sense:

$$\begin{aligned}\langle k\pi(i),\nu|T_0{}^{\pi}|l\pi(j),\nu\rangle &= \sum_\mu C(t_k,1,t_i|\mu,\,\nu-\mu,\,\nu)C(t_l,1,t_j|\mu,\,\nu-\mu,\,\nu) \\ &\quad\times\langle k,\mu|l,\mu\rangle\langle\pi,\,\nu-\mu|T_0{}^{\pi}|\pi,\,\nu-\mu\rangle \\ &= \delta_{kl}\sum_\mu C(t_k,1,t_i|\mu,\,\nu-\mu,\,\nu)C(t_l,1,t_j|\mu,\,\nu-\mu,\,\nu)(\nu-\mu)\,. \end{aligned} \tag{B8}$$

Taking expectation values as in Secs. 2 and 3, and noting (B5), (B7), and (B8), we find at once

$$\begin{aligned}\langle i||\delta||j\rangle C(t_j,1,t_i|\nu,0,\nu) &= \sum_{k,l}\alpha_{ik}{}^*\alpha_{jl} \\ &\times\sum_\mu C(t_k,1,t_i|\mu,\,\nu-\mu,\,\nu)C(t_l,1,t_j|\mu,\,\nu-\mu,\,\nu) \\ &\quad\times\{\langle k||\delta||l\rangle C(t_l,1,t_k|\mu,0,\mu)+\tfrac{1}{2}C\delta_{kl}(\nu-\mu)\}\,. \end{aligned} \tag{B9}$$

On the right, the term containing $(C/2)\delta_{kl}\nu$ simplifies by virtue of (B6) and (B3), and one gets

$$\begin{aligned}\langle i||\delta||j\rangle C(t_j,1,t_i|\nu,0,\nu) &= \tfrac{1}{2}C\nu\delta_{ij} \\ &+\sum_{k,l}\alpha_{ik}{}^*\alpha_{jl}\sum_\mu C(t_k,1,t_i|\mu,\,\nu-\mu,\,\nu)C(t_l,1,t_j|\mu,\,\nu-\mu,\,\nu) \\ &\quad\times\{\langle k||\delta||l\rangle C(t_l,1,t_k|\mu,0,\mu)-\tfrac{1}{2}C\mu\delta_{kl}\}\,. \end{aligned} \tag{B10}$$

This is a set of simultaneous inhomogeneous linear equations for the $\langle i||\delta||j\rangle$ whose solution is unique. By inspection, the solution is given by

$$\langle i||\delta||j\rangle C(t_j,1,t_i|\nu,0,\nu)=\tfrac{1}{2}C\nu\delta_{ij},\quad \text{(for all } i,\,j).\tag{B11}$$

To check, notice that by virtue of (B11), the first term on the right of (B10) equals the left-hand side, and the second term on the right vanishes because the contents of the curly brackets vanish; here again one relies on (B7).

Evaluating the mass difference between substates of $|i\rangle$ with equal and opposite values of ν (to which the tensor splitting does not contribute), we find from (B5), (B7), and (B11),

$$[\langle i,\nu|\delta|i,\nu\rangle-\langle i,\,-\nu|\delta|i,\,-\nu\rangle]=C\nu,\quad \text{Q.E.D.},\tag{B12}$$

independently of i, and independently of the dynamical coefficients α_{ij}. The results (2.11) (with $M=0$) and (3.8) are special cases of (B12).

PHYSICAL REVIEW VOLUME 150, NUMBER 4 28 OCTOBER 1966

Generalized Nonet Representation of $SU(3)\otimes SU(3)$ and Its Applications

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(Received 31 May 1966)

A class of representations of the nonchiral $SU(3)\otimes SU(3)$ is worked out. These consist of a sequence of self-conjugate representations of $SU(3)$, starting always with a singlet and with each $SU(3)$ representation occurring once. An analog of the Gell-Mann–Okubo mass formula, valid for these representations of $SU(3)\otimes SU(3)$, is obtained. When applied to the lowest nontrivial representation, this formula correctly explains ω-ϕ mixing, thus providing a justification of Okubo's ansatz. Possible use of the next higher representation is indicated. From the same construction, the corresponding unitary irreducible representations of $SL(3,C)$ and $T_8\times SU(3)$ are simultaneously obtained.

I. INTRODUCTION

In this paper we describe the explicit construction of a class of irreducible representations of the nonchiral $SU(3)\otimes SU(3)$ and discuss their possible experimental relevance.[1] Let G_i and F_i denote the infinitesimal generators of the two commuting $SU(3)$'s. We now define a third $SU(3)$ whose infinitesimal generators are G_i+F_i. The special representations we have in mind are those in which this last $SU(3)$ is diagonal and which consist of a finite sequence of self-conjugate representations of this $SU(3)$, starting always with a singlet and with each representation occurring once. These representations are characterized by a single parameter which can take up odd integral values, and which is essentially a measure of the dimensionality of the representation.

Our construction also yields the corresponding representations of the noncompact $SL(3,C)$. In this case, the diagonal $SU(3)$ can be identified with the maximum compact subgroup. The single parameter that labels the irreducible representations can now take up real, odd-integral values or purely imaginary values. In the former case we get finite-dimensional nonunitary representations (unitary trick). In the latter case we get infinite-dimensional unitary representations. For the sake of completeness we also describe a similar representation of $T_8\times SU(3)$—the semidirect product of $SU(3)$ with eight mutually commuting translations [see Eq. (38)].

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It may be noted that the above-mentioned class of representations are labelled by a single parameter. Hence, out of the four Casimir invariants of $SU(3)\otimes SU(3)$ [or $SL(3,C)$], only one is independent. This is the maximal degenerate situation and is analogous to the corresponding representations used in classifying the states of the nonrelativistic hydrogen atom and other dynamical systems,[2] as well as the isobar states of the charge-independent, pseudoscalar strong-coupling theory.[3] These representations are thus of intrinsic theoretical interest. Apart from this, there remains the possibility of being able to group together distinct $SU(3)$ multiplets of particles, with the same spin and parity, in a single degenerate representation of $SU(3)\otimes SU(3)$. The lowest nontrivial representation [$SU(3)$ singlet plus octet] is already realized experimentally. It is interesting to see if the next representation might also be realized. In this connection the most promising candidate seems to be a boson 27-plet (see Sec. IV).

In Sec. II we work out the desired representation. In Sec. III we consider the problem of symmetry breaking. With a simple assumption about the symmetry breaking interaction, we obtain an analog of the Gell-Mann–Okubo mass formula for the present case. This new formula expresses masses within multiplets of $SU(3)$ as well as transition masses between adjacent multiplets in terms of two unknown parameters. As an application of these ideas, we consider in Sec. IV the lowest nontrivial representation. In this case we obtain a rigorous justification of the "nonet ansatz" of Okubo.[4] In Sec. V we make concluding remarks.

II. REPRESENTATION

Let us start with a pair of commuting $SU(3)$'s so that their infinitesimal generators obey the commutation relations:

$$[F_i,F_j]=if_{ijk}F_k\,, \qquad (1)$$

$$[G_i,G_j]=if_{ijk}G_k\,, \qquad (2)$$

$$[G_i,F_j]=0\,. \qquad (3)$$

In the above, f_{ijk} are the structure constants of $SU(3)$ and i, j, k take values 1, 2, 3 $\cdots$ 8. We now define new operators A_i and B_i as

$$A_i=F_i+G_i\,, \qquad (4)$$

$$B_i=F_i-G_i\,, \qquad (5)$$

so that their commutation relations are

$$[A_i,A_j]=if_{ijk}A_k\,, \qquad (6)$$

$$[A_i,B_j]=if_{ijk}B_k\,, \qquad (7)$$

$$[B_i,B_j]=if_{ijk}A_k\,. \qquad (8)$$

Equations (6)–(8) define the algebra of $SW(3)$. If instead of Eq. (5) we define B_i as

$$B_i=i(F_i-G_i)\,, \qquad (9)$$

then the right-hand side of Eq. (8) will acquire a negative sign and we will have the algebra of noncompact $SL(3,C)$. First notice that operators A_i generate algebra of $SU(3)$; we shall construct representations of $SW(3)$ or $SL(3,C)$ in which this $SU(3)$ is diagonal. We shall follow a method which is a straightforward extension of that used by Naimark[5] in obtaining the representations of $SL(2,C)$. We first rewrite Eqs. (6)–(8) on a spherical basis:

$$\begin{aligned} &I_\pm=A_1\pm iA_2\,, &&\hat{I}_\pm=B_1\pm iB_2\,,\\ &K_\pm=A_4\pm iA_5\,, &&\hat{K}_\pm=B_4\pm iB_5\,,\\ &L_\pm=A_6\pm iA_7\,, &&\hat{L}_\pm=B_6\pm iB_7\,,\\ &I_3=A_3\,,\quad Y=\tfrac{2}{3}\sqrt{3}A_8\,, &&\hat{I}_3=B_3\,,\quad \hat{Y}=\tfrac{2}{3}\sqrt{3}B_8\,. \end{aligned} \qquad (10)$$

Let us now denote by $f_{I,I_3,Y}{}^{m,n}$ a set of vectors which provide a unitary irreducible representation of $SU(3)$, so that $I_\pm$, I_3, $K_\pm$, $L_\pm$, and Y are represented in this basis in the usual way.[6] We have now to obtain representation of operators $\hat{I}_\pm, \hat{K}_\pm \cdots \hat{Y}$. Clearly it is sufficient to obtain $\hat{Y}f_{I,I_3,Y}{}^{m,n}$, as this yields the form of every other generator through Eqs. (7) and (10). First we note that according to Eqs. (7) and (10) the operator $\hat{Y}$ transforms as the $I=0$, $Y=0$ component of a regular $SU(3)$ tensor. Hence, acting on $f_{I,I_3,Y}{}^{m,n}$ it can only change m and n, the allowed values of this transition being $(m,n)\to(m\pm1,n\pm1)$, $(m,n)\,(m\to\mp1,n\pm2)$, $(m,n)\to(m\mp2,n\pm1)$, and $(m,n)\to(m,n)$. To obtain the desired representation we seek to construct an invariant vector space for our operators using only those vectors for which $m=n$. Thus we have to consider only transitions $m\to m\pm1$ and $m\to m$. Hence we write, putting $I_3=I$,

$$\begin{aligned}\hat{Y}f_{I,Y}{}^{m}=\mathfrak{z}(m,I,Y;m-1)f_{I,Y}{}^{m-1}&\\ +A(m,I,Y)f_{I,Y}{}^{m}+B(m,I,Y)f_{I,Y}{}^{m}&\\ -\mathfrak{z}(m,I,Y;m+1)f_{I,Y}{}^{m+1}\,.&\end{aligned} \qquad (11)$$

Remembering once again that $\hat{Y}$ is the component of a regular tensor operator of subgroup $SU(3)$ generated by A_i and applying the Wigner-Eckart theorem, we conclude that each of the (unknown) functions $\mathfrak{z}(m,I,Y;m\pm1)$, $A(m,I,Y)$, and $B(m,I,Y)$ will factor into a suitable Clebsch-Gordan coefficient and a reduced matrix element independent of I and Y. The Clebsch-Gordan coefficients corresponding to the $m\to m\pm1$ transitions have been given in an explicit form by Lurié and Macfarlane[7] and those corresponding to the diagonal terms $m\to m$ are given by Okubo[8] in the form

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of his celebrated mass formula. Combining these results we obtain from Eq. (11)

$$\hat{Y}f_{I,Y}{}^{m}=j(m,I,Y;m-1)C_{m}f_{I,Y}{}^{m-1}+\{Ya_{m}+[I(I+1)-\tfrac{1}{4}Y^{2}-\tfrac{1}{3}m(m+2)]b_{m}\}f_{I,Y}{}^{m}-j(m,I,Y;m+1)C_{m+1}f_{I,Y}{}^{m+1}. \quad (12)$$

The Clebsch-Gordan coefficients occurring above are given by

$$j(m,I,Y;m-1)=-\sqrt{2}\{[(m-I)^{2}-\tfrac{1}{4}Y^{2}][(m+I+1)^{2}-\tfrac{1}{4}Y^{2}]\}^{1/2}, \quad (13)$$

$$j(m,I,Y;m+1)=-\sqrt{2}\{[(m-I+1)^{2}-\tfrac{1}{4}Y^{2}][(m+I+2)^{2}-\tfrac{1}{4}Y^{2}]\}^{1/2}. \quad (14)$$

It remains to determine the parameters a_m, b_m, and c_m, which depend on m only. First we obtain from (7), (10), and (12):

$$\begin{aligned}\hat{K}_{+}f_{I,Y}{}^{m}&=[\hat{Y},K_{+}]f_{I,Y}{}^{m}\\&=[j(m,I+\tfrac{1}{2},Y+1;m-1)b_{+}-j(m,I,Y;m-1)b_{+1}]c_{m}f_{I+1/2}{}^{m-1}{}_{,Y+1}\\&\quad-[j(m,I+\tfrac{1}{2},Y+1;m+1)b_{+}-j(m,I,Y;m+1)b_{+2}]c_{m+1}f_{I+1/2}{}^{m+1}{}_{,Y+1}\\&\quad+[a_{m}+\tfrac{1}{2}(2I+1-Y)b_{m}]b_{+}f_{I+1/2}{}^{m}{}_{,Y+1},\end{aligned} \quad (15)$$

when b_+ is given by

$$b_{+}=\left[\frac{(I+1+\frac{1}{2}Y)(m+I+2+\frac{1}{2}Y)(m-I-\frac{1}{2}Y)}{2(I+1)}\right]^{1/2}, \quad (16)$$

and b_{+1} and b_{+2} are obtained from b_+ by the substitution $m\to m-1$ and $m\to m+1$, respectively. We now use the remaining commutation relation (8). From Eqs. (8) and (10) we get

$$[\hat{Y},\hat{K}_{+}]=\pm K_{+}. \quad (17)$$

The plus and minus signs on the right-hand side of (17) refer to $SW(3)$ and $SL(3,C)$, respectively. Applying (17) to $f_{I,Y}{}^{m}$ and using Eqs. (12)–(16), we get equations for a_m, b_m, and c_m.

$$(m+2)a_{m}-ma_{m-1}+\tfrac{1}{6}(m+2)(2m+3)b_{m}-\tfrac{1}{6}(m+2)(2m-1)b_{m-1}+(I-\tfrac{1}{2}Y)[a_{m}-a_{m-1}+\tfrac{1}{2}(2m+3)b_{m}-\tfrac{1}{2}(2m-1)b_{m-1}]=0, \quad (18)$$

$$4(m-I+1+\tfrac{1}{2}Y)(m+I+2-\tfrac{1}{2}Y)(m+2)c_{m+1}{}^{2}-4(m-I+\tfrac{1}{2}Y)(m+I+1-\tfrac{1}{2}Y)mc_{m}{}^{2}+[a_{m}+(I-\tfrac{1}{2}Y+\tfrac{1}{2})b_{m}]^{2}=\pm1. \quad (19)$$

We now solve Eqs. (18) and (19). First consider $SW(3)$, i.e., plus the sign on the right-hand side of Eq. (19). From Eq. (18) we get

$$(m+2)a_{m}-ma_{m-1}+\tfrac{1}{6}(m+2)[(2m+3)b_{m}-(2m-1)b_{m-1}]=0, \quad (20)$$

$$a_{m}-a_{m-1}+\tfrac{1}{2}[(2m+3)b_{m}-(2m-1)b_{m-1}]=0. \quad (21)$$

From Eqs. (20) and (21), we get a recursion relation for a_m,

$$(m+2)a_{m}-(m-1)a_{m-1}=0, \quad (22)$$

whose solution is given by

$$a_{m}=a/m(m+1)(m+2), \quad (23)$$

where a is a constant. In order that a_m remain well defined for $m=0$, we define a new constant c as

$$a=m_{0}c. \quad (24)$$

Clearly, m_0 is to be interpreted as the least value of m in an irreducible representation. We defer the determination of b_m to a later occasion and pass on to examine Eq. (19), which is equivalent to a set of three simultaneous equations:

$$4(m+1)(m+2)^{2}c_{m+1}{}^{2}-4m^{2}(m+1)c_{m}{}^{2}+(a_{m}+\tfrac{1}{2}b_{m})^{2}=1, \quad (25)$$

$$(m+2)c_{m+1}{}^{2}-mc_{m}{}^{2}-\tfrac{1}{2}b_{m}(a_{m}+\tfrac{1}{2}b_{m})=0, \quad (26)$$

$$(m+2)c_{m+1}{}^{2}-mc_{m}{}^{2}-\tfrac{1}{4}b_{m}{}^{2}=0. \quad (27)$$

From Eqs. (26) and (27) we get $a_m b_m=0$, so that there are two cases to consider:

(a) $a_m=0,\ b_m\neq0$,

(b) $a_m\neq0,\ b_m=0$.

However, from Eqs. (20) and (21) we see that $b_m=0$ will automatically imply $a_m=0$ as well, so that case (b) is not possible. Hence we obtain

$$a_{m}=0 \text{ for all } m, \quad (28)$$

which implies that $m_0=0$. Thus the desired representation starts with a $SU(3)$ singlet. With (28), Eqs. (20) and (21) reduce to a recursion relation,

$$(2m+3)b_{m}-(2m-1)b_{m-1}=0, \quad (29)$$

whose solution is

$$b_{m}=2b/(2m+1)(2m+3), \quad (30)$$

where b is a real constant independent of m. With a_m and b_m thus determined, Eqs. (25) and (26) acquire now the final form

$$4(m+1)(m+2)^2c_{m+1}^2-4m^2(m+1)c_m^2+b^2/(2m+1)^2(2m+3)^2=1\,, \quad (31)$$

$$(m+2)c_{m+1}^2-mc_m^2-b^2/(2m+1)^2(2m+3)^2=0\,. \quad (32)$$

To solve the above equations we define

$$\sigma_m=4m^2(m+1)^2c_m^2\,, \quad (33)$$

so that Eq. (31) becomes

$$\sigma_{m+1}-\sigma_m=(m+1)-b^2(m+1)/(2m+1)^2(2m+3)^2. \quad (34)$$

Hence

$$\sigma_m=\sum_{n=0}^{m-1}(\sigma_{n+1}-\sigma_n)$$
$$=\sum_{n=0}^{m-1}(n+1)-\tfrac{1}{8}b^2\sum_{n=0}^{m-1}\left\{\frac{1}{(2n+1)^2}-\frac{1}{(2n+3)^2}\right\}$$
$$=\frac{m(m+1)}{2(2m+1)^2}[(2m+1)^2-b^2]. \quad (35)$$

From Eqs. (33) and (35) we get the final solution

$$c_m=\frac{1}{2(2m+1)}\left[\frac{(2m+1)^2-b^2}{2m(m+1)}\right]^{1/2}. \quad (36)$$

Equation (32) is also now satisfied, as can be verified by direct substitution. Equations (12), (13), (14), (28), (30), and (36) give the desired class of representations. These are characterized by a single parameter b which is real because the group generators are Hermitian [see Eq. (30)]. This parameter has a simple meaning, i.e., it is the highest value of m in a given representation so that b is related to the highest dimensional $SU(3)$ multiplet in a given representation of $SW(3)$. Thus if we have a representation with a $SU(3)$ singlet and octet, then $c_2=0$ and $|b|=5$. Similarly, in a representation with a singlet, octet, and 27-plet of $SU(3)$, we have $c_3=0$ so that $|b|=7$, and so on. We summarize our results on $SW(3)$. The class of representations under discussion consists of the following sequence of $SU(3)$ representations:

$$D(0,0),\,D(1,1),\,D(2,2),\,D(3,3),\,\cdots, \quad (37)$$

with each representation occurring once. The sequence (37) always starts with $D(0,0)$ [$SU(3)$ singlet] and ends with a definite $SU(3)$ representation depending on the value of the parameter b.

In terms of representations of the two commuting $SU(3)$'s generated by G_i and F_i, respectively, the sequence (37) means (d,d^*). Thus $|b|=5$ corresponds to $(3,3^*)$, $|b|=7$ to $(6,6^*)$ $(=1\oplus 8\oplus 27)$, $|b|=9$ to $(10,10^*)$ $(=1\oplus 8\oplus 27\oplus 64)$, and so on. The explicit values of matrix elements of generators are given by Eqs. (12)–(14), (28), (30), and (36).

A similar construction can be done for $SL(3,C)$. We state final results. The right-hand side of Eq. (30) now undergoes the substitution $b\rightarrow ib$, and the right-hand side of Eq. (36) is multiplied by i. The unitary representations of $SW(3)$ discussed above thus become nonunitary representations of $SL(3,C)$. However, if in this case b is chosen as purely imaginary we have a unitary representation [Hermitian generator, see Eq. (30) with $b\rightarrow ib$] of $SL(3,C)$. In this case we see from Eq. (36) that $c_m\neq 0$ for any m so that we have an infinite-dimensional representation. Finally, we might mention that considerations identical to those above can be made to obtain a similar representation of the group $T_8\times SU(3)$ (semidirect product of $SU(3)$ with eight mutually commuting translation generators), which appears in strong-coupling theory. In this case the right-hand side of Eq. (17) is zero and the final solution is exactly as above except that Eq. (36) is now replaced by

$$c_m=[b/2(2m+1)][2m(m+1)]^{-1/2}, \quad (38)$$

so that the representation is infinite dimensional unless $b=0$, in which case all translation operators are identically zero.

III. MASS FORMULA

We consider the problem of mass splitting. We assume that the Hamiltonian consists of two pieces:

$$H=H_0+H_1. \quad (39)$$

H_0 is invariant under $SW(3)$ and H_1 transforms as the $I=0$, $Y=0$ component of a certain tensor. This tensor, of course, is a linear combination of a finite number of $SU(3)$ tensors. We now require that within a given $SU(3)$ multiplet the usual Gell-Mann-Okubo mass formula be left undisturbed. It then follows that the noninvariant part H_1 transforms as the component of an octet tensor [under $SU(3)$ generated by A_i]. Because the generator $\hat{Y}$ has exactly the same properties, we conclude that H_1 transforms as $\hat{Y}$,

$$H_1\sim\hat{Y}. \quad (40)$$

Using Eqs. (12), (30), and (36), we can obtain from (40) the desired mass formulas. For diagonal matrix elements, valid within a $SU(3)$ multiplet, we get

$$M=\langle mIY|H|mIY\rangle$$
$$=M_0+[2b/(2m+1)(2m+3)]\times[I(I+1)-\tfrac{1}{4}Y^2-\tfrac{1}{3}m(m+2)]M'. \quad (41)$$

For transition masses between adjacent $SU(3)$ multiplets, we get

$$M_T=|\langle m-1,IY|H|mIY\rangle|=\frac{1}{2(2m+1)}\left|\left\{\frac{[(m-I)^2-\frac{1}{4}Y^2][(m+I+1)^2-\frac{1}{4}Y^2][(2m+1)^2-b^2]}{m(m+1)}\right\}^{1/2}\right|M'. \quad (42)$$

Equations (41) and (42) give the mass formula for the maximal degenerate representation of $SW(3)$. In these formulas M_0 and M' are unknown parameters, I and Y denote the isotopic spin and hypercharge, respectively, m denotes the $SU(3)$ representation (dimensionality $=(1+m)^3$), and b is the label of $SW(3)$ representation. Using Eqs. (41) and (42), we can diagonalize the mass matrix and obtain sum rules between physical masses. Because of the absence of the term proportional to hypercharge, Eq. (41) gives very bad results for fermions, and in what follows we apply it to bosons only.

Finally, we mention that for $SL(3,C)$ Eq. (42) remains valid, but in Eq. (41) we have to replace $b\rightarrow ib$.

IV. APPLICATIONS

We consider possible applications of the above scheme. The simplest nontrivial representation of $SW(3)$ is for $|b|=5$. The $SU(3)$ content of this is a singlet plus an octet. The observed nonet of particles with the same spin-parity may be put in this representation. The general mass formulas (41) and (42) reduce in this case to

$$M_8=M_0+\tfrac{2}{3}[I(I+1)-\tfrac{1}{4}Y^2-1]M', \quad (43)$$

$$M_T(1\leftrightarrow 8)=\tfrac{2}{3}\sqrt{2}M', \quad M_1=M_0. \quad (44)$$

We apply Eqs. (43) and (44) to vector mesons. Diagonalizing the mass matrix we get [particle label = particle (mass)^2]

$$\omega=\rho, \quad (45)$$

$$2K^*-\omega=\phi. \quad (46)$$

Equations (45) and (46) are in excellent agreement with experiments. These results were first derived by Okubo[4] on the basis of his "nonet ansatz," so that we have found a rigorous justification for his ansatz. For pseudoscalar particles, Eqs. (45) and (46) are in bad agreement with observed masses, but agreement for 2^+ particles is again good.

The next higher representation is $|b|=7$, which contains a singlet, octet, and a 27-plet. The mass formula becomes

$$M_{27}=M_0+(14/35)\times[I(I+1)-\tfrac{1}{4}Y^2-8/3]M', \quad (47)$$

$$M_8=M_0+(14/15)\times[I(I+1)-\tfrac{1}{4}Y^2-1]M', \quad (48)$$

$$M_1=M_0, \quad M_T(1\leftrightarrow 8)=(\tfrac{2}{3}\sqrt{5})M', \quad (49)$$

$$M_T(8\leftrightarrow 27)=\tfrac{1}{5}\{[(2-I)^2-\tfrac{1}{4}Y^2]\times[(3+I)^2-\tfrac{1}{4}Y^2]\}^{1/2}M'. \quad (50)$$

If a boson 27-plet is ever observed, then the empirical masses can be used to test Eqs. (47)–(50).

V. CONCLUDING REMARKS

We have constructed a class of representations of $SW(3)$ consisting of a sequence of zero-triality $SU(3)$ representations, each occurring once. We have a mass formula for this class of representations. When restricted to the lowest nontrivial representation, this formula correctly explains ω-ϕ mixing and provides justification of "Okubo ansatz." We have also listed results for the next higher representation in case a boson 27-plet is ever observed.

Techniques similar to those used in this paper may be used to obtain other classes of representations of $SW(3)$, $SL(3,C)$, and $T_8\times SU(3)$. For instance, we can compute isobar spectrum of scalar, $SU(3)$-symmetric strong-coupling theory [group $T_8\times SU(3)$]. By explicit calculation using conventional methods, Dullemond[9] has reported the isobar spectrum of this theory to consist of **8**, **10**, **10***, **27**, **35**, **35***, $\cdots$, etc. A representation of this type may be obtained if in Eq. (11) we admit, instead, the transitions $(m,n)\rightarrow(m\pm2,n\mp1)$, $(m,n)\rightarrow(m\mp1,n\pm2)$. However, in this case it seems impossible to have a representation containing only a single **27**. The spectrum is given by **8**, **10**, **10***, **27**(2), **35**, **35***, **64**(2). The multiplicity of each representation is shown in the bracket. It is possible that these two 27-plets, degenerate in mass to start with, split owing to representation mixing so that one of them is pushed high up, which might explain why the second 27 has not been noticed in actual calculations.

ACKNOWLEDGMENTS

The author is deeply indebted to Professor S. Okubo and Professor Abdus Salam for much helpful advice. He thanks Professor Y. Ne'eman for a useful discussion. He also thanks Dr. A. Böhm and Dr. M. P. Khanna for discussions and criticisms. He is grateful to the IAEA, to Professor Abdus Salam, and to Professor P. Budini for hospitality at the International Centre for Theoretical Physics, Trieste. Finally, he thanks UNESCO for financial support.

[9] C. Dullemond, Ann. Phys. (N.Y.) 33, 214 (1965).