Near-Peripheral p-p Scattering from Simple One-Boson-Exchange Contributions*†

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We fit the non-S-wave p-p amplitude over the 0- to 345-MeV range with three pole terms utilized in a simple fashion; the pole terms are equated with the real part of the scattering amplitude. The imaginary part is then specified by elastic unitarity. The pole terms which give the fit are due to vector-meson exchange ($g_V^2=2.3$, $f_V/g_V=2.3$, $m_V=770$ MeV), scalar-meson exchange ($g_\sigma^2=1.6$, $m_\sigma=390$ MeV), and pion exchange ($g_\pi^2=12.3$, $m_\pi=135$ MeV). The coupling constants and masses were found with an automatic search routine on an IBM 7094 or CDC 3600 computer. For experiment we took the single-energy phase-shift analyses of Arndt and MacGregor at 25, 50, 95, 142, 210, and 310 MeV. The pole parameters are consistent with other independent experiments to within the accuracy of the model. The scalar meson is postulated, but may be responsible for the $T=0$, S-wave π-π correlation observed in the reaction $\pi^-+p \to \pi^++\pi^-+n$ at low energies.

I. INTRODUCTION

THIS is the first of a series of articles which will treat proton-proton scattering from the standpoint of multipion resonances. The experiments in question range from 22 to 345 MeV in laboratory kinetic energy of the projectile proton. It is well known that this interaction is very complicated; nearly every kind of force allowed by general invariance principles is present: central, spin-spin, tensor, spin-orbit, and perhaps quadratic spin-orbit. Yet recently there has appeared the so-called "pole" model which succeeds in fitting the major part of the data with just a few properly chosen one-meson-exchange Born terms or cross-channel (non-Regge) pole contributions.

A minimum of three pole terms is required: the pion pole term, a vector-meson ($J=1^-$) pole term, and a scalar-meson ($J=0^+$) pole term. Of these, the pion pole contribution is well understood. The vector pole term is also pretty well understood, particularly in regard to the strong central repulsion and the spin-orbit attraction which comprise part of the interaction. One recalls that these features led Breit[1] and Sakurai[2] to predict a vector meson in the first place. Our $J=1^-$ pole term may be considered an average of the ω, ρ, and ϕ pole terms.

The necessity for a third pole term is also clear; the central repulsion occasioned by the $J=1^-$ pole term is far too strong at impact parameters exceeding 1 F. A $J=0^+$ pole term is postulated—let us call the field quantum "σ"—in order to provide a strong central attraction at these larger distances. The σ pole term could be replaced by other physical processes leading to an attraction of the same strength and range—for example, the pole contribution of a $J=2^+$ meson—but we are led to choose the 0^+ pole contribution because the effective mass appears to lie between 2 and 4 pion masses, and such low mass favors the lower angular-momentum quantum number. Actually, not even a $J=0^+$ pole term, or resonance, is required *per se*; Scotti and Wong[3] fit the data in one version of their work with just a $T=0$, π-π S-wave attraction parametrized by an effective-range formula. But if a resonance is hypothesized, the width is immaterial to the fit over a very large range; only the resonance position is critical.

On the experimental side, there appears to be some evidence for a strong π-π S-wave interaction. Principally there is the correlation of the outgoing pions in the reaction[4] $\pi^-+p\to\pi^++\pi^-+n$. A peak occurs in the $T=0$ π-π effective mass spectrum near 400 MeV. However, this peak occurs only when the incident pion momentum is low, say less than 700 MeV/c. When the pion momentum is greater the peak disappears. Thurnauer[5] suggests that this disappearance is only because of competition with the (3,3) resonance which emerges at 600 MeV/c. Assuming the existence of the P_{11} resonance he considers the extra pion to be produced via either $N_{1/2}^*(1480)\to N+\sigma$ or $N_{1/2}^*(1518)\to N_{1/2}^*(1236)+\pi$. The σ parameters which best fit the data are $m_\sigma=490$ MeV and $\Gamma=110$ MeV.

The fact that Thurnauer's σ is nearly 500 MeV and not 400 MeV may explain why the σ resonance has not shown up in K_{e4} decay. Birge *et al.*[6] have studied $K_{e4}^+\to\pi^-+\pi^++e^++\nu$ and find that the invariant π-π mass spectrum resembles S-wave phase space and not the peaked curve that results from the 400 MeV σ, width 100 MeV, proposed by Faier and Brown.[7] If the σ resonance position were moved to 490 MeV, however, the π-π mass distribution should again resemble phase space.

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[6] R. W. Birge, R. P. Ely, Jr., G. Gidal, G. E. Kalmus, A. Kernan, W. M. Powell, U. Camerini, D. Cline, W. F. Fry, J. G. Gaidos, D. Murphree, and C. T. Murphy, Phys. Rev. 139, B1600 (1965).

[7] H. Faier and L. M. Brown, Bull. Am. Phys. Soc. 10, 467 (1965).

Another possible $T=0$ $J=0^+$ π-π resonance which has appeared on the experimental scene is the ϵ^0, at or near 700 MeV. However, 700 MeV is too high a resonance position to allow a fit to the data according to our calculations. But there is nothing to say that the ϵ^0 could not contribute along with a lower effective mass σ.

The basic pole terms have been treated in a variety of ways in the literature. A common requirement in all attempts is the necessity of generating a unitary amplitude from the Born terms which are of course real. One method has been to treat the Born terms as a potential in a Schrödinger equation[8]; it turns out that the resulting amplitude can differ drastically from the potential in P states. This is due to the ladder diagrams. While this need not be incorrect in itself, there may be a problem of double-counting, as the multipion resonances appear in both the ladder terms and the pole terms. A different approach has been to introduce unitarity corrections via partial-wave dispersion relations, either by solving the N/D equation for the partial wave amplitudes[3,9] or by simply including in some fashion the integral over the "physical," or "unitary" cut.[10–12] In so doing, however, it is necessary to preserve the threshold behavior inherent in the pole terms and there is considerable arbitrariness in how this may be carried out. Furthermore, in all treatments, a cutoff has to be introduced to which the results can be quite sensitive.

To avoid these ambiguities and yet learn as much as possible about the problem, we have elected to study p-p scattering in a first approximation, namely, to represent the real part of the amplitude by the pole terms alone, and let the imaginary part be specified solely by elastic unitarity. This procedure has been termed "geometric unitarization" by Moravcsik.[13] The results obtained will subsequently be used as a first approximation to a dispersion-theoretic treatment which will appear in later publications.

With geometric unitarization we must exclude the $^1S_0$ state at the outset as clearly this approximation is inadequate to treat a nearly bound state. For P states and higher, however, the approximation may prove to be much more successful; $l\geq 1$ phase shifts in p-p scattering are rather small on the average, and only the $^3P_1$ phaseshift ever exceeds 20^0.

Sawada, Ueda, Watari, and Yonezawa[14] have treated p-p scattering using a different version of geometric unitarization, equating the pole terms with the K matrix ($B_l=\tan\delta_l$). Because the phase shifts are small, this treatment is similar to the one we employ ($B_l=\cos\delta_l \sin\delta_l$).

In Sec. II partial-wave projections of the pseudoscalar, scalar, and vector pole terms are given. The physical consequences of the pole model are presented and discussed in Sec. III. A summary of principal findings is given in Sec. IV. The reader disinterested in details may skip to Sec. III with little loss in continuity.

II. POLE PROJECTIONS

In this section we present a method for projecting the pole contributions into the individual angular-momentum states. One may begin with Stapp's expansion of the nucleon-nucleon M-matrix elements into partial wave amplitudes as in Table III of Stapp, Ypsilantis and Metropolis (SYM).[15] However, we prefer to employ the unsymmetrized form of the M matrix; this is obtained by multiplying Stapp's formulas by $\frac{1}{2}$ and taking the sum over all l values. One then inverts these formulae and obtains expressions for the partial wave amplitudes in terms of the M's.

In carrying out this inversion, one projects out l by multiplying through by $P_l^{|m_s'-m_s|}(\cos\theta)$ and integrating over θ; this gives terms like

$$M_{l,m_s,m_s'}=\int_{-1}^{1}\tfrac{1}{2}d(\cos\theta)\times P_l^{|m_s-m_s'|}(\cos\theta)M_{m_s,m_s'}(\cos\theta,\phi=0).$$

One then projects out j by summing over the $M_{l,m_s,m_s'}$ with the appropriate Clebsch-Gordan coefficients. This yields the following formulas;

$$\begin{aligned}
p^{-1}\alpha_l&=M_{l,ss},\\
p^{-1}\alpha_{ll}&=M_{l,11}+l^{-1}(l+1)^{-1}\sqrt{2}M_{l,01}-l^{-1}(l+1)^{-1}M_{l,-11},\\
p^{-1}(2l+3)\alpha_{l,l+1}&=(l+2)M_{l,11}-(l+1)^{-1}(l+2)\sqrt{2}M_{l,01}+(l+1)^{-1}M_{l,-11}+\sqrt{2}M_{l,01}+(l+1)M_{l,00},\\
p^{-1}(2l+3)(l+2)^{-1/2}(l+1)^{1/2}\alpha^{l+1}&=-(l+1)M_{l,11}+\sqrt{2}M_{l,01}-(l+2)^{-1}M_{l,-11}+\sqrt{2}M_{l,01}+(l+1)M_{l,-11},\\
p^{-1}(2l-1)\alpha_{l,l-1}&=(l-1)M_{l,11}+l^{-1}(l-1)\sqrt{2}M_{l,01}+l^{-1}M_{l,-11}-\sqrt{2}M_{l,01}+lM_{l,00},\\
p^{-1}(2l-1)(l-1)^{-1/2}l^{1/2}\alpha^{l-1}&=-lM_{l,11}-\sqrt{2}M_{l,01}-(l-1)^{-1}M_{l,-11}-\sqrt{2}M_{l,10}+lM_{l,00}.
\end{aligned}$$

[8] R. A. Bryan, C. R. Dismukes, and W. Ramsay, Nucl. Phys. **45**, 353 (1963). For a later potential model treating the complete p-p and n-p amplitude (except S waves) see R. A. Bryan and B. L. Scott, Phys. Rev. **135**, B434 (1964), or R. A. Bryan, Colloq. Intern. Phys. Théor. Bordeau, **1**, 1 (1964).

[9] A. Scotti and D. Y. Wong, Phys. Rev. Letters **10**, 142 (1963); and *International Conference on Nucleon Structure* (Stanford University Press, Palo Alto, California, 1963), p. 298.

[10] R. A. Arndt, thesis, University of California Radiation Laboratory Report 12211 (unpublished), and Phys. Rev. (to be published).

[11] M. H. MacGregor, Phys. Rev. Letters **12**, 403 (1964).

[12] P. B. Kantor, Phys. Rev. Letters **12**, 52 (1964).

[13] M. J. Moravcsik, Ann. Phys. (N. Y.) **30**, 10 (1964).

[14] S. Sawada, T. Ueda, W. Watari, and M. Yonezawa, Progr. Theoret. Phys. (Kyoto) **28**, 991 (1962). For a pole fit to just the higher partial waves, see W. Ramsey, Phys. Rev. **130**, 1552 (1963). For a later treatment of p-p plus n-p scattering, see S. Sawada *et al.*, Progr. Theoret. Phys. (Kyoto) **32**, 380 (1964); T. Ino, M. Matsuda, and S. Swada, *ibid.* **33**, 489 (1965).

[15] H. P. Stapp, T. J. Ypsilantis, and N. Metropolis, Phys. Rev. **105**, 302 (1957).

$M_{l,ss}$ results from integrating over the singlet amplitude M_{ss}. The α_{lj}, α^j, and α_l are defined to be the α's of SYM divided by $2i$; p is the magnitude of the center-of-mass momentum of either nucleon. Thus

$$\begin{aligned} \alpha_l &= \exp(i\delta_l)\sin\delta_l\,, \\ \alpha_{ll} &= \exp(i\delta_{ll})\sin\delta_{ll}\,, \\ \alpha_{lj} &= (2i)^{-1}[\cos 2\epsilon_j \exp(2i\delta_{lj})-1], \quad l=j\pm 1, \qquad (2.1) \\ \alpha^j &= \tfrac{1}{2}\sin 2\epsilon_j \exp(i\delta_{j-1,j}+i\delta_{j+1,j})\,. \end{aligned}$$

To project a given pole term into the α's, one may expand the pole term into 2-particle products of 2×2 Pauli spin matrices and thence deduce the $M_{m_s,m_s}{}^l$ terms. One then computes the α's using the above formulas. In practice this results in very complicated expressions which largely cancel at the end of the calculation.

The mathematics is simpler if one starts with amplitudes similar to those introduced by Wolfenstein.[16] We define the set

$$\begin{aligned} M(x) = A(x)\cdot 1 &+ B(x)\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2 + iC(x)(\boldsymbol{\sigma}_1+\boldsymbol{\sigma}_2) \\ &\cdot\hat{N} + G(x)\mathcal{S}_{12} + H(x)(\boldsymbol{\sigma}_1\cdot\hat{N})(\boldsymbol{\sigma}_2\cdot\hat{N}), \qquad (2.2) \end{aligned}$$

where $\hat{N}=\hat{p}\times\hat{p}'$ and $\mathcal{S}_{12}=3(\boldsymbol{\sigma}_1\cdot\hat{q})(\boldsymbol{\sigma}_2\cdot\hat{q})-\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2$; $\mathbf{p}$ and $\mathbf{p}'$ are the initial and final three-momenta in the center-of-mass system; $\mathbf{q}=\mathbf{p}'-\mathbf{p}$; $\boldsymbol{\sigma}_1$ and $\boldsymbol{\sigma}_2$ are 2×2 Pauli spin matrices for particles 1 and 2, and 1 is the unit matrix.

Equation (2.2) projects most naturally into what we shall call "spin-space" amplitudes; $\alpha_{l,C}$, $\alpha_{l,\sigma\sigma}$, $\alpha_{l,T}$, $\alpha_{l,LS}$ and $\alpha_{l,Q}$; $\alpha_{lsj}=\alpha_l$ for $S=0$ and α_{lj} for $S=1$;

$$\begin{aligned} \alpha_{lsj} = \alpha_{l,C} &+ \alpha_{l,\sigma\sigma}\langle lsj|\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2|lsj\rangle + \alpha_{l,T}\langle lsj|\mathcal{S}_{12}|lsj\rangle \\ &+ \alpha_{l,LS}\langle lsj|\mathbf{L}\cdot\mathbf{S}|lsj\rangle + \alpha_{l,Q}\langle lsj|Q|lsj\rangle, \qquad (2.3a) \end{aligned}$$

where

$$p^{-1}\alpha_{l,C} = \int_{-1}^{1} \tfrac{1}{2}dx\, P_l(x)A(x)\,,$$

$$p^{-1}\alpha_{l,\sigma\sigma} = \int \tfrac{1}{2}dx\, P_l(x)B(x)\,,$$

$$\begin{aligned} p^{-1}\alpha_{l,T} = \int \tfrac{1}{2}dx\{G(x)[P_l(x)-\tfrac{3}{2}(1-x)^{-1}\mathcal{P}_l(x)] \\ + H(x)(1-x^2)^{-1}x\mathcal{P}_l(x)\}\,, \end{aligned}$$

$$p^{-1}\alpha_{l,LS} = \int \tfrac{1}{2}dx\, \mathcal{P}_l(x)2(\sin\theta)^{-1}C(x)\,,$$

and

$$p^{-1}\alpha_{l,Q} = \int \tfrac{1}{2}dx\, P_l(x)H(x)\,.$$

$P_l(x)$ is the Legendre polynomial of argument $x=\cos\theta$, and $\mathcal{P}_l=(2l+1)^{-1}(P_{l-1}-P_{l+1})$; $\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2$, $\mathcal{S}_{12}$, and $\mathbf{L}\cdot\mathbf{S}$ have the usual definitions[17]; Q has diagonal matrix elements

$$\langle lsj|Q|lsj\rangle = -(-1)^{l-j}(2l+1)(2j+1)^{-1}.$$

Its off-diagonal matrix elements vanish.

The fifth and last partial-wave amplitude α^j is given by

$$\alpha^j = \alpha_T{}^j\langle l=j\pm 1, s, j|\mathcal{S}_{12}|l=j\pm 1, s, j\rangle, \qquad (2.3b)$$

where

$$\begin{aligned} p^{-1}\alpha_T{}^j = \int_{-1}^{1} \tfrac{1}{2}dx\{G(x)(1-x)^{-1} \\ \times[P_j(x)-\tfrac{1}{2}P_{j-1}(x)-\tfrac{1}{2}P_{j+1}(x)] \\ +\tfrac{1}{3}H(x)(1-x^2)^{-1}\mathcal{P}_j(x)\}\,. \end{aligned}$$

A. Pseudoscalar Pole

One may now compute the pseudoscalar pole projections by inserting the pole-term contribution in Eqs. (2.3a) and (2.3b). The M matrix as defined by Stapp is related to the Feynman T matrix by the formula

$$\langle p'n'|T|pn\rangle = (i/\pi E)\delta^{(4)}(p'+n'-p-n)\langle p'n'|M|pn\rangle;$$

p and n are the initial four-momenta and p' and n' are the final four-momenta; E is the total energy of any nucleon in the c.m. system.

Let us call the pseudoscalar pole-term contribution to the M matrix B^{PS}. Then by the standard Feynman rules

$$\begin{aligned} \langle p'n'|B^{PS}|pn\rangle \equiv B^{PS} - (m_N{}^2/E)g_{PS}{}^2[\bar{u}(p')\gamma_5 u(p)] \\ \times(q^2+m_{PS}{}^2)^{-1}[\bar{u}(n')\gamma_5 u(n)]; \end{aligned}$$

the Dirac spinors u are normalized so that $\bar{u}u=1$; m_{PS} is the mass of the exchanged meson and m_N is the mass of the nucleon; γ_5 and all other 4×4 Dirac matrices are defined according to Schweber, Bethe, and de Hoffmann.[18]

In terms of field theory this Born term may be derived from the interaction Lagrangian

$$\mathcal{L}^{\text{int}} = (4\pi)^{1/2}g_{PS}\bar{\Psi}\gamma_5\phi^{(PS)}\Psi\,.$$

B^{PS} expanded in 2×2 Pauli spin matrices yields

$$B^{PS} = (g_{PS}{}^2/12E)(x_0-x)^{-1}(1-x)(\mathcal{S}_{12}+\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2)\,,$$

where

$$q^2+m_{PS}{}^2 = 2p^2(x_0-x)\,, \quad \text{or} \quad x_0 = 1+m_{PS}{}^2/2p^2$$

with

$$p = |\mathbf{p}| = |\mathbf{p}'|\,.$$

[16] L. Wolfenstein and J. Ashkin, Phys. Rev. 85, 947 (1952).

[17] Thus for $s=1$ and for $j=l+1$, l, and $l-1$, $\langle lsj|\mathcal{S}_{12}|lsj\rangle = -2l(2j+1)^{-1}$, 2, and $-2(l+1)(2j+1)^{-1}$, respectively, and $\langle lsj|\mathbf{L}\cdot\mathbf{S}|lsj\rangle = l$, -1, and $-(l+1)$, respectively; $\langle l=j\pm 1, s, j|\mathcal{S}_{12}|l=j\mp 1, s, j\rangle = 6(2j+1)^{-1}[j(j+1)]^{1/2}$; for $s=0$, all matrix elements of $\mathcal{S}_{12}$ and $\mathbf{L}\cdot\mathbf{S}$ vanish. $\langle lsj|\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2|lsj\rangle = 2s(s+1)-3$.

[18] S. S. Schweber, H. A. Bethe, and F. de Hoffmann, *Mesons and Fields* (Row, Peterson and Company, Evanston, Illinois, 1956), Vol. I.

Carrying out the integration over x called for by Eq. (2.3a), one obtains the pseudoscalar pole contribution to α_{lsj}; let us call this $B_{lsj}^{(PS)}$;

$$p^{-1}B_{lsj}^{(PS)}=-(g_{PS}^2/12E)\times\{[(x_0-1)Q_l(x_0)-\delta_{l,0}]\langle lsj|\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2|lsj\rangle+[(x_0-1)Q_l(x_0)+\tfrac{3}{2}\mathcal{Q}_l(x_0)]\langle lsj|S_{12}|lsj\rangle\}.$$

Similarly, Eq. (2.3b) defines the pole contribution to the off-diagonal element α^j; let us call this $B^{j(PS)}$;

$$p^{-1}B^{j(PS)}=-(g_{PS}^2/24E)[Q_{j-1}(x_0)-2Q_j(x_0)+Q_{j+1}(x_0)]\times\langle j\mp1,s,j|S_{12}|j\pm1,s,j\rangle.$$

In these expressions

$$Q_l(x_0)=\int_{-1}^{1}\tfrac{1}{2}dx(x_0-x)^{-1}P_l(x)$$

as in Jahnke and Emde.[19] $\mathcal{Q}_l$ is defined to be $(2l+1)^{-1}\times(Q_{l-1}-Q_{l+1})$; $\delta_{l,0}$ is the Kronecker δ.

B. Scalar Pole

The one scalar meson exchange Born term is

$$\langle p'n'|B^{(S)}|pn\rangle\equiv B^{(S)}=(m_N^2/E)g_S^2[\bar{u}(p')u(p)]\times(q^2+m_S^2)^{-1}[\bar{u}(n')u(n)],$$

with m_S the mass of the exchanged quantum. The interaction Lagrangian in field theory is

$$\mathcal{L}^{\text{int}}=(4\pi)^{1/2}g_S\bar{\Psi}\Psi\phi^{(S)}.$$

The expansion of $B^{(S)}$ in Pauli spin matrices is

$$B^{(S)}=(g_S^2/8aE)(x_0-x)^{-1}[(1-ax)^2+i(\sin\theta)a(1-ax)\times(\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2)\cdot\hat{N}-a^2(1-x^2)(\boldsymbol{\sigma}_1\cdot\hat{N})(\boldsymbol{\sigma}_2\cdot\hat{N})],$$

where

$$x_0=1+m_S^2/2p^2\quad\text{and}\quad a=p^2(E+M)^{-2}.$$

The quantity a goes like v^2/c^2 and remains quite small for the range of incident proton energies of interest to us. (For $T_{\text{lab}}=320$ MeV, $a=0.04$.)

Projection of $B^{(S)}$ into angular-momentum states yields

$$p^{-1}B_{lsj}^{(S)}=(g_S^2/8aE)\{[(1-ax_0)^2Q_l(x_0)+a(2-ax_0)\delta_{l,0}-\tfrac{1}{3}a^2\delta_{l,1}]+a[2(1-ax_0)\mathcal{Q}_l(x_0)+\tfrac{2}{3}a\delta_{l,1}]\langle lsj|\mathbf{L}\cdot\mathbf{S}|lsj\rangle+a^2[(x_0^2-1)Q_l(x_0)-x_0\delta_{l,0}-\tfrac{1}{3}\delta_{l,1}]\langle lsj|Q|lsj\rangle-a^2[x_0\mathcal{Q}_l(x_0)-\tfrac{1}{3}\delta_{l,1}]\langle lsj|S_{12}|lsj\rangle\}\quad(2.4a)$$

and

$$p^{-1}B^{j(S)}=-(g_S^2/8aE)fa^2\mathcal{Q}_j(x_0)\langle j\pm1,s,\tfrac{1}{3}|S_{12}|j\mp1,s,j\rangle.\quad(2.4b)$$

The quadratic (Q) and tensor terms are exceedingly small as they go as a^2. Thus the scalar pole behaves essentially as a sum of central and spin-orbit terms.

C. Vector Pole

The vector-meson pole term is much more complicated than the pseudoscalar or scalar pole term because the vector-meson–nucleon–antinucleon vertex admits both Dirac and Pauli coupling. Thus

$$\mathcal{L}^{\text{int}}=(4\pi)^{1/2}\bar{\Psi}[g\gamma_\mu\phi_\mu^{(V)}+(f/4m_N)\sigma_{\mu\nu}(\partial_\nu\phi_\mu^{(V)}-\partial_\mu\phi_\nu^{(V)})]\Psi.$$

However, the pole term can be evaluated somewhat more easily by using the equivalent Lagrangian

$$\mathcal{L}^{\text{int}}=(4\pi)^{1/2}\bar{\Psi}[(g+f)\gamma_\mu\phi_\mu^{(V)}+(f/2m_N)(\phi^{(V)}i\overrightarrow{\partial}_\mu-i\overleftarrow{\partial}_\mu\phi_\mu^{(V)})]\Psi.$$

The equivalence can be demonstrated through explicit use of the Dirac equation.

The pole term is thus

$$\langle p'n'|B^{(V)}|pn\rangle\equiv B^{(V)}=-(m_N^2/E)\{\bar{u}(p')[(g+f)\gamma_\mu-(f/2m_N)(p_\mu'+p_\mu)]u(p)\}\times(q^2+m_V^2)^{-1}\{\bar{u}(n')[(g+f)\gamma_\mu-(f/2m_N)(n_\mu'+n_\mu)u(n)\},$$

where m_V is the mass of the exchanged vector meson. $B^{(V)}$ expands in Pauli spin matrices as follows:

$$\begin{aligned}B^{(V)}=(8Ea)^{-1}(x_0-x)^{-1}\{&1\cdot[-g^2(1+2a+4ax+a^2x^2)+\\&+gf(1-a)^{-1}4a(1-x)(1+a+2ax)-f^2(1-a)^{-2}2a^2(1-x)^2(2+a+ax)]\\&-(2/3)a(S_{12}-2\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2)(g+f)^2(1-x)+ai\sin\theta(\boldsymbol{\sigma}_1+\boldsymbol{\sigma}_2)\\&\times\hat{N}[g^2(3+ax)+4gf(1-a)^{-1}(1-a+2ax)-f^22a(1-a)^{-2}(1-x)(3+ax)]\\&+a^2\sin^2\theta(\boldsymbol{\sigma}_1\cdot\hat{N})(\boldsymbol{\sigma}_2\cdot\hat{N})[g^2+8gf(1-a)^{-1}+2f^2(1-a)^{-2}(4-a+ax)]\}\end{aligned}$$

with $x_0=1+m_V^2/2p^2$.

Note that the central term goes as 1, that the spin-orbit, spin-spin, and tensor terms go as a, and that the quadratic term goes as a^2.

[19] E. Jahnke and F. Emde, *Tables of Functions* (Dover Publications, Inc., New York, 1945).

$B^{(V)}$ projects as follows into partial waves:

$$
\begin{aligned}
(8aE)p^{-1}B_{lsj}{}^{(V)}=\{&g^2[(1+4ax_0+a^2x_0{}^2+2a)Q_l(x_0)-(4a+a^2x_0)\delta_{l,0}-\tfrac{1}{3}a^2\delta_{l,1}]\\
&+gf4a(1-a)^{-1}[(x_0-1)(1+2ax_0+a)Q_l(x_0)-(1-a+2ax_0)\delta_{l,0}-\tfrac{2}{3}\delta_{l,1}]\\
&+f^24a^2(1-a)^{-2}[(x_0-1)^2(1+\tfrac{1}{2}ax_0+\tfrac{1}{2}a)Q_l(x_0)-(x_0-2+\tfrac{1}{2}ax_0{}^2-\tfrac{1}{2}ax_0-\tfrac{1}{3}a)\delta_{l,0}\\
&-\tfrac{1}{3}(1+\tfrac{1}{2}ax_0-\tfrac{1}{2}a)\delta_{l,1}-(1/15)a\delta_{l,2}]\}-\tfrac{4}{3}a\langle lsj|\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2|lsj\rangle(g+f)^2[(x_0-1)Q_l(x_0)-\delta_{l,0}]\\
&+\tfrac{2}{3}a\langle lsj|S_{12}|lsj\rangle(g+f)^2[(x_0-1)Q_l(x_0)+\tfrac{3}{2}\mathcal{Q}_l(x_0)]+2a\langle lsj|\mathbf{L}\cdot\mathbf{S}|lsj\rangle\{g^2[(3+ax_0)\mathcal{Q}_l(x_0)-\tfrac{1}{3}a\delta_{l,1}]\\
&+4gf(1-a)^{-1}[(1+2ax_0-a)\mathcal{Q}_l(x_0)-\tfrac{2}{3}a\delta_{l,1}]+f^2(1-a^{-2}[2a(x_0-1)(3+ax_0)\mathcal{Q}_l(x_0)\\
&-(2a+\tfrac{2}{3}a^2x_0-\tfrac{2}{3}a^2)\delta_{l,1}-(2/15)a^2\delta_{l,2}]\}-a^2\langle lsj|Q|lsj\rangle\{[g^2+8gf(1-a)^{-1}+8f^2(1-a)^{-2}]\\
&+2f^2(1-a)^{-2}[a(x_0-1)(x_0{}^2-1)Q_l(x_0)-a(x_0{}^2-x_0-\tfrac{2}{3})\delta_{l,0}-\tfrac{1}{3}a(x_0-1)\delta_{l,}-(2/15)a\delta_{l,2}]\}\\
&-a^2\langle lsj|Q|lsj\rangle\{[g^2+8gf(1-a)^{-1}+8f^2(1-a)^{-2}][(x_0{}^2-1)Q_l(x_0)-x_0\delta_{l,0}-\tfrac{1}{3}\delta_{l,1}]\\
&+2f^2(1-a)^{-2}[a(x_0-1)(x_0{}^2-1)Q_l(x_0)-a(x_0{}^2-x_0-\tfrac{2}{3})\delta_{l,1}-\tfrac{1}{3}a(x_0-1)\delta_{l,1}-(2/15)a\delta_{l,2}]\}\\
&+a^2\langle lsj|S_{12}|lsj\rangle\{[g^2+8gf(1-a)^{-1}+8f^2(1-a)^{-2}][x_{0,l}(x_0)-\tfrac{1}{3}\delta_{l,1}]\\
&\qquad+2f^2(1-a)^{-2}[ax_0(x_0-1)\mathcal{Q}_l(x_0)-\tfrac{1}{3}a(x_0-1)\delta_{l,1}-(1/15)a\delta_{l,2}]\}. \qquad (2.5a)
\end{aligned}
$$

The coefficient for $\langle lsj|S_{12}|lsj\rangle$ is divided into two terms; the second, due to integration over $H(x)$, is much smaller than the first. The off-diagonal matrix element $B^{j(V)}$ is

$$
\begin{aligned}
p^{-1}B^{j(V)}=(24E)^{-1}\{&(g+f)^2[Q_{j-1}(x_0)+Q_{j+1}(x_0)-2Q_j(x_0)]+a[g^2+8gf(1-a)^{-1}+8f^2(1-a)^{-2}(1+\tfrac{1}{4}ax_0-\tfrac{1}{4}a)]\\
&\times\mathcal{Q}_j(x_0)-\tfrac{2}{3}f^2a^2(1-a)^{-2}\delta_{l,1}\}\langle j\pm1,s,j|S_{12}|j\mp1,s,j\rangle. \qquad (2.5b)
\end{aligned}
$$

It should be pointed out that despite the nonrelativistic appearance of the operators $\mathbf{L}\cdot\mathbf{S}$, S_{12}, etc., the pole projections of $B^{(PS)}$, $B^{(S)}$, and $B^{(V)}$ are fully relativistic. For pole projections α_{lsj} rather than $\alpha_{l,C}$, $\alpha_{l,T}$, etc. one may refer to calculations by Sawada *et al.*[14] and by Perring and Phillips.[20]

The pole projections apply to $T=0$ mesons. For $T=1$ mesons, replace in the interaction Lagrangians, ϕ by $\boldsymbol{\tau}\cdot\boldsymbol{\phi}$, and in the Born terms, g^2 by $\boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_2 g^2$.

III. COMPARISON WITH p-p EXPERIMENTAL AMPLITUDES

A. Three-Pole Fit Using Geometric Unitarization

Let B be the sum of the pole terms identified with the pseudoscalar, scalar, and vector mesons; i.e.,

$$B=\sum_\nu B^{(\nu)}, \quad \nu=\pi, V, \sigma. \qquad (3.1)$$

Then we shall physically identify B with the real part of the scattering amplitude, as sketched in Fig. 1. This applies only to states with $l\geq1$ as explained in the introduction. In terms of these partial waves, Eq. (1) may be written

$$
\begin{aligned}
&B_l=\mathrm{Re}\alpha_l, \quad l\neq0 \quad \text{(singlet)}\\
&B_{ll}=\mathrm{Re}\alpha_{ll}, \qquad \text{(triplet uncoupled)} \qquad (3.2)
\end{aligned}
$$

and

$$\|B_j\|=\mathrm{Re}\|\alpha_j\|, \quad \text{(triplet uncoupled)}$$

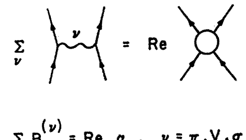

FIG. 1. Geometric unitarization; the sum of pole terms is taken to be equal to the real part of the scattering amplitude.

where

$$\|B_j\|=\begin{pmatrix}B_{j-1,j} & B^j\\ B^j & B_{j+1,j}\end{pmatrix}$$

and

$$\|\alpha_j\|=\begin{pmatrix}\alpha_{j-1,j} & \alpha^j\\ \alpha^j & \alpha_{j+1,j}\end{pmatrix}.$$

Elastic unitarity defines the complete amplitude once the real part is assumed to be equal to B; thus

$$
\begin{aligned}
\mathrm{Im}\alpha_l&=(\mathrm{Im}\alpha_l)^2+(B_l)^2\\
\mathrm{Im}\alpha_{ll}&=(\mathrm{Im}\alpha_{ll})^2+(B_{ll})^2
\end{aligned}
$$

and

$$\mathrm{Im}\|\alpha_j\|=(\mathrm{Im}\|\alpha_j\|)^2+\|B_j\|^2. \qquad (3.3)$$

In the case of the uncoupled states, the complex amplitude α_l can conveniently be visualized as in Fig. 2. Geometric unitarization means constructing $e^{i\delta_l}\sin\delta_l=\alpha_l$, line OQ, from $B_l=\mathrm{Re}\alpha_l=\cos\delta_l\,\sin\delta_l$, line PQ. Other forms of geometric unitarization are immediately apparent, such as setting $B_l=\delta_l$, arc OQ, or $B_l=\cos\delta_l$ (the K-matrix method of Sawada *et al.*[14]). A diagram analogous to Fig. 2 for the case of the coupled states is given in the article by Moravcsik.[13]

The following pole parameters are adjusted for a best fit to experiment: $g_V{}^2$, f_V/g_V, $g_\pi{}^2$, $g_\sigma{}^2$, and m_σ. "Experiment" is represented by phase shifts determined through phase-shift analysis of the p-p scattering data. The

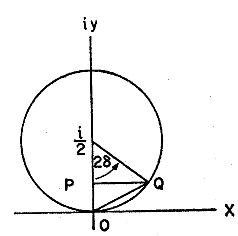

FIG. 2. Geometry in the complex plane; if the phase shift is δ_l, then the complex amplitude $\alpha_l=\exp(i\delta_l)\sin\delta_l$ is line OQ, and the real part, line PQ.

[20] J. K. Perring and R. J. N. Phillips, Atomic Energy Research Establishment, Harwell, England, Report No. R-*4077*, 1962 (unpublished).

TABLE I. Experimental values for the real part of the p-p scattering amplitudes α_l, α_{ll}, α_{lj}, and α^j, where $\mathrm{Re}\alpha_l=\cos\delta_l\,\sin\delta_l$, $\mathrm{Re}\alpha_{ll}=\cos\delta_{ll}\sin\delta_{ll}$, and for $j=l\pm1$, $\mathrm{Re}\alpha_{lj}=\cos2\epsilon_j\cos\delta_{lj}\sin\delta_{lj}$ and $\mathrm{Re}\alpha^j=\frac{1}{2}\sin2\epsilon_j\cos(\delta_{j-1,j}+\delta_{j+1,j})$; the δ's and ϵ's are Stapp's "bar" phase shifts. The values of Reα have been calculated from the bar phase shifts found by Arndt and MacGregor through analysis of the data at discrete energy bands centered near 25, 50, 95, 142, 210, and 310 MeV. These phase shifts are an earlier version of those reported in Ref. 21.

	25 MeV	50 MeV	95 MeV	142 MeV	210 MeV	310 MeV
$\mathrm{Re}\alpha_{1,0}^{(\mathrm{exp})}$	0.095 ± 0.016	0.204 ± 0.013	0.217 ± 0.030	0.109 ± 0.010	-0.012 ± 0.010	-0.193 ± 0.027
$\mathrm{Re}\alpha_{1,1}^{(\mathrm{exp})}$	-0.063 ± 0.008	-0.139 ± 0.005	-0.218 ± 0.008	-0.281 ± 0.006	-0.342 ± 0.008	-0.418 ± 0.012
$\mathrm{Re}\alpha_{1,2}^{(\mathrm{exp})}$	0.033 ± 0.003	0.105 ± 0.004	0.179 ± 0.009	0.230 ± 0.003	0.262 ± 0.004	0.269 ± 0.010
$\mathrm{Re}\alpha_{2}^{(\mathrm{exp})}$	0.012 ± 0.004	0.037 ± 0.004	0.061 ± 0.007	0.090 ± 0.004	0.122 ± 0.006	0.155 ± 0.012
$\mathrm{Re}\alpha^{2(\mathrm{exp})}$	-0.020 ± 0.009	-0.037 ± 0.006	-0.047 ± 0.006	-0.049 ± 0.003	-0.046 ± 0.003	-0.050 ± 0.008
$\mathrm{Re}\alpha_{3,2}^{(\mathrm{exp})}$	0.005 ± 0.003	0.011 ± 0.007	0.022 ± 0.013	0.013 ± 0.006	0.027 ± 0.006	0.017 ± 0.012
$\mathrm{Re}\alpha_{3,3}^{(\mathrm{exp})}$	-0.004 ± 0.009	-0.006 ± 0.008	-0.028 ± 0.010	-0.036 ± 0.004	-0.045 ± 0.004	-0.053 ± 0.012
$\mathrm{Re}\alpha_{3,4}^{(\mathrm{exp})}$	0.000 ± 0.009	0.003 ± 0.003	0.011 ± 0.004	0.016 ± 0.003	0.040 ± 0.004	0.048 ± 0.007
$\mathrm{Re}\alpha_{4}^{(\mathrm{exp})}$	0.001 ± 0.009	0.003 ± 0.009	0.006 ± 0.009	0.010 ± 0.002	0.018 ± 0.003	0.024 ± 0.006
$\mathrm{Re}\alpha^{4(\mathrm{exp})}$	-0.001 ± 0.009	-0.003 ± 0.009	-0.008 ± 0.009	-0.001 ± 0.001	-0.016 ± 0.002	-0.017 ± 0.006
$\mathrm{Re}\alpha_{5,4}^{(\mathrm{exp})}$	0.000 ± 0.009	0.000 ± 0.009	0.002 ± 0.009	0.003 ± 0.001	0.008 ± 0.006	0.021 ± 0.007
$\mathrm{Re}\alpha_{5,5}^{(\mathrm{exp})}$	-0.000 ± 0.009	-0.001 ± 0.009	-0.005 ± 0.009	-0.008 ± 0.002	-0.011 ± 0.004	-0.025 ± 0.009

particular phase shifts we use have been found by Arndt and MacGregor.[21] These are very similar to phase shifts found by other groups in analyzing p-p data over the same range of energies.[22–25] The Arndt-MacGregor phase shifts are really a collection of six sets of phase shifts, each set independently determined from scattering data within a different energy band. Solutions are provided at 25, 50, 95, 142, 210, and 310 MeV.

We shall compare our predictions with $\mathrm{Re}\alpha^{(\mathrm{exp})}$ rather than $\delta^{(\mathrm{exp})}$, since it is easier to compute $\mathrm{Re}\alpha^{(\mathrm{exp})}$ once from the phase shifts and compare with B, than it is to compute $\delta^{(\mathrm{theor})}$ each time the pole parameters are changed and compare with $\delta^{(\mathrm{exp})}$. Thus we minimize the quantity

$$\chi^2=\sum_{ik}\frac{[\mathrm{Re}\alpha_k(T_i)-\mathrm{Re}\alpha_k^{(\mathrm{exp})}(T_i)]^2}{[\Delta_k^{(\mathrm{exp})}(T_i)]^2}. \tag{3.4}$$

α_k is a generic symbol for any of the α_l, α_{lj}, or α^j. The sum on k is taken over α_{10}, α_{11}, α_{12}, α_2, α^2, α_{32}, α_{33}, α_{34}, α_4, α^4, α_{54}, and α_{55}; the sum on i is taken for $T_i=25$, 50, 95, 142, 210, and 310 MeV. The $\mathrm{Re}\alpha_k^{(\mathrm{exp})}$ have been calculated using Stapp's definitions [Eq. (2.1)] and are listed with the experimental errors, $\Delta_k^{(\mathrm{exp})}$, in Table I. They are plotted in Fig. 3. These $\mathrm{Re}\alpha_k^{(\mathrm{exp})}$ exhibit a smooth dependence with changing energy: however, they have not been determined through energy-dependent forms; the phase shift analysis at each energy is independent from that at any other energy; the smoothness apparently reflects the consistency of the data.

It will be noted that Eq. (4) neglects correlations between the phase shifts. We feel that this is not serious because we are only concerned with qualitative predictions of the pole model. However, we have investigated fits to the actual data (not phase shifts) and find that the pole parameters are only slightly affected when correlations are included. This will be reported in a later publication.[26]

The quantity in Eq. (4) was minimized by an automatic search code which incorporated both grid and matrix search routines. The code could be run on either an IBM 7094 or a CDC 3600. Minimum χ^2 was achieved with the parameters listed in Table II. The predictions[27] for the $\mathrm{Re}\alpha_k$ are graphed in Fig. 3. The qualitative agreement with experiment is good.

The pole parameters of Table II also seem reasonable. The vector-pole contribution represents some average of the ω, ϕ, and ρ-pole contributions; f_V/g_V, found to be 2.3, seems consistent with $f_\omega/g_\omega\approx0$, $f_\phi/g_\phi\approx0$ and $f_\rho/g_\rho\approx4$ as estimated from the nucleon electromagnetic form factors. g_V^2, found to be 2.3, is small and in this sense consistent with $g_\rho^2\approx0.5$ estimated from $\rho\to\pi\pi$

TABLE II. Values for g_ν^2 and m_ν ($\nu=\pi$, V, σ) which provide the best agreement between theoretical $\mathrm{Re}\alpha_k=\sum_\nu B_k^{(\nu)}$ and ex-experimental $\mathrm{Re}\alpha_k^{(\mathrm{exp})}$; the $\mathrm{Re}\alpha_k^{(\mathrm{exp})}$ are listed in Table I. m_V and m_π are prefixed.

$g_\pi^2=12.3$	$g_\sigma^2=1.6$	$g_V^2=2.3$
$m_\pi=135$ MeV	$m_\sigma=390$ MeV	$f_V/g_V=2.3$
		$m_V=770$ MeV

[21] R. A. Arndt and M. H. MacGregor, Phys. Rev. **141**, 873 (1966).

[22] G. Breit, M. H. Hull, Jr., K. E. Lassila, and K. D. Pyatt, Jr., Phys. Rev. **120**, 2227 (1960); and G. Breit, M. H. Hull, Jr., K. E. Lassila, K. D. Pyatt, Jr., and H. M. Ruppel, *ibid.* **128**, 826 (1962).

[23] P. Signell, Phys. Rev. **139**, B315 (1965), and earlier work cited therein.

[24] C. J. Batty and J. K. Perring, Nucl. Phys. **59**, 141 (1964), and earlier work cited therein.

[25] Yu. M. Kazarinov, V. S. Kiselev, and V. I. Satarov, Zh. Eksperim. i Teor. Fiz. **46**, 920 (1964) [English transl.: Soviet Phys.—JETP **19**, 627 (1964)], and earlier work cited therein.

[26] R. A. Arndt, R. A. Bryan, and M. H. MacGregor, University of California Radiation Laboratory Report No. UCRL-14807 (unpublished).

[27] The V and σ are taken in a zero-width approximation. Calculations, not reported here, were undertaken to establish the effect of considering the V and σ exchanges in a non-zero-width approximation, that is, to give them structure corresponding to a Breit-Wigner resonance in the N-$\bar{N}$ channel with some finite width. It was determined that, so far as the effect on N-N scattering is concerned, one cannot distinguish between a "wide" (say 150-MeV) resonance and a zero-width pole at a slightly lesser (25-MeV) mass.

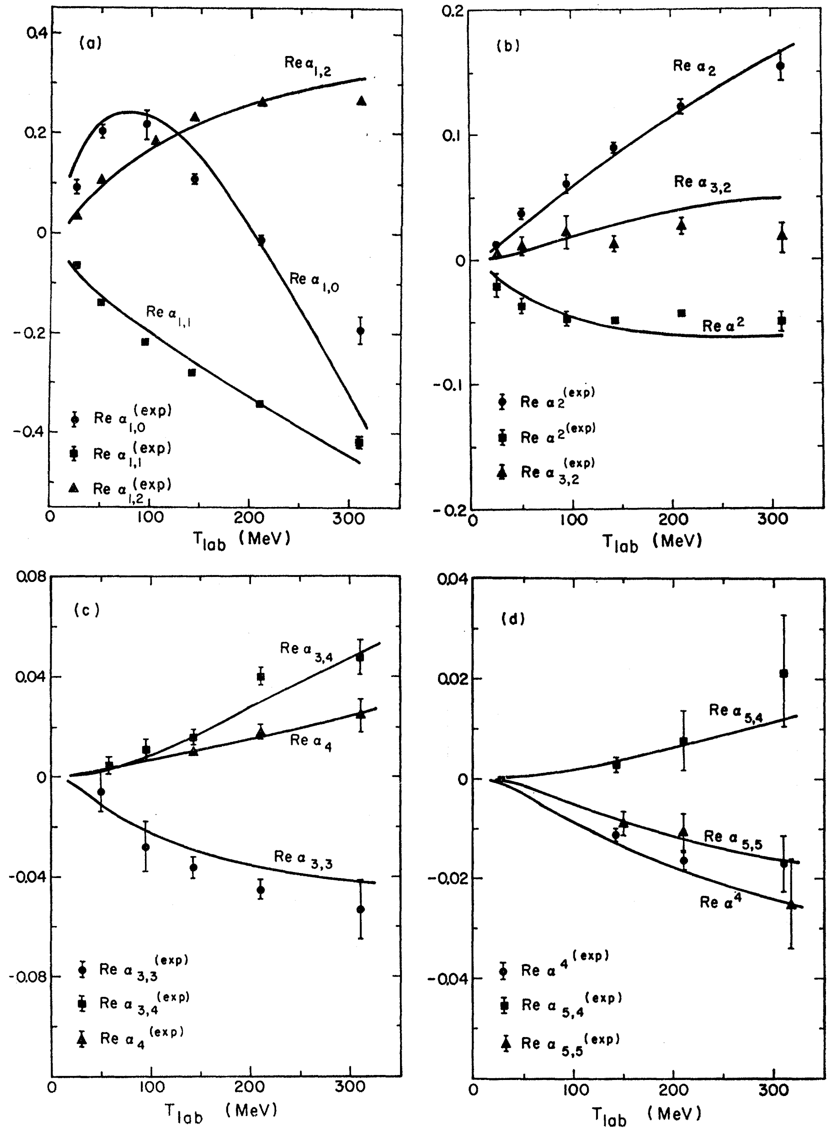

FIG. 3. Theoretical and experimental values for the p-p scattering amplitude, real part, plotted versus laboratory scattering energy. The solid lines depict $\sum_\nu B_k^{(\nu)}=\mathrm{Re}\alpha_k$, $\nu=\pi$, V, σ, for pole parameters listed in Table II. The experimental points are plots of the $\mathrm{Re}\alpha_k^{(\mathrm{exp})}$ listed in Table I.

decay and universal ρ coupling to the isospin current, and $g_{\omega 8}{}^2=3g_\rho{}^2$ (pure F coupling between the baryon and the vector meson octets); ω_8 refers to the "8" member of the mixed octet.

The pseudoscalar coupling constant also appears reasonable. If we assume $g_\eta{}^2 \ll g_\pi{}^2$, as predicted by SU_3 and borne out by fits to the combined p-p and n-p data, then Table II predicts $g_\pi{}^2 \sim 12$. This agrees with 14 to within the accuracy one should expect from geometric unitarization.

For the scalar meson there is no well-established experimental resonance with which to make a comparsion, but m_σ of Table II falls in the same energy range as the π-π enhancement in the $\pi^-+p\rightarrow\pi^-+\pi^++n$ reaction; 390 MeV is certainly lower than Thurnauer's estimate of 490 MeV for the di-pion mass,[5] but in fact the fit to the data is not all that sensitive to m_σ and even a value as high as 490 MeV still yields a pretty good fit. However, 700 MeV for the σ mass is definitely too high; using this mass the fit to the experimental amplitudes is very poor. Thus the π-π resonance observed at this energy cannot be identified with the σ of the model; however, it might contribute along with a low mass or other π-π effect to yield something equivalent to $\sigma(390)$.

B. Spin-Space Structure of Pole Terms

The pole parameters of Table II are similar to those found by Sawada, Ueda, Watari, and Yonezawa using their K-matrix version of geometric unitarization. These parameters are also similar to those of Scotti and Wong.[3] Interestingly, however, they are very different from those found by Bryan, Dismukes and Ramsay[8] using the Schrödinger equation; in that case, one finds $g_V{}^2 \sim 20$,

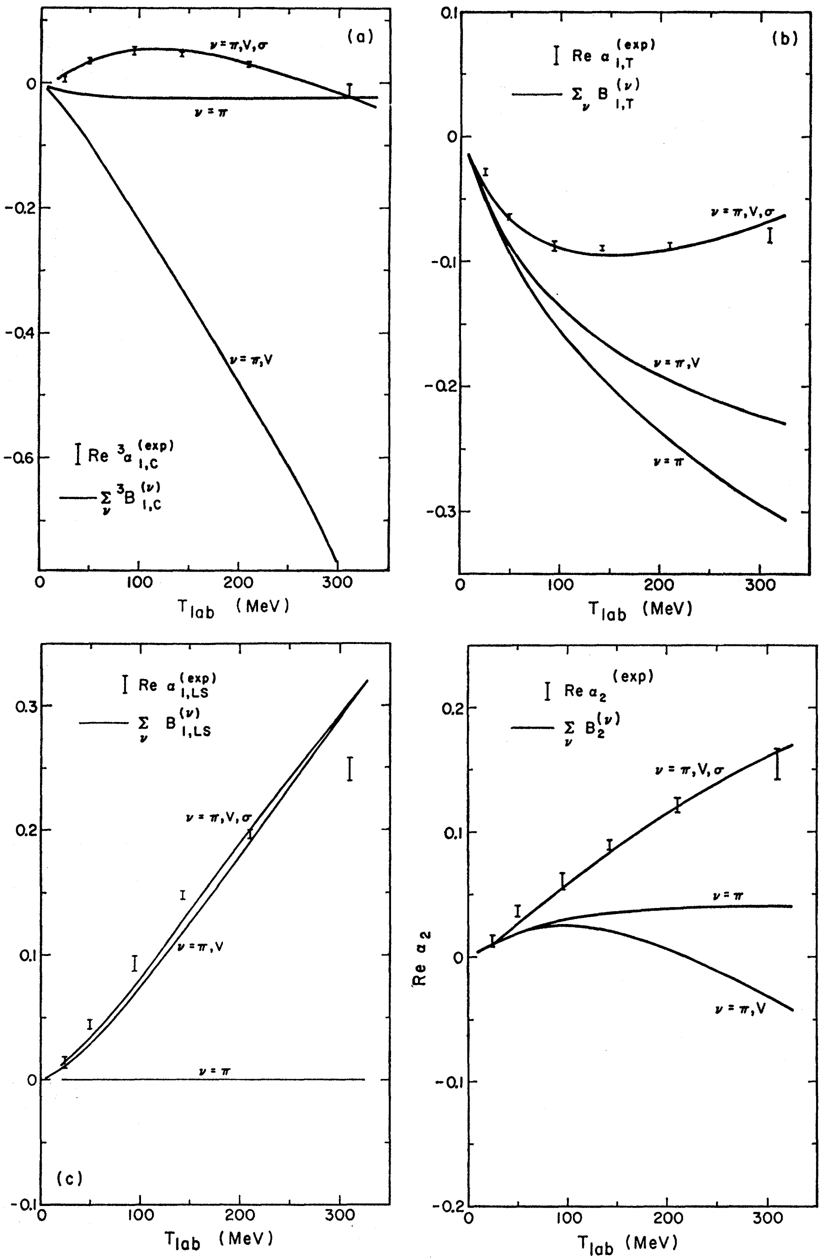

FIG. 4. Plots versus energy of the real part of the spin-space amplitudes $^3\alpha_{l,C}$, $\alpha_{l,T}$, and $\alpha_{l,LS}$ for $l=1$, and α_l for $l=2$; the experimental amplitudes have been calculated from the $\mathrm{Re}\alpha_k^{(\mathrm{exp})}$ listed in Table I using formula (3.5) of the text. Geometric unitarization is again assumed for the theoretical amplitudes; $\mathrm{Re}\alpha_k=\sum_\nu B_k^{(\nu)}$; theoretical cases graphed in solid lines are for (a) $\nu=\pi$, with $g_\pi^2=14$, (b) $\nu=\pi$, V, with $g_\pi^2=14$, $g_V^2=9.5$, and (c) $\nu=\pi$, V, σ with $g_\pi^2=12.8$, $g_V^2=2.2$, $f_V/g_V=2.5$, $g_\sigma^2=1.6$, and $m_\sigma=380$ MeV. Prefixed are $m_V=770$ MeV and $m_\pi=135$ MeV.

$f_V/g_V\sim0.4$, $g_\sigma^2\sim15$, $m_\sigma\sim560$ MeV, and $g_\pi^2\sim14$. Since g_V^2 is observed to vary by a factor of 10, one might ask how such different treatments of the pole terms could result in fits to the same data. Since it is pretty well understood how the three pole terms suffice in the case of the Schrödinger equation,[8] we will present a parallel investigation for the case of geometric unitarization.

Only the $^3P_0$, $^3P_1$, $^3P_2$, and $^1D_2$ states will be considered since it is in these states that the differences in unitarization are greatest. The experimental amplitudes will be put in the form of the spin-space amplitudes of Sec. II, Eq. (2.3), because certain features of the pole terms stand out better then. We take for the P states

$$\alpha_{l,j}^{(\mathrm{exp})}={}^3\alpha_{l,C}^{(\mathrm{exp})}+\alpha_{l,T}^{(\mathrm{exp})}\langle lj|S_{12}|lj\rangle+\alpha_{l,LS}^{(\mathrm{exp})}\langle lj|\mathbf{L}\cdot\mathbf{S}|lj\rangle,\quad(3.5)$$

where $^3\alpha_{l,C}=\alpha_{l,C}+\alpha_{l,\sigma\sigma}$. [We neglect $\alpha_{l,Q}$ as the pole term contributions to this amplitude are very small.] For the $^1D_2$ state, we take $\alpha_2^{(\mathrm{exp})}$ as before. In Fig. 4 are plotted $\mathrm{Re}^3\alpha_{1,C}^{(\mathrm{exp})}$, $\mathrm{Re}_{1,T}^{(\mathrm{exp})}$, $\mathrm{Re}\alpha_{1,LS}^{(\mathrm{exp})}$, and $\mathrm{Re}\alpha_2^{(\mathrm{exp})}$ at each of the six energies.

The role of the pion pole term is much the same as before. The tensor term dominates and is long-ranged.

$$B_{ljs}^{(\pi)}\approx-g_\pi^2(M/2p)\{(m^2/12M^2)Q_l(x_0)\langle lsj|\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2|lsj\rangle+[(m^2/12M^2)Q_l(x_0)+(p^2/4M^2)\mathcal{Q}_l(x_0)]\times\langle lsj|S_{12}|lsj\rangle\}.\quad(3.6)$$

Values of ${}^3B_{1,C}{}^{(\pi)}$, $B_{1,T}{}^{(\pi)}$, $B_{1,LS}{}^{(\pi)}$, and $B_2{}^{(\pi)}$ for $g_\pi{}^2=14$ are graphed in Fig. 4. The B's are defined analogous to the α's of Eq. (5). Comparison of the $B_k{}^{(\pi)}$ with the $\mathrm{Re}\alpha_k{}^{(\mathrm{exp})}$ (k standing for any of the four amplitudes) shows that, as before, the tensor contribution is much too strong and a spin-orbit contribution is missing for the pion pole taken alone.

One now adds the pole contribution of a vector meson to provide the spin-orbit term. This time, however, a $g_V{}^2$ of only 9.5 suffices to fit $\mathrm{Re}\alpha_{1,LS}{}^{(\mathrm{exp})}$ instead of $g_V{}^2=30$ or more in the case of the Schrödinger equation (assuming $f_V=0$). The values of $B_k{}^{(\pi)}+B_k{}^{(V)}$ are graphed in Fig. 4; $g_\pi{}^2$ is again 14. It may be seen that $B_{1,LS}{}^{(V)}$ agrees qualitatively very well with $\mathrm{Re}\alpha_{1,LS}{}^{(\mathrm{exp})}$. The fit to $\mathrm{Re}\alpha_{1,T}{}^{(\mathrm{exp})}$ also shows improvement. However, the predicton for $\mathrm{Re}\,{}^3\alpha_{1,C}$ is now far too repulsive; this is a consequence of the leading term in the expansion of $B^{(V)}$;

$$\begin{aligned}B_{lsj}{}^{(V)}=(M/2p)\{&-g_V{}^2Q_l(x_0)+\cdots\\&-(f_V+g_V)^2(m_V{}^2/6M^2)Q_l(x_0)\langle lsj|\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2|lsj\rangle\\&+(f_V+g_V)^2[(m_V{}^2/12M^2)Q_l(x_0)+(p^2/4M^2)\mathcal{Q}_l(x_0)]\\&\qquad\times\langle lsj|S_{12}|lsj\rangle\\&-(3g_V{}^2+4g_Vf_V)(p^2/2M^2)Q_l(x_0)\langle lsj|\mathbf{L}\cdot\mathbf{S}|lsj\rangle\\&+\cdots\}.\end{aligned}\tag{3.7}$$

One adds the pole term of a scalar meson, since the leading term bears the opposite sign;

$$\begin{aligned}B_{lsj}{}^{(\sigma)}=g_\sigma{}^2(M/2p)\{&Q_l(x_0)+\cdots\\&+(p^2/2M^2)\mathcal{Q}_l(x_0)\langle lsj|\mathbf{L}\cdot\mathbf{S}|lsj\rangle+\cdots\}.\end{aligned}$$

Let $m_\sigma=m_V$ for exact cancellation. Actually, exact cancellation is not desired, because while $\mathrm{Re}\,{}^3\alpha_{1,C}{}^{(\mathrm{exp})}$ is close to zero, $\mathrm{Re}\alpha_2{}^{(\mathrm{exp})}$ is positive, i.e., indicates attraction. Thus one chooses a somewhat lower σ mass to provide long-range attraction in the D state concurrent with near cancellation in the P states.

Finally only the pole prediction for $\mathrm{Re}\alpha_{1,T}{}^{(\mathrm{exp})}$ remains poor. This is greatly improved by adjusting f_V. A best fit to the four experimental amplitudes is achieved with $g_V{}^2=2.2$, $f_V/g_V=2.5$, $g_\sigma{}^2=1.6$, $m_\sigma=380$ MeV and $g_\pi{}^2=12.8$. The values of $\sum B_k{}^{(\nu)}=\mathrm{Re}\alpha_k$ are plotted in Fig. 4. They now agree with the experimental $\mathrm{Re}\,{}^3\alpha_{1,C}$, $\mathrm{Re}\alpha_{1,T}$, $\mathrm{Re}\alpha_{1,LS}$, and $\mathrm{Re}\alpha_2$ rather well.

As mentioned earlier, the chief difference between the geometric and Schrödinger equation treatments of the pole terms is the order-of-magnitude difference in $g_V{}^2$. On the other hand, $(g_V+f_V)^2$ is rather stable, since in the former case $g_V=1.5$ and $f_V=3.5$ while in the latter case $g_V\sim 4.5$ and $f_V\sim 1.5$. That is to say, the tensor and spin-spin contributions remain much the same, since these are weighted by $(f_V+g_V)^2$, as in Eq. (7). It is the spin-orbit contribution which is weaker (by a factor of 3) in the case of geometric unitarization. Equation (7) then dictates that g_V be reduced and f_V increased. The central term is weighted by $g_V{}^2$ and is therefore very much smaller; still a fit to experiment may be obtained because $g_\sigma{}^2$ is reduced correspondingly. But now m_σ must be lower. (To see this consider the opposite case where g_σ and g_V approach ∞. Then m_σ must approach m_V to yield a finite difference.)

There is apparently sufficient flexibility in the pole contributions to allow a fit to experiment whether Schrödinger equation or geometric unitarization (or a partial wave dispersion relation[3]) is employed. In one sense this is disappointing, as one might have hoped that one method would prove definitely superior to the others. In another sense, however, it is pleasing, in that the same basic pole terms are required irrespective of the different corrections.

IV. SUMMARY

We have assumed that the real part of the non-S-wave p-p scattering amplitude is given by a sum of three pole terms, corresponding to vector (ρ,ω,ϕ), scalar (σ), and pseudoscalar (π) exchange. The imaginary parts are dictated by elastic unitary (geometric unitarization). Five parameters are searched (g_V, f_V, $g_\sigma{}^2$, m_σ, and $g_\pi{}^2$) and a good qualitative fit to the experimental amplitudes is obtained.

The spin-space structure of the individual pole terms is examined in some detail in order to provide quantitative measure of the arguments leading to the vector and scalar-meson hypotheses.

The fact that the simple pole contributions very nearly fit the physical amplitudes suggests that corrections may prove to be small. However, this is by no means certain, as there exists the example of the Schrödinger equation treatment where corrections to the pole terms are very large. However, the pole parameters obtained through Schrödinger iteration seem less reasonable than those obtained through geometric unitarization.

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