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Conditions on Pion-Nucleon Scattering Derived from Current Commutation Relations*

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Using the assumptions of current commutation relations and of a partially conserved axial-vector current, new relations between pion-nucleon scattering parameters, the axial-vector renormalization constant g_A, and the nucleon-isovector charge and magnetic moment are derived. Calculating the nucleon total magnetic moment, we obtain a result which is in good agreement with experiment.

I. INTRODUCTION

ASSUMING that the equal-time commutators of the integrated time components of the hadron currents obey the algebra of $SU(3)\times SU(3)$, as well as assuming a form of the partially conserved axial-vector-current (PCAC) hypothesis, Adler and Weisberger[1] have shown that there follows a relation between the axial-vector coupling constant in β decay, $g_A=G_A/G_V$, and the pion-nucleon scattering amplitude. Furthermore, this relation, together with Adler's consistency condition,[2] which is a relation involving only strong-interaction parameters, have been derived simultaneously[3] using the form of PCAC analogous to that of Bernstein *et al.*[4] That is to say, one assumes that the divergence of the axial-vector current is a highly convergent operator whose matrix elements obey unsubtracted dispersion relations. The connection between the compatibility of these two relations and the existence of various dispersion theoretic subtraction constants was discussed by the author.[3] In this paper we will show that from the same starting point, and with only slightly more restrictive assumptions on the existence of subtraction-free dispersion relations, further consequences may be deduced which involve the pion-nucleon scattering parameters g_A and the nucleon isovector magnetic moment.

II. DERIVATION OF THE RELATIONS

We consider the matrix element of the time-ordered product of two components of the axial-vector current between one-proton states of momenta p_1, p_2.

$$R_{\alpha\beta}=\int d^4x\, e^{iq_1x}\langle p_2|T(A_\alpha^+(x)A_\beta^-(0))|p_1\rangle, \qquad (1)$$

where $A_\alpha^\pm=A_\alpha^1\pm iA_\alpha^2$. The currents A^i, $i=1, 2, 3$, are the isovector members of the octet of axial-vector currents. The quantity $R_{\alpha\beta}$ is related to the amplitude for scattering protons by an axial-vector field, and we define q_2 such that

$$p_1+q_1=p_2+q_2 \qquad (2)$$

to be able to use the scattering amplitude description more conveniently. Equation (2) then represents energy-momentum conservation for the process. From invariance arguments, $R_{\alpha\beta}$ may be written as a sum of second-rank tensors formed from combinations of the p_i, q_j, and γ matrices evaluated between Dirac spinors, each multiplied by a Lorentz-invariant scalar function. The arguments of the scalar functions are the invariant variables in the problem, which are chosen to be

$$\nu=Q\cdot P/M\,,\quad t=\Delta^2,\quad q_1^2, \qquad (3)$$

where

$$P=\tfrac{1}{2}(p_1+p_2)\,,\quad Q=\tfrac{1}{2}(q_1+q_2)\,,\quad \Delta=q_1-q_2=p_2-p_1\,, \qquad (4)$$

and M is the proton mass. We restrict the initial and final proton and the final meson momenta to satisfy

$$p_1^2=p_2^2=M^2\,,\quad q_2^2=q_1^2. \qquad (5)$$

Finally, we note that

$$2q_1p_1=2M\nu-\tfrac{1}{2}t\,,\quad 2q_1p_2=2M\nu+\tfrac{1}{2}t\,,\quad 2q_1q_2=-t+2q_1^2. \qquad (6)$$

From Eq. (1) we obtain

$$q_1^\alpha q_2^\beta R_{\alpha\beta}=\int d^4x\, e^{-iq_1x}[\langle p_2|T(\partial^\alpha A_\alpha^+(0)\partial^\beta A_\beta^-(x))|p_1\rangle - \delta(x_0)\langle p_2|[\partial^\alpha A_\alpha^+(0),A_0^-(x)]|p_1\rangle + \delta(x_0)iq_2^\beta\langle p_2|[A_0^+(0),A_\beta^-(x)]|p_1\rangle] \equiv R(q_1^2,\nu,t)-C(t)+iN_pJ(\nu,t)\,, \qquad (7)$$

where

$$N_p=(2\pi)^{-3}M_p/E_p.$$

It has been shown that, by using the fact that $q_1^\alpha q_2^\beta R_{\alpha\beta}$ vanishes in the limit q_1^2, ν, $t\to 0$ and by assuming pion-pole-dominated dispersion relations in

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[2] S. L. Adler, Phys. Rev. 137, B1022 (1965).
[3] N. H. Fuchs, Phys. Rev. 149, 1145 (1966); referred to as I.
[4] J. Bernstein, S. Fubini, M. Gell-Mann, and W. Thirring, Nuovo Cimento 17, 757 (1960); W. I. Weisberger, Phys. Rev. 143, 1302 (1966).

q_1^2 for fixed $t=0$, $\nu\cong0$, one may derive two conditions on the pion-nucleon scattering amplitude. The reason that two relations emerged is that for fixed $q_1^2=t=0$, the left-hand side of Eq. (7) vanishes like ν^2. Hence the term independent of ν gives rise to Adler's consistency condition

$$f_\pi^2\tilde{A}^{\pi N(+)}(\nu=t=0)=2Mg_A^2 \tag{8}$$

while the term proportional to ν gives rise to the Adler-Weisberger relation, which we write as

$$g_A^2=2F_1^V(0)-f_\pi^2\left(\frac{\partial\tilde{A}^{\pi N(-)}}{\partial\nu}+\tilde{B}^{\pi N(-)}\right),\quad (\nu=t=0) \tag{9}$$

where F_1^V is the isovector Dirac form factor for the nucleon, and the tilde indicates that the nucleon Born terms have been extracted. All quantities involve on-mass-shell variables. Although $2F_1^V(0)=1$, we write it as above to emphasize the relation of Eq. (9) to the results we will obtain below. We are using the usual conventions that the scattering amplitude T which appears between Dirac spinors may be written in the form

$$T=-A+i\gamma\cdot QB \tag{10}$$

and that each term may be decomposed into symmetric and antisymmetric isotopic-spin parts,

$$A_{\alpha\beta}{}^{\pi N}=A^{\pi N(+)}\delta_{\alpha\beta}+\tfrac{1}{2}[\tau_\alpha,\tau_\beta]A^{\pi N(-)}, \tag{11}$$

where α, β are isotopic-spin indices for the pions.

Now we return to the consideration of Eq. (7) with the momentum-transfer variable t taken nonzero. The term $C(t)$ is expected to be small for $t=0$. This function and its relation to a subtraction constant in the dispersion relation for the even isotopic-spin amplitude $A^{\pi N(+)}$ were discussed in I and in the work of Mahanthappa and Riazuddin.[5] In order to relate $C(t)$ to some quantity we have more information about, we might assume that the commutator defining $C(t)$ could be evaluated by using the $SU(6)$ algebra which includes, in addition to the vector and axial-vector currents, the scalar and pseudoscalar densities which are all constructed according to a quark model. The resulting scalar form factor should be slowly varying as a function of t, since a dispersion relation in t would be determined by scalar meson poles, which do not exist for low masses.[6] Therefore we assume that $\partial C/\partial t$ may be neglected in the following.

The remaining equal-time commutator $J(\nu,t)$ determined by the chiral $SU(3)\times SU(3)$ current algebra, is given by

$$\bar{u}(p_2)2\tau_3\{-\nu F_2^V(t)+i\gamma\cdot Q[F_1^V(t)+F_2^V(t)]\}u(p_1), \tag{12}$$

where F_1^V, F_2^V are the Dirac and Pauli isovector form factors for the nucleon

$$F_1^V(0)=\tfrac{1}{2},\quad F_2^V(0)=\tfrac{1}{2}(\mu_p-\mu_n)=1.85. \tag{13}$$

One easily sees that in the limit of $t=0$, the expression (12) reduces to essentially $2\nu F_1^V(0)=\nu$; it is this which gives the one in the Adler-Weisberger relation.

The expression for $R_{\alpha\beta}$ may be expanded in kinematic tensors, as described above; if one takes the limit $q_1\cdot q_2$, $\nu\to0$, it may be easily seen that the left-hand side of Eq. (9) must vanish like t^2 as $t\to0$. We are thus led to new relations if we set the term in the right-hand side of Eq. (9) which is proportional to t equal to zero. The term independent of t, of course, has already been considered above; it gives rise to Adler's consistency condition, Eq. (8).

Finally we turn to the first term on the right-hand side of Eq. (7), $R(q_1^2,\nu,t)$. We follow the procedure employed in the derivation of Eqs. (8) and (9) and assume that for fixed $\nu\cong0$, $t\cong0$, the quantity $\tilde{R}(q_1,\nu,t)$ satisfies an unsubtracted dispersion relation in q_1^2 which is dominated for $q_1^2\cong0$ by the double pion pole at $q_1^2=\mu^2$. We have defined $\tilde{R}$ as R with the one-neutron pole term R_B extracted. It is the one-neutron pole term which gives the g_A^2 in the Adler-Weisberger relation. Then $\tilde{R}(q_1^2=0,\nu,t)$ is proportional to the amplitude for π^-p scattering on the mass shell

$$\tilde{R}(q_1^2=0,\nu,t)=-f_\pi^2\tilde{T}_{\pi^-p}(\nu,t) \tag{14}$$

with ν, $t\cong0$. We would now like to extract the term proportional to t in the expressions for $\tilde{R}$, R_B, and J which we have obtained. However, this is not as simple as it was for the analogous problem of the term proportional to ν since we have not yet displayed the complete t dependence of these quantities. In order to do this, we project the helicity amplitudes. If we denote the nucleon's helicity by subscripts $+$ or $-$, we have the helicity amplitude[7]

$$\begin{aligned}M_{++}&=(\cos\tfrac{1}{2}\theta)(f_1+f_2)\\&=(\cos\tfrac{1}{2}\theta)[A(\nu,t)+(\nu-t/4M)B(\nu,t)]\end{aligned} \tag{15}$$

and the kinematic relations

$$\begin{aligned}\cos\tfrac{1}{2}\theta&=[(S^2+st/S^2]^{1/2},\\S^2&=[s-(m+\mu)^2][s-(m-\mu)^2],\\s&=(p_1+q_1)^2=M^2+\mu^2+2M\nu-\tfrac{1}{2}t.\end{aligned} \tag{16}$$

Now, if we project the $++$ helicity amplitude out of all the terms in Eq. (7) and take the derivative with respect to t, there will be two terms. One will be $\partial(\cos\frac{1}{2}\theta)/\partial t$ multiplied by an undifferentiated expression, while the other will be $\cos\frac{1}{2}\theta$ multiplied by the derivative of the expression with respect to t. It is clear that the expression multiplying $\partial(\cos\frac{1}{2}\theta)/\partial t$ will vanish owing to the Adler consistency condition, when we let ν, t, $q_1^2\to0$. However, the second term, multiplying

[5] K. T. Mahanthappa and Riazuddin, University of Pennsylvania Report (unpublished).

[6] Riazuddin and Fayyazuddin, Nuovo Cimento 44, 546 (1966).

[7] M. Jacob and G. C. Wick, Ann. Phys. (N.Y.) 7, 404 (1959).

$\cos\frac{1}{2}\theta$, now has all the t dependence exposed and will in the limit of ν, t, $q_1^2 \to 0$ give some new information. The relations that appear after these manipulations are

$$f_\pi^2 \partial \tilde{A}^{(+)}(\nu=t=0)/\partial t = g_A^2/4M\,, \tag{17}$$

$$f_\pi^2 \tilde{B}^{(-)}(\nu=t=0) = 2[F_1^V(0)+F_2^V(0)]\,, \tag{18}$$

where the tilde as usual denotes that the nucleon pole term has been extracted. It should be noted that we have assumed that $R_{\alpha\beta}$ has no pole at $\nu=t=q_1^2=0$; this is true if we remember the nonzero neutron-proton mass difference. However, then $A^{(+)}$ has a contribution from the one-nucleon pole and thus the tilde in Eq. (17) has some meaning. (See discussion in I.) Again, as in the case of the Adler-Weisberger relation, the right-hand side of Eq. (18) comes from the current commutation relations at equal times. Since $A^{(-)}$ is an odd function of ν, the fact that $\partial A^{(-)}/\partial t=0$ at $\nu=t=0$ is trivial; contrariwise, Eq. (17) is nontrivial. Also, since $B^{(+)}$ is odd in ν it does not contribute to Eq. (17).

III. CONCLUSIONS

In order to compute $B^{(-)}(\nu=t=0)$ we may assume that $B^{(-)}$ satisfies an unsubtracted dispersion relation in ν for fixed $t=0$, and then extract the Born term. This gives

$$\tilde{B}^{(-)}(\nu=t=0) = \frac{2}{\pi}\int_\mu^\infty \frac{d\nu}{\nu}\,\mathrm{Im}B^{(-)}(\nu,0)\,. \tag{19}$$

The function $\mathrm{Im}B^{(-)}$ may be taken from the paper of Hamilton and Woolcock.[8] For the range $\mu \leq \nu \leq 11\,\mu$ we use their Fig. 7, which is based on a phase-shift analysis of π-p scattering data. For very high energies, Hamilton and Woolcock estimate

$$\mathrm{Im}B^{(-)}(\nu,0) \simeq \tfrac{1}{2}(\sigma_- - \sigma_+)\,, \tag{20}$$

where $\sigma_\pm$ is the $\pi^\pm p$ total cross section. We find that the integral up to a pion lab kinetic energy of 300 MeV contributes 4.3; the integral over the range 300 MeV to 1400 MeV contributes 1.0; the integral over the range 1400 MeV to ∞ contributes about 0.10. Thus we see that the 3-3 resonance is the dominant factor in determining $\tilde{B}^{(-)}$, and we obtain

$$\tilde{B}^{(-)}(\nu=t=0) = 5.4\,. \tag{21}$$

If we put this value into Eq. (18), we find that the left-hand side is 4.9, while the right-hand side is 4.7. The agreement is excellent.

[8] J. Hamilton and W. S. Woolcock, Rev. Mod. Phys. 35, 737 (1963).

Similar relations involving $F_2^V(0)$, the isovector charge radius, and the amplitude for isovector photon production on protons have been derived by several authors.[9–13] We may use these relations to compute $F_2^V(0)$. If we do this, we find values for $F_2^V(0)$ which are consistently larger than the experimental number, although roughly in agreement among themselves. These relations involve production amplitudes which are not experimentally determined at present, and so their evaluation depends upon theoretical models of photoproduction. On the other hand, the relations derived in this paper may be checked by using only a phase-shift analysis of pion-nucleon scattering data. If one is confident of the phase-shift analysis up to say 1.4 GeV, then the contribution from the second and third pion-nucleon resonances are then known, and the value of $\tilde{B}^{(-)}(\nu=t=0)$ is fixed quite accurately.

We have not here attempted to check Eq. (17) by computing $\partial\tilde{A}^{(+)}(\nu=t=0)/\partial t$. It would be of interest to evaluate this expression by use of either the experimental phase-shift analysis of pion-nucleon scattering or by a model of pion-nucleon scattering with 3-3 resonance dominance. There is however one point of uncertainty here, and that is our assumption that the scalar form factor $C(t)$ may be neglected. Although $C(t)$ may be small, it is possible that the t dependence may be such that this term becomes non-negligible.

In the above we have used only the chiral $SU(2)\times SU(2)$ current algebra, and the *PCAC* hypothesis applied in the case of pions. However, we might point out here that if one uses the chiral $SU(3)\times SU(3)$ current algebra and the *PCAC* hypothesis for kaons, one would obtain relations involving kaon-baryon scattering parameters, the d/f ratio for the axial current, the baryon isovector *and* isoscalar magnetic moments, and of course the axial-vector renormalization constant g_A.

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