expected structure, and the

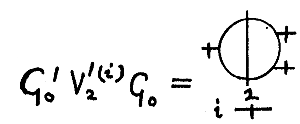


FIG. 30. The partial contribution $V_2'^{(i)}$ to V_2'.

solution of Fig. 24 will generate T directly if $V_3=0$, as we have seen from (4). Also, (4) becomes Fig. 25. It is possible to separate out the disconnected parts of each function $N^{(i)}$, but this separation is so evident from Fig. 24 that we will not perform it here.

We perform a similar removal of the two-particle potential in Fig. 10 proceeding as before, with G_0' as in Fig. 26, $G_0'T'G_0$ as in Fig. 27, $G_0'V_3G_0$ in Fig. 28, and $G_0'V_2'G_0$ in Fig. 29, so Fig. 10 becomes

$$T'=V_2'+V_3+T'G_0V_3+T'G_0V_2. \tag{5}$$

As before, this becomes

$$\begin{aligned} T'&=(V_2'+V_3+T'G_0V_3)(1-G_0V_2)^{-1}\\ &=(V_3+T'G_0V_3)(\sum G_0N^{(i)}+I)\\ &\qquad+V_2'(\sum G_0N^{(i)}+I). \end{aligned} \tag{6}$$

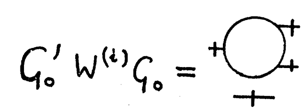


FIG. 31. The partial disconnected two → three amplitude $W^{(i)}$.

We may rewite $V_2'(\sum G_0N^{(i)}+I)$ to remove V_2' as follows. We have $G_0'V_2'G_0=\sum_i G_0'V_2'^{(i)}$, where $G_0'V_2^{(i)}G_0$ is as in Fig. 30. Then if

$$V_2'(1-G_0V_2)^{-1}=H,$$

we have

$$V_2'+HG_0V_2=H.$$

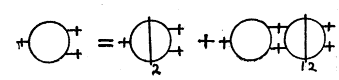


FIG. 32. The two-particle exposure in $W^{(i)}$

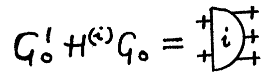


FIG. 33. The partial amplitude $H^{(i)}$ for two → three particle scattering.

We let $H=\sum H^{(i)}$, where $H^{(i)}$ satisfies

$$V_2'^{(i)}+HG_0V_2^{(i)}=H^{(i)},$$

so

$$H^{(i)}=V_2'^{(i)}G^{(i)}G_0^{-1}+[H^{(j)}+H^{(k)}]G_0t^{(i)},$$

where $G_0^{(i)}=(1-G_0V_2^{(i)})^{-1}$ is the Green's function when the ith particle is unscattered. But $V_2'^{(i)}G^{(i)}G_0^{-1}=W^{(i)}G_0^{-1}$ satisfies the equation

$$W^{(i)}=W^{(i)}G_0V_2^{(i)}=V_2'^{(i)},$$

or graphically, with $G_0'W^{(i)}G_0$ as in Fig. 31, we have the equation expressed by Fig. 32. But this is just the equation for the complete vertex function (which we

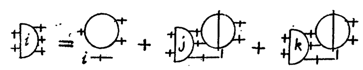


FIG. 34. The equation for the partial two → three-particle scattering amplitudes containing only the two-particle amplitude.

had anticipated by the graphical notation we gave to $W^{(i)}$). Hence in (6)

$$H^{(i)}=W^{(i)}+[H^{(j)}+H^{(k)}]G_0t^{(i)}. \tag{7}$$

If we denote $G_0'H^{(i)}G_0$ as in Fig. 33, then (7) becomes as in Fig. 34, while (5) graphically is as in Fig. 35. As with the three → three particle amplitude we have completely removed the two-particle potential from (5), as is readily seen in the graphical equations of Figs. 34 and 35. We have also preserved the linearity of the equations. Again we could separate out connected parts specifically in Figs. 34 and 35, but we think they are too evident to do so.

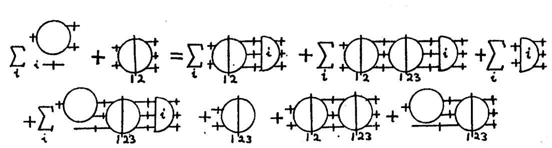


FIG. 35. The graphical representation of Fig. 8.

We also remark that when the three-particle potential is neglected the value of T' given by (6) is just $\sum_i H^{(i)}$, as would be expected.

We have now solved the problem of exposing three-particle intermediate states so that the resulting equations have kernels which depend only on the complete propagator, the vertex function, and the two-particle scattering amplitude. In the next section we will specify in detail the form of these equations for 3π, $\pi\pi N$, and $3N$ scattering, so that they will be ready for the computations to be attempted.

IV. SPECIFIC PROCESSES

We will not spell out the details of equations expressed by Figs. 24, 25, 34, and 35 for processes of particular interest. We will also neglect three-body potentials.

1. Three-Pion Scattering

In this case Figs. 34 and 35 become trivial, while Fig. 24 acquires a factor $\frac{1}{2}$ in front of the first two terms

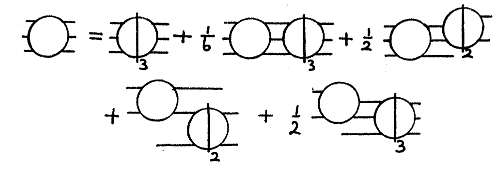


FIG. 36. The exposure of three bare particles in three-particle scattering.

on the right, while the last term on the right has to be summed over all permutations of the momenta on the left, and i denotes a given value of the momentum of one of the pions on the right. The remaining equation is given in Fig. 11.

There is one essential difference between the structure of these equations and those considered by Lovelace *et al.*[12] It is in the removal of the single-pion intermediate state initially in our equation of Fig. 11 as

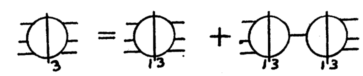


FIG. 37. The expression for the contribution to the three-particle scattering amplitude containing no three-pion intermediate stage.

compared to its being left in the three-particle amplitude in Ref. 12. In order to allow this, one can no longer put to zero the three-particle potential term in (1), since it now contains the single bare pion. Thus we have the graphical equation of Fig. 36 where the first term on the right of Fig. 36, A_3, contains no three-pion intermediate state, but may contain a one-pion intermediate state. Thus we have Fig. 37, where $(-)$ denotes the bare pion propagator, and the term B_3 of Fig. 38 is a reduced vertex function. In the approximation of neglecting three-body potentials we have $B_3=$ constant, and $A_3=B_3(-)B_3$, and thus is separable. However, this term gives explicit renormalization contributions which, taken together with single-particle contributions in the second and third terms on the right-hand side of Fig. 36, produces just the single-particle pole term exposed in the last term of Fig. 6. Since the actual process of renormalization has to be

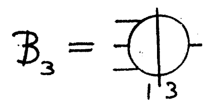


FIG. 38. The coupling constants B_3.

performed, and will lead back to Figs. 6 and 11, we consider it better to start from the one-particle irreducible amplitude in the first place. It is essential to do this, for example, in πNN scattering, for as the example in Fig. 1 of Sec. 2 shows that the first term on the right-hand side of Fig. 36 cannot be uniquely defined for this case.

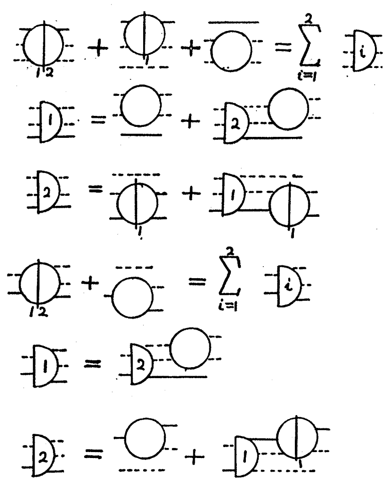


FIG. 39. The equations for $\pi\pi N$ scattering.

2. $\pi\pi N$ Scattering

Again we take out the single-particle (nucleon) pole and then we have (with --- denoting a pion, —— a nucleon) as in Fig. 39, where the identical particles have been correctly accounted for. We remark again here that we will not have a single-nucleon pole contribution arising in the solutions of Fig. 34; such a pole has been sub-

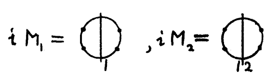


FIG. 40. The two particle amplitude iM_1 and potential iM_2.

tracted out. It is not consistent with this procedure to attempt to force a solution which has this pole, so apparently making the nucleon a bound state.

The equations for three-nucleon scattering are identical to those for three-pion scattering, so we will not write them down here.

V. UNITARITY

We finally discuss the manner in which unitarity is satisfied. We do this first for two-particle unitarity in the B.S. representation of Fig. 7. If we denote the two-particle amplitude by iM_1 and the potential by iM_2 as in Fig. 40, then Fig. 7 may be written

$$M_1 = M_2 + iM_1KM_2. \tag{8}$$

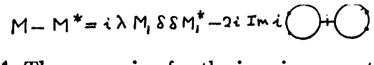


FIG. 41. The expression for the imaginary part of the two-particle amplitude M.

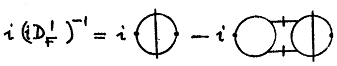


FIG. 42. The general inverse propagator equation.

The kernel K denotes the product of the two complete propagators. We may replace these complete propagators by the free-particle propagators in (8), if we are only concerned with two-particle unitarity. We assume M_2 is real below the single-particle production threshold, and is an analytic function of its energy variables in the expected cut planes[22]; then we expect M_1 to be analytic in the same regions. Arguing formally, we have

$$M_1 = M_2(1 - iKM_2)^{-1},$$

so

$$\begin{aligned} M_1 - M_1^* &= M_2[(1 - iKM_2)^{-1} - (1 + iK^*M_2)^{-1}] \\ &= iM_1(K + K^*)M_1^*. \end{aligned} \tag{9}$$

If we split up each propagator in K into a principal-value term plue $(i\pi)$ times a delta-function term then

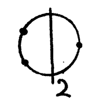


FIG. 43. The "coupling constant."

$(K^* + K)$ will be the sum of the product of the two principal-value terms and the product of the delta-function terms; the assumed analyticity of M_1 implies that the first of these terms gives no contribution, so $K^* + K = \lambda\delta\delta$, for some real constant λ, and

$$M_1 - M_1^* = i\lambda M_1 \delta\delta M_1^*, \tag{10}$$

which is the usual expression for unitarity, though as yet without the single-particle intermediate-state contribution.[23]

We finally have to add in this single-particle intermediate-state contribution; in other words, to the right-hand side of (10) must be added the imaginary

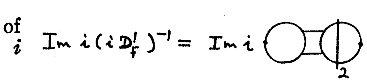


FIG. 44. The value of the imaginary part of i $(iD'_F)^{-1}$.

part of this single-particle term, in order to calculate $M - M^*$, as in Fig. 41. We will have to calculate the imaginary part of the complete propagator iD_F':

$$\mathrm{Im}(iD_F') = \mathrm{Im}\, i(iD_F')^{-1}(D_F')(D_F')^*.$$

Now the general inverse propagator equation is[15] given in Fig. 42. We are interested only in two-particle intermediate states in Fig. 41, so we may take the

[23] The value of this constant is not given correctly by splitting each propagator into its principal part and δ-function contributions, due to the singular nature of the product ot the two principal-part contributions. This singular contribution can only be some constant times the product of the two δ functions; it may be shown to be equal to that arising from the original product of δ functions. The author would like to thank A. Pagnamenta for pointing this out, and for giving a simple derivation of the correct contribution.

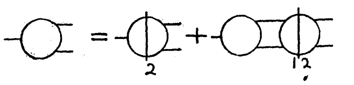


FIG. 45. The two-particle approximation to the vertex function.

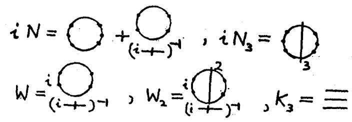


FIG. 47. The various terms entering in the equation of Fig. 36.

coupling content of Fig. 43 to be real; in this approximation we may also replace the complete propagators by free propagators. Also the first term on the right of Figs. 42 is (p^2-m^2),[24] so we have Fig. 44.

The two-particle approximation to the vertex function is given in Fig. 45 so if Γ is as in Fig. 46, and Γ_2 is the coupling constant of Fig. 43, then $\Gamma=\Gamma_2(1-iKM_2)^{-1}$, and

$$\begin{aligned}&-2i\,\mathrm{Im}i(+)^{-1}\\&=i\Gamma_2[(1-iKM_2)^{-1}K+(1+iK^*M_2)^{-1}K^*]\Gamma_2\\&=-\Gamma_2(1+iK^*M_2)^{-1}(K+K^*)M_2(1-iKM_2)^{-1}K\Gamma_2\\&\quad+i\Gamma_2(1+iK^*M_2)^{-1}(K+K^*)\Gamma_2\\&=-\lambda\Gamma^*\delta\delta\pi_1K\Gamma_2+\lambda i\Gamma^*\delta\delta\Gamma_2=\lambda i\Gamma\delta\delta\Gamma^*.\end{aligned}\tag{11}$$

In the last step of (11) we have used the alternative vertex-function equation $\Gamma=iM_1K\Gamma_2+\Gamma_2$. Thus $-(D_F')-(D_F')^*=-\lambda i\Gamma\delta\delta\Gamma^*(D_F')(D_F')^*$. Also

$$\begin{aligned}\Gamma-\Gamma^*&=-\Gamma_2(1-iKM_2)^{-1}i(K+K^*)M_2(1+iK^*M_2)^{-1}\\&=-\lambda i\Gamma^*\delta\delta M_1.\end{aligned}$$

If we write

$$\Gamma=\Gamma_r+i\Gamma_i,\quad \Gamma_i=\mathrm{Im}\Gamma,$$

then

$$\begin{aligned}[-\Gamma(D_F')\Gamma-\Gamma^*(D_F')^*\Gamma^*]&=+\lambda iM_1\delta\delta\Gamma^*(D_F')^*\Gamma^*\\&\quad-\lambda i\Gamma(D_F')\Gamma\delta\delta\Gamma^*(D_F')^*\Gamma^*+\lambda i\Gamma(D_F')\Gamma\delta\delta\Gamma_1^*.\end{aligned}$$

So altogether

$$\begin{aligned}M-M^*&=i\lambda[M_1\delta\delta\Gamma_1^*-M_1\delta\delta\Gamma^*(D_F^1)^*\Gamma^*-\Gamma(D_F^1)\Gamma\delta\delta\Gamma_1\\&\quad-\Gamma(D_F^1)\Gamma\delta\delta\Gamma^*(D_F^1)^*\Gamma^*]=i\lambda M\delta\delta M^*,\end{aligned}$$

which is the required two-particle unitarity equation.

It is not possible to regard the above as a full proof, since we have assumed kernels that involved conditionally convergent integrals were invertible. However, we hope elsewhere to give a rigorous analysis of this problem by means of the energy-analytic representation.[22] Our "result" on unitarity gives the details of how unitarity obtains its correct contributions from the different off-mass-shell terms, and shows under what conditions we expect unitarity to be valid. This will be useful in later discussions of approximations.

We should add here that there is a rigorous proof of unitarity for each partial-wave B.S. equation[25]; this proof has not yet been extended to the full B.S. equation, though the methods of Ref. 22 should give such a proof. We now turn to three-particle unitarity. We wish to prove that three-particle unitarity is satisfied (along with two-particle unitarity) provided certain "potentials" or irreducible amplitudes are real. We do this for the three-pion problem; the extension to the general case discussed in Secs. 2 and 3 is lengthier but does not add anything essentially new to the problem. We also use the nonremormalized equation of Fig. 36, though, again, the proof may be given for the renormalized equation of Fig. 11, with added complexity, as we elucidated in the two-particle case. Those details are not given here. We may write Fig. 36, with iN, iN_3, W, W_2, and K_3 as denoted in Fig. 47,

$$N=(N_3+W_2)(1-iK_3N_3-iK_3W_2)^{-1}.$$

Then when N_3 and W_2 are real, we have

$$N-N^*=iN(K_3+K_3^*)N^*.$$

The kernel $K_3+K_3^*=+[P^2\delta+\delta^3]$ (to within constants), where P denotes a principal-value operator and δ a delta function. Assuming analyticity in the energy variables of N, then the part of $(K_3+K_3^*)$ involving $P^2\delta$ will give zero, since one of the two independent energy variables can be chosen to have only these principal-value terms, and not appear in the delta function. Hence

$$N-N^*=\mu iN\delta^3N^*,$$

which is the expected three-particle unitarity[26]; the disconnected term in N gives no contribution on the mass shell. Again this "proof" is not rigorous (we will present a more rigorous proof elsewhere); however, it shows that we always preserve three-particle unitarity provided the three-body "potential" is real below the five-particle threshold.

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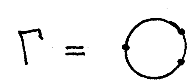

FIG. 46. The vertex function.

[24] See paper III, p. 964 of Ref. 15.

[25] See paper V of Ref. 15, Appendix I.

[26] The constant μ is again given incorrectly by the (principal-part+δ-function) splitting of propagators; the correct value may be reached by similar methods to those needed for two-particle unitarity mentioned in Ref. 23.