where $f_\Xi$ is the renormalized $\pi\Xi\Xi$ coupling constant:

$$\frac{f_\Xi^2}{f_N^2}=\left(\frac{m_N}{m}\right)^2\frac{g_{\pi\Xi\Xi}^2}{g_{\pi NN}^2}\approx\tfrac{1}{2}(1-2\alpha_p)^2\,,\quad (f_N^2=0.08)\,; \qquad (16)$$

or

$$\delta m=\frac{0.88}{(1-2\alpha_p)^2}=3.5\text{ MeV}\quad\text{for}\quad \alpha_p=0.75 \quad \text{(Martin's value)}\,, \qquad (17)$$

as compared to the experimental value $\delta m=(6.5\approx 1)$ MeV. The expression (17) becomes catastrophically large (88 MeV) if $\alpha_p=0.55$, which, as was noted in Sec. 2, corresponds to the highly unlikely case $g_{\pi\Xi\Xi}^2/g_{\pi NN}^2=0.01$. For $0.55\leq\alpha_p\leq 0.75$, Eq. (17) gives 3.5 MeV $\leq\delta m\leq 88$ MeV, which is consistent with experiment. In view of the crudity of the $D$ function used and the drastic assumptions made in extracting the form factors, an uncertainty on the order of a factor of 2 in the coefficient of (15) is probably not an overestimate. With the choice $D(W)\approx W-m$, however, the sign of the mass difference appears quite stable[12] within the framework of the Dashen-Frautschi method.

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[12] To estimate the uncertainity in the mass difference due to variations in the numbers $a$, $b$, $c$, etc., we obtained values of the latter using the older Cornell data [K. Berkelman, in *Proceedings of the 1963 International Conference on Nucleon Structure* (Stanford University Press, Stanford, California, 1964), p. 45]. Some of these values differ considerably from those given in (5′) and (9). Upon evaluating the mass difference, however, there is much cancellation, with the result that $\delta m$ suffers only minor changes.

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# Conservation Laws and Symmetries. II

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The reciprocal relationship between conservation laws and symmetries is established for those theories wherein the equations of motion are derivable from a variational principle. It is shown, for a general variational problem with arbitrary number of independent and dependent variables, that to every divergenceless vector there corresponds another which differs from it, in general, by terms that vanish when the Euler-Lagrange equations are satisfied and which has the structure obtained by applying Noether's theorem to some symmetry transformation. Thus existence of a continuity equation implies some invariance property of the variational problem (converse of Noether's theorem). The Lagrangian is invariant, in general, up to a divergence. Derivatives of dependent variables of any arbitrary finite order are allowed to appear in the Lagrangian; it is assumed, however, that it does not contain independent variables explicitly. A systematic procedure is formulated to deduce the invariance property associated with a given conservation law and is illustrated by some examples.

## I. INTRODUCTION

IN a previous paper[1] an attempt was made to prove, in the Lagrangian formalism of local field theory, that every conservation law has associated with it some symmetry property of the (coupled) field system. An analogous proof can, of course, also be worked out[2] in particle mechanics. The proofs given in Ref. 1 and in a previous work by Horn[3] have rather severe limitations. Apart from assuming the existence of the space integrals of the time components of the conserved currents, they involve very restrictive assumptions about the structure of the conserved quantities and of the Lagrangian. In this Paper we shall give a more general proof whose scope has been outlined in the abstract.

Instead of specializing to specific dynamical systems, we shall speak of a general variational problem leaving the nature of the variables unspecified. Indeed, Noether's theorem[4–6] itself deals with variational problems in general, without having any physics associated with it. Like every other theorem in mathematics, it becomes a statement of a physical law only when the variables are identified with the dynamical variables of some physical system.

In Sec. II we collect some useful formulas from the calculus of variation. The next section contains the above-mentioned proof of the converse of Noether's

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theorem. In Sec. IV we formulate a systematic procedure to deduce the symmetry associated with a given conservation law, and to illustrate this we consider as examples the zilch tensor[7,8] and the two conservation laws proposed by Fairlie[9] as counterexamples to the converse of Noether's theorem.

## II. NOETHER'S THEOREM AND CONSERVATION LAWS

Consider a general variational problem in which there are $m$ independent variables $x_\mu$ and $N$ dependent variables $Q_A(x)$. When the Lagrange function contains derivatives of the $Q$'s up to, say, $n$th order, the Euler-Lagrange equations take the form[10]

$$[\mathfrak{L}]_A \equiv \sum_{i=0}^{n} (-1)^i \left(\frac{\partial \mathfrak{L}}{\partial Q_{A,\nu_1\cdots\nu_i}}\right)_{,\nu_1\cdots\nu_i} = 0, \tag{1}$$

where $Q_{A,\nu} \equiv \partial_\nu Q_A \equiv \partial Q_A/\partial x_\nu$ and the summation convention has been used. If the infinitesimal transformation

$$\begin{aligned} x_\mu &\to x_\mu' = x_\mu + \delta x_\mu, \\ Q_A(x) &\to Q_A'(x') = Q_A(x) + \delta Q_A(x) \end{aligned} \tag{2}$$

is a symmetry transformation,[5,6] the following identity must hold[6]:

$$\bar{\delta}\mathfrak{L} + (\bar{\delta}\Omega_\mu + \mathfrak{L}\delta x_\mu)_{,\mu} \equiv 0, \tag{3}$$

where $\bar{\delta}$ denotes the local variation

$$\begin{aligned} \bar{\delta}Q_A &= Q_A'(x) - Q_A(x) = \delta Q_A(x) - Q_{A,\lambda}\delta x_\lambda, \\ \bar{\delta}\mathfrak{L} &= \mathfrak{L}(x,Q'(x)) - \mathfrak{L}(x,Q(x)), \end{aligned} \tag{4}$$

$\bar{\delta}\Omega_\mu$ is an arbitrary infinitesimal vector which vanishes if the Lagrangian is form invariant under the transformation (2).

Now, it is easy to show that

$$\bar{\delta}\mathfrak{L} \equiv [\mathfrak{L}]_A \bar{\delta}Q_A + (\bar{\delta}t_\mu)_{,\mu}, \tag{5}$$

where

$$\bar{\delta}t_\mu = \sum_{i=1}^{n} \sum_{j=0}^{i-1} (-1)^j \left(\frac{\partial \mathfrak{L}}{\partial Q_{A,\nu_1\cdots\nu_{i-1}\mu}}\right)_{,\nu_1\cdots\nu_j} (\bar{\delta}Q_A)_{,\nu_{j+1}\cdots\nu_{i-1}}. \tag{6}$$

Substituting (5) in (3), we get

$$[\mathfrak{L}]_A \bar{\delta}Q_A + (\bar{\delta}\Omega_\mu + \bar{\delta}t_\mu + \mathfrak{L}\delta x_\mu)_{,\mu} \equiv 0. \tag{7}$$

This identity is known as Noether's theorem. When the $Q_A$'s satisfy the Euler-Lagrange equations, this gives

$$(\bar{\delta}\Omega_\mu + \bar{\delta}t_\mu + \mathfrak{L}\delta x_\mu)_{,\mu} = 0. \tag{8}$$

Expressing the variations in terms of a discrete set of parameters $\epsilon_r$ ($r=1, \cdots f$) by writing

$$\begin{aligned} \delta x_\mu &= \epsilon_r X_\mu{}^r, \\ \delta Q_A &= \epsilon_r \Psi_A{}^r(Q), \\ \bar{\delta}Q_A &= \epsilon_r[\Psi_A{}^r - Q_{A,\lambda}X_\lambda{}^r] \equiv \epsilon_r \Phi_A{}^r, \\ \bar{\delta}\Omega_\mu &= \epsilon_r F_\mu{}^r, \end{aligned} \tag{9}$$

one obtains from Eq. (8) the conservation equations

$$\Theta_\mu{}^r{}_{,\mu} = 0, \tag{10}$$

where

$$\Theta_\mu{}^r = F_\mu{}^r + \mathfrak{L}X_\mu{}^r + \sum_{i=1}^{n} \sum_{j=0}^{i-1} (-1)^j \left(\frac{\partial \mathfrak{L}}{\partial Q_{A,\nu_1\cdots\nu_{i-1}\mu}}\right) \Phi_A{}^r{}_{,\nu_{j+1}\cdots\nu_{i-1}}. \tag{11}$$

With $n=1$, Eqs. (1) and (11) take the familiar form

$$\frac{\partial \mathfrak{L}}{\partial Q_A} - \left(\frac{\partial \mathfrak{L}}{\partial Q_{A,\nu}}\right)_{,\nu} = 0, \tag{12}$$

and

$$\begin{aligned} \Theta_\mu{}^r &= F_\mu{}^r + \mathfrak{L}X_\mu{}^r + \frac{\partial \mathfrak{L}}{\partial Q_{A,\mu}} \Phi_A{}^r \\ &= F_\mu{}^r + \left[\mathfrak{L}\delta_{\mu\nu} - \frac{\partial \mathfrak{L}}{\partial Q_{A,\mu}} Q_{A,\nu}\right] X_\nu{}^r + \frac{\partial \mathfrak{L}}{\partial Q_{A,\mu}} \Psi_A{}^r. \end{aligned} \tag{13}$$

## III. CONVERSE OF NOETHER'S THEOREM

We are given a quantity $J_\mu(x,Q(x))$ satisfying the conservation equation

$$J_{\mu,\mu} = 0. \tag{14}$$

Apart from the index $\mu$, the quantity $J_\mu$ may carry an arbitrary number of indices representing its transformation properties in various spaces. These indices will be suppressed in the following discussion.

Now, Eq. (14) holds, in general, by virtue of the equations of motion (1), some given subsidiary conditions, and certain identities that may be applicable. Calculating $(J_{\mu,\mu})$ and substituting the identities and the subsidiary conditions (which we treat as identities) at appropriate places, we will be left with an identity of the form

$$J_{\mu,\mu} \equiv f([\mathfrak{L}]_A), \tag{15}$$

where the function $f$ vanishes with its argument. When $J_\mu$ is a differential expression involving $Q$'s and their derivatives (with a possible explicit dependence on $x$), this identity will take the form

$$J_{\mu,\mu} \equiv (G_A + G_{A\mu}\partial_\mu + G_{A\mu\nu}\partial_\mu\partial_\nu + \cdots)[\mathfrak{L}]_A, \tag{16}$$

where the $G$'s may in general be functions of the $x$'s and of the $Q$'s and their derivatives. The form (16) does not necessarily imply the assumption that the divergence of $J_\mu$ vanishes linearly with the equations of motion (1); this is because the $G$'s have been left completely

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[9] D. B. Fairlie, Nuovo Cimento **37**, 897 (1965).
[10] R. Courant and D. Hilbert, *Methods of Mathematical Physics* (Interscience Publishers, Inc., New York, 1962), Vol. I.

arbitrary and may themselves contain powers of $[\mathfrak{L}]_A$. Some of the $G$'s may be singular when equations of motion (1) are satisfied; however, this does not concern us so long as each term on the right-hand side is well behaved. Now

$$G_{A\mu}\partial_\mu[\mathfrak{L}]_A=(G_{A\mu}[\mathfrak{L}]_A)_{,\mu}-G_{A\mu,\mu}[\mathfrak{L}]_A.$$

The second term on the right can be absorbed in the term $G_A[\mathfrak{L}]_A$; the first term when moved to the left amounts to deducting a term $G_{A\mu}[\mathfrak{L}]_A$ from $J_\mu$. This term vanishes when the equations of motion are satisfied. A similar treatment can be given to the third and higher terms on the right of (16). We are therefore left with the simpler identity

$$\Theta_{\mu,\mu}\equiv G_A[\mathfrak{L}]_A, \tag{17}$$

where the vector $\Theta_\mu$ differs from $J_\mu$ by terms that vanish when the equations of motion (1) are satisfied.

Now, the right-hand side of (17) is a special case of the general structure.

$$[a+a_\mu\partial_\mu+a_{\mu\nu}\partial_\mu\partial_\nu+\cdots]\mathfrak{L}+[b_A+b_{A\mu}\partial_\mu+b_{A\mu\nu}\partial_\mu\partial_\nu+\cdots]\frac{\partial\mathfrak{L}}{\partial Q_A}+[c_{A\nu}+c_{A\nu\mu}\partial_\mu+\cdots]\frac{\partial\mathfrak{L}}{\partial Q_{A,\nu}}+\cdots. \tag{18}$$

Operating on the left of (18) by $\partial_\lambda$, we will again get a similar structure. The converse, however, may not be true, i.e., if the derivative of a differential expression has a structure (18) the expression itself may not always have this structure. We should write, therefore, the most general structure of $\Theta_\mu$ satisfying the identity (17) in the form

$$\Theta_\mu=F_\mu+[A_\mu+A_{\mu\sigma}\partial_\sigma+A_{\mu\sigma\tau}\partial_\sigma\partial_\tau+\cdots]\mathfrak{L}+[B_{A\mu}'+B_{A\mu\sigma}'\partial_\sigma+\cdots]\frac{\partial\mathfrak{L}}{\partial Q_A}+[C_{A\mu\nu}'+C_{A\mu\nu\sigma}'\partial_\sigma+\cdots]\frac{\partial\mathfrak{L}}{\partial Q_{A,\nu}}+\cdots. \tag{19}$$

The function $F_\mu$ and the coefficients $A$, $B'$, $C'$, $\cdots$ will, in general, be functions of the $x$'s and of $Q$'s and their derivatives.

We assume that $\mathfrak{L}$ does not contain $x$ explicitly so that we can write

$$\partial_\sigma\mathfrak{L}=\frac{\partial\mathfrak{L}}{\partial Q_A}Q_{A,\sigma}+\frac{\partial\mathfrak{L}}{\partial Q_{A,\nu}}Q_{A,\nu\sigma}+\cdots+\frac{\partial\mathfrak{L}}{\partial Q_{A,\nu_1\cdots\nu_n}}Q_{A,\nu_1\cdots\nu_n\sigma}.$$

This gives

$$\Theta_\mu=F_\mu+A_\mu\mathfrak{L}+[B_{A\mu}''+B_{A\mu\sigma}''\partial_\sigma+\cdots]\frac{\partial\mathfrak{L}}{\partial Q_A}+[C_{A\mu\nu}''+C_{A\mu\nu\sigma}''\partial_\sigma+\cdots]\frac{\partial\mathfrak{L}}{\partial Q_{A,\nu}}+\cdots. \tag{20}$$

Since we are interested in the structure of $\Theta_\mu$ only when Eqs. (1) are satisfied, we are free to use these equations to make simplifications. Eliminating $\partial\mathfrak{L}/\partial Q_A$ from (20) in this manner, we obtain

$$\Theta_\mu=F_\mu+A_\mu\mathfrak{L}+\sum_{i=1}^{n}\sum_{j=0}^{m}C_{A\mu(\nu_1\cdots\nu_i)(\sigma_1\cdots\sigma_j)}\left(\frac{\partial\mathfrak{L}}{\partial Q_{A,\nu_1\cdots\nu_i}}\right)_{,\sigma_1\cdots\sigma_j}, \tag{21}$$

where $m$ is some finite positive integer.

Now,

$$\begin{aligned}\Theta_{\mu,\mu}&=F_{\mu,\mu}+(A_\mu\mathfrak{L})_{,\mu}+\sum_{i=1}^{n}\sum_{j=0}^{m}C_{A\mu(\nu_1\cdots\nu_i)(\sigma_1\cdots\sigma_j),\mu}\left(\frac{\partial\mathfrak{L}}{\partial Q_{A,\nu_1\cdots\nu_i}}\right)_{,\sigma_1\cdots\sigma_j}\\&\quad+\sum_{i=1}^{n}\sum_{j=1}^{m}C_{A\mu(\nu_1\cdots\nu_i)(\sigma_1\cdots\sigma_j)}\left(\frac{\partial\mathfrak{L}}{\partial Q_{A,\nu_1\cdots\nu_i}}\right)_{,\sigma_1\cdots\sigma_j\mu}\\&=F_{\mu,\mu}+(A_\mu\mathfrak{L})_{,\mu}+\sum_{i=1}^{n}C_{A\mu(\nu_1\cdots\nu_i),\mu}\left(\frac{\partial\mathfrak{L}}{\partial Q_{A,\nu_1\cdots\nu_i}}\right)\\&\quad+\sum_{i=1}^{n}\sum_{j=1}^{m+1}R_{A(\nu_1\cdots\nu_i)(\sigma_1\cdots\sigma_j)}\left(\frac{\partial\mathfrak{L}}{\partial Q_{A,\nu_1\cdots\nu_i}}\right)_{,\sigma_1\cdots\sigma_j},\end{aligned} \tag{22}$$

where

$$R_{A(\nu_1\cdots\nu_i)(\sigma_1\cdots\sigma_j)}=C_{A\mu(\nu_1\cdots\nu_i)(\sigma_1\cdots\sigma_j),\mu}+C_{A\sigma_j(\nu_1\cdots\nu_i)(\sigma_1\cdots\sigma_{j-1})}\quad\text{for}\quad i=1,\cdots n;\ j=1,\cdots,m \tag{23}$$

and

$$R_{A(\nu_1\cdots\nu_i)(\sigma_1\cdots\sigma_{m+1})}=C_{A\sigma_{m+1}(\nu_1\cdots\nu_i)(\sigma_1\cdots\sigma_m)}\quad\text{for}\quad i=1,\cdots,n. \tag{24}$$

The identity (17) now gives

$$F_{\mu,\mu}+(A_\mu\mathfrak{L})_{,\mu}+\sum_{i=1}^{n}C_{A\mu(\nu_1\cdots\nu_i),\mu}\frac{\partial\mathfrak{L}}{\partial Q_{A,\nu_1\cdots\nu_i}}+\sum_{i=1}^{n}\sum_{j=1}^{m+1}R_{A(\nu_1\cdots\nu_i)(\sigma_1\cdots\sigma_j)}\left(\frac{\partial\mathfrak{L}}{\partial Q_{A,\nu_1\cdots\nu_i}}\right)_{,\sigma_1\cdots\sigma_j}\equiv\sum_{i=1}^{n}(-1)^iG_A\left(\frac{\partial\mathfrak{L}}{\partial Q_{A,\nu_1\cdots\nu_i}}\right)_{,\nu_1\cdots\nu_i}. \tag{25}$$

Now, the choice of $Q_A$'s is in general arbitrary. The functional form of the quantities $(\partial\mathfrak{L}/\partial Q_{A,\nu_1\cdots\nu_i})_{,\sigma_1\cdots\sigma_j}$

will change when a different choice of the $Q$'s is made. In order that (25) may hold as an identity, we must have

$$m=n-1$$

and

$$R_{A(\nu_1\cdots\nu_i)(\sigma_1\cdots\sigma_j)}=(-1)^i\delta_{ij}G_A\delta_{\nu_1\sigma_1}\cdots\delta_{\nu_j\sigma_j}. \quad (26)$$

Substituting this in Eqs. (23) and (24), we obtain

$$C_{A\mu(\nu_1\cdots\nu_i)(\sigma_1\cdots\sigma_j),\mu}+C_{A\sigma_j(\nu_1\cdots\nu_i)(\sigma_1\cdots\sigma_{j-1})}=(-1)^i\delta_{ij}G_A\delta_{\nu_1\sigma_1}\cdots\delta_{\nu_j\sigma_j}$$
$$\text{for } i=1,\cdots,n;\ j=1,\cdots,n-1 \quad (27)$$

and

$$C_{A\mu(\nu_1\cdots\nu_i)(\sigma_1\cdots\sigma_{n-1})}=(-1)^n\delta_{in}G_A\delta_{\nu_1\sigma_1}\cdots\delta_{\nu_{i-1}\sigma_{i-1}}\delta_{\nu_i\mu}$$
$$\text{for } i=1,\cdots,n. \quad (28)$$

Putting $j=n-1$ in (27), we obtain

$$\begin{aligned}C_{A\mu(\nu_1\cdots\nu_i)(\sigma_1\cdots\sigma_{n-2})}&=-C_{A\lambda(\nu_1\cdots\nu_i)(\sigma_1\cdots\sigma_{n-2}\mu),\lambda}\\&\quad+(-1)^{n-1}\delta_{i,n-1}G_A\delta_{\nu_1\sigma_1}\cdots\delta_{\nu_{i-1}\sigma_{i-1}}\delta_{\mu\nu_i}\\&=-(-1)^n\delta_{in}G_{A,\nu_i}\delta_{\nu_1\sigma_1}\cdots\delta_{\nu_{i-1}\mu}\\&\quad+(-1)^{n-1}\delta_{i,n-1}G_A\delta_{\nu_1\sigma_1}\cdots\delta_{\nu_{i-1}\sigma_{i-1}}\delta_{\mu\nu_i}\\&=-(-1)^{n-2}\delta_{\nu_1\sigma_1}\cdots\delta_{\nu_{n-2}\sigma_{n-2}}\delta_{\mu\nu_{n-1}}[\delta_{in}G_{A,\nu_i}+\delta_{i,n-1}G_A].\end{aligned} \quad (29)$$

Substituting successively lower values for $j$ in Eq. (27), we obtain the following general expression for the $C$'s:

$$C_{A\mu(\nu_1\cdots\nu_i)(\sigma_1\cdots\sigma_j)}=-(-1)^i\theta(i-j-1)\delta_{\mu\nu_i}\delta_{\nu_1\sigma_1}\cdots\delta_{\nu_j\sigma_j}G_{A,\nu_{j+1}\cdots\nu_{i-1}}$$
$$\text{for } i=1,\cdots n;\ j=1,\cdots,n-1. \quad (30)$$

For $j=1$, Eq. (30) gives

$$C_{A\mu(\nu_1\cdots\nu_i)\sigma_1}=\theta(i-2)\delta_{\mu\nu_i}\delta_{\nu_1\sigma_1}G_{A,\nu_2\cdots\nu_{i-1}}. \quad (31)$$

Putting $j=1$ in Eq. (27), we get

$$\begin{aligned}C_{A\mu(\nu_1\cdots\nu_i)}&=-C_{A\lambda(\nu_1\cdots\nu_i)\mu,\lambda}-\delta_{i,1}G_A\delta_{\nu_1\mu}\\&=-\delta_{\mu\nu_1}G_{A,\nu_2\cdots\nu_i}\quad\text{for } i=1,\cdots,n,\end{aligned} \quad (32)$$

where Eq. (31) has been used in the second step. This gives

$$C_{A\mu(\nu_1\cdots\nu_i),\mu}=-G_{A,\nu_1\cdots\nu_i}. \quad (33)$$

Now, on substituting from Eqs. (26) and (33), Eq. (25) gives

$$F_{\mu,\mu}+(A_\mu\mathcal{L})_{,\mu}+\sum_{i=0}^{n}\frac{\partial\mathcal{L}}{\partial Q_{A,\nu_1\cdots\nu_i}}\Phi_{A,\nu_1\cdots\nu_i}\equiv 0, \quad (34)$$

where we have put

$$\Phi_A\equiv -G_A. \quad (35)$$

The identity (34) is analogous to the identity (3), indicating invariance of the Lagrangian (up to the divergence of the vector $F_\mu$) under the transformations

$$\delta x_\mu=\epsilon A_\mu,\qquad \bar\delta Q_A=\epsilon\Phi_A. \quad (36)$$

Substituting (30) in Eq. (21), we obtain

$$\Theta_\mu=F_\mu+A_\mu\mathcal{L}+\sum_{i=1}^{n}\sum_{j=0}^{i-1}(-1)^j\left(\frac{\partial\mathcal{L}}{\partial Q_{A,\nu_1\cdots\nu_{i-1}\mu}}\right)_{,\nu_1\cdots\nu_j}\Phi_{A,\nu_{j+1}\cdots\nu_{i-1}}, \quad (37)$$

which is of the same form as determined by applying Noether's theorem to the transformation (36).

## IV. SOME EXAMPLES

We have seen that in all theories wherein the equations of motion are derivable from a Lagrangian, every conservation law has associated with it some invariance property of the equations of motion. Proceeding along the lines of the proof given in the previous section, one can directly deduce the symmetry transformation associated with a given conservation law. The following systematic procedure could be followed:

(i) Calculate the divergence of the conserved quantity and, making use of appropriate identities and subsidiary conditions, obtain an identity of the type (17). This determines $\Phi_A$ [see Eqs. (9) through (35)]. If the expression for $\Phi_A$ contains no term proportional to $Q_{A,\lambda}$, then $X_\mu=0$, i.e., no transformation of the independent variables is involved. If such a term is present, however, then it may not always allow an unambiguous determination of $X_\mu$ through the relation

$$\Phi_A=\Psi_A-Q_{A,\lambda}X_\mu$$

because transformations mixing $Q_A$ wlth their derivatives cannot be excluded.

(ii) Write the conserved quantity in the form of the right-hand side of Eq. (11). In most of the cases this will determine $X_\mu$ and $F_\mu$ unambiguously.

(iii) If some ambiguity remains, it can be removed by actual verification that the Lagrangian is invariant under the transformation (36) up to the divergence of the vector $F_\mu$.

We shall now consider two examples from field dynamics.

### (a) The "Zilch"

An interesting example is the zilch tensor of the free electromagnetic field whose conservation, first discovered by Lipkin,[7] initiated an interesting discussion of the conservation laws and invariance properties of linear field theories. We shall employ Kibble's expression[8] (with a modified notation) for the zilch, i.e.,

$$Z_{\mu\nu\rho}={}^*F_{\mu\lambda}\overleftrightarrow{\partial}_\rho F_{\lambda\nu}+{}^*F_{\nu\lambda}\overleftrightarrow{\partial}_\rho F_{\lambda\mu}, \quad (38)$$

where

$$F_{\mu\nu}=A_{\mu,\nu}-A_{\nu,\mu},$$
$${}^*F_{\mu\nu}=\tfrac{1}{2}\epsilon_{\mu\nu\alpha\beta}F_{\alpha\beta}.$$

A straightforward calculation shows that

$$Z_{\mu\nu\rho,\rho} \equiv G_{\mu\nu\alpha\beta\sigma\tau} A_{\beta,\tau} (\Box A_\alpha)_{,\sigma}, \tag{39}$$

where

$$G_{\mu\nu\alpha\beta\sigma\tau} = \tfrac{1}{2}(\epsilon_{\mu\alpha\beta\sigma}\delta_{\nu\tau} + \epsilon_{\mu\alpha\beta\tau}\delta_{\nu\sigma} + \epsilon_{\nu\alpha\beta\sigma}\delta_{\mu\tau} + \epsilon_{\nu\alpha\beta\tau}\delta_{\mu\sigma}). \tag{40}$$

The quantity $G_{\mu\nu\alpha\beta\sigma\tau}$ is symmetric in $\mu\nu$, antisymmetric in $\alpha\beta$, and symmetric in $\sigma\tau$. Now, (39) gives

$$[Z_{\mu\nu\rho} - G_{\mu\nu\alpha\beta\rho\tau} A_{\beta,\tau} \Box A_\alpha]_{,\rho} \equiv -G_{\mu\nu\alpha\beta\sigma\tau} A_{\beta,\sigma\tau} \Box A_\alpha. \tag{41}$$

The symmetry transformation associated with the conservation of zilch is, therefore,

$$\bar{\delta} A_\alpha = \lambda_{\mu\nu} G_{\mu\nu\alpha\beta\sigma\tau} A_{\beta,\sigma\tau}, \quad \lambda_{\mu\nu} = \lambda_{\nu\mu}. \tag{42}$$

Since there is no term in $\bar{\delta} A_\alpha$ proportional to $A_{\alpha,\lambda}$, no transformation of the space-time variables is involved. Now, it was found by Steudel[11] that the zilch is contained in the conservation laws associated with the following 60-parametric transformation of the free electromagnetic field:

$$\delta A_\mu = \epsilon a_{\alpha\beta} b_{\mu\nu} A_{\nu,\alpha\beta}, \quad a_{\alpha\beta} = a_{\beta\alpha}, \quad b_{\mu\nu} = -b_{\nu\mu}. \tag{43}$$

Our deduced symmetry transformations (42) are indeed a subset of these transformations.

The verification that the Lagrangian of the free electromagnetic field is invariant, up to a divergence, under the transformations (42) is straightforward.

### (b) The Counter-Examples of Fairlie

It is well known that electromagnetism and other massless free-field theories are invariant under the conformal group.[12] This invariance yields, apart from the conservation of the energy-momentum and angular momentum tensors, the following additional conservation laws:

$$R_{\mu,\mu} \equiv (x_\nu T_{\mu\nu})_{,\mu} = 0, \tag{44}$$

$$S_{\mu\rho,\mu} \equiv [(x_\rho x_\nu) T_{\mu\nu} - \tfrac{1}{2}(x_\lambda x_\lambda) T_{\mu\rho}]_{,\mu} = 0. \tag{45}$$

These conservation laws can also be deduced from the conservation and tracelessness properties of the energy-momentum tensor, i.e.,

$$T_{\mu\nu,\mu} = 0, \tag{46}$$

$$T_{\mu\mu} = 0. \tag{47}$$

Fairlie[9] showed that, for a free massive vector field, a certain tensor $Z'_{\mu\nu\rho}$ satisfies equations analogous to (46) and (47), and holds good by virtue of these the following conservation laws:

$$(x_\nu Z'_{\mu\rho\nu})_{,\mu} = 0, \tag{48}$$

$$[(x_\lambda x_\nu) Z'_{\mu\rho\nu} - \tfrac{1}{2}(x_\sigma x_\sigma) Z'_{\mu\rho\lambda}]_{,\mu} = 0. \tag{49}$$

Then he contends that since the method of construction of these conservation laws is appropriate to conformal invariant theories, which a massive field theory is not, these conservation laws do not follow from any invariance property of the equations of motion.

The point is that this method of construction does *not* always correspond to conformal invariance. For example, if a quantity $T_{\mu\nu}^{(r)}$, where $(r)$ is an arbitrary set of internal or space-time indices, satisfies equations analogous to (46) and (47), then corresponding $R_\mu^{(r)}$ and $S_{\mu\rho}^{(r)}$ will also satisfy equations analogous to (44) and (45). The invariance properties associated with the conservation of $R_\mu^{(r)}$ and $S_{\mu\rho}^{(r)}$ can be easily determined in terms of those associated with the conservation of $T_{\mu\nu}^{(r)}$. Suppose this latter symmetry transformation is

$$\bar{\delta} Q_A = \epsilon_{(r)\nu} \Phi_{\nu A}^{(r)}, \quad \delta x_\mu = \epsilon_{(r)\nu} X_{\mu\nu}^{(r)}. \tag{50}$$

We have

$$R_\mu{}^{(r)}{}_{,\mu} \equiv T_{\mu\mu}^{(r)} + x_\nu T_{\mu\nu}{}^{(r)}{}_{,\mu} \equiv T_{\mu\mu}^{(r)} - x_\nu \Phi_{\nu A}^{(r)} [\mathcal{L}]_A. \tag{51}$$

Now if

$$T_{\mu\mu}^{(r)} = g_A^{(r)} [\mathcal{L}]_A, \tag{52}$$

then

$$R_\mu{}^{(r)}{}_{,\mu} \equiv [g_A^{(r)} - x_\nu \Phi_{\nu A}^{(r)}][\mathcal{L}]_A, \tag{53}$$

so that the relevant symmetry transformation is

$$\bar{\delta} Q_A = \epsilon_r [-g_A^{(r)} + x_\nu \Phi_{\nu A}^{(r)}]. \tag{54}$$

The same is true for $S_{\mu\rho}^{(r)}$. It is clear that the symmetry associated with the conservation of $R_\mu^{(r)}$ and $S_{\mu\rho}^{(r)}$ will not be the conformal group in general; this latter symmetry appears in the particular case when the symmetry associated with $T_{\mu\nu}^{(r)}$ is that of translations, e.g., for the energy-momentum tensor $T_{\mu\nu}$.

Proceeding along the same lines, one can easily deduce and verify the invariance property associated with the conversation equations (48) and (49).

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