Some Implications of the Bootstrap Model for the Electromagnetic Properties of Baryons

G. BARTON AND D. DARE
University of Sussex, Brighton, Sussex, England
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If baryons are baryon-meson bound states, their electromagnetic (EM) mass splittings result from (i) long-range EM forces between their constituents; (ii) EM mass splittings of their constituents; (iii) charge-dependent corrections (short-range) to the strong forces. The model implies many self-consistency conditions, some of which we exploit to estimate the total baryon magnetic moments in terms of the orbital g values of the mesons, and the baryon EM splittings in terms of Coulomb and magnetic energies and of the observed meson EM splittings. The baryons are treated as static, and effects of type (iii) are assumed not to be crucial. Though our assumptions may constitute a poor model of reality, the results of the model follow immediately from its physics, and should be a reliable qualitative guide to the output of correct but more sophisticated calculations based on similar physical ideas. Comparison with experiment shows that it is essential to include strange particles even when considering the nucleons, and that the motion is not essentially relativistic. Agreement is satisfactory for splittings and for the isovector moment, and less so for the isoscalar moments. We make some comments on alternative approaches to these problems.

1. INTRODUCTION

THREE new ideas have recently been applied to the old problem of the electromagnetic (EM) mass splittings whose classical representative is the neutron-proton mass difference[1] δN. First, that it is not enough to include only the directly obvious EM effects, i.e., the direct differences in EM field energies, the so-called "driving" terms; but that one should take into account also indirect effects such as, for instance, the influence of the EM mass splittings themselves on the self-energies due to the strong interactions. We shall call such effects "feedback." Because of the strong interactions, this idea entails the second, that the splittings of the various isomultiplets should not be considered in isolation from each other. Third, that the observed particles might not be elementary, but, in the spirit of bootstrap theory, bound states of each other[2]; then their splittings are due partly to those of their constituents, and partly to EM contributions to the binding forces.

The first two ideas draw strength from the variety of the observed signs and magnitudes of the splittings, combined with the validity of the Coleman-Glashow formula[3] deduced from $SU(3)$ symmetry: $\delta\Sigma=(\delta N+\delta\Xi)$ (for definitions see Sec. 4.3). Unless this is an accident, a calculation even of δN is suspect if it ignores strange particles. The most recent exploration taking extensive account of $SU(3)$ is that of Wojtaszek, Marshak, and Riazuddin.[4] The first idea by itself has been applied to δN by Pagels.[5] To avoid confusion one should bear in mind that the first two ideas are independent of the third, which is due to Dashen and Frautschi[6]; its vicissitudes have motivated the present paper. Here we shall be concerned only with the bound-state model; although it works with the same ingredients as the elementary-particle approach, we emphasize that it can combine them very differently.

It is convenient to distinguish from the outset between long-range and short-range effects. Being carried directly by the EM field, all the driving terms are basically long range; as such we can handle them with some confidence even when the solution of the underlying strong-binding problem is only assumed to exist without being available in detail. Here we are helped further by experimental information about the EM form factors. By contrast, there are feedback terms both of short and of long range. If the baryons are baryon-meson bound states, then the EM mass shifts of their actual constituents could be regarded as long-range effects and are fairly easy to take into account. On the other hand, once the driving terms upset charge independence, the strong forces themselves suffer charge-dependent corrections, which are short range. To visualize this we construct, as a useful schematic analog, a Schrödinger equation with an energy- (i.e., eigenvalue-)

[1] A fourth clue, less of an explanation than a consistency check, lies in the remarkable work of N. Cottingham [Ann. Phys. (N. Y.) **25**, 424 (1963)]. He expresses δN in terms of certain integrals over the observed electron-nucleon scattering cross sections. His representation holds independently of the detailed origins of δN; but it should not yield the true result until it incorporates data at energies high enough to produce, in the laboratory, the intermediate states that are in fact responsible for δN. With present data the integrals appear to converge well, but to a wrong (negative) value; this is legitimate cause for pessimism about most currently popular approaches, including those considered in the present paper.

[2] Here we shall not consider the possibility that they are made of quarks. If they are, the problem merely transfers to the quarks, and one has to explain the splitting of the quark isodoublet. Cottingham's results (Ref. 1) are compatible with the quark model. The evidence has been analyzed by us from this point of view, Nuovo Cimento (to be published).

[3] S. Coleman and S. L. Glashow, Phys. Rev. Letters **6**, 423 (1961).

[4] J. H. Wojtaszek, R. E. Marshak, and Riazuddin, Phys. Rev. **136**, B1053 (1964).

[5] H. R. Pagels, Phys. Rev. **144**, 1261 (1966). See also H. M. Fried and T. N. Truong, Phys. Rev. Letters **16**, 559 (1966); Phys. Rev. (to be published).

[6] R. F. Dashen and S. C. Frautschi, Phys. Rev. **135**, B1190 (1964).

dependent potential; this dependence reflects the usual self-consistency requirement on the bootstrap model. The unperturbed equation is

$$[-\nabla^2+V(E,r)]\psi(r)=E\psi(r). \tag{1.1}$$

(For the purpose in hand we ignore the question of the reduced mass.) Now add to V a perturbation $\delta V(r)$ representing the driving terms and therefore independent of the eigenvalue, which changes from E to $E+\delta E$. Then it follows straightforwardly that to first order δE is given by

$$\delta E=\langle\psi|\left[\delta V(r)+\frac{\partial V(E,r)}{\partial E}\delta E\right]|\psi\rangle,$$
$$\delta E=\frac{\langle\psi|\delta V|\psi\rangle}{[1-\langle\psi|\partial V(E,r)/\partial E|\psi\rangle]}. \tag{1.2}$$

The denominator in (1.2) embodies the short-range feedback, since the range of $\partial V/\partial E$ is comparable to that of V: for instance, when baryon exchange contributes to V, the baryon EM mass shifts and coupling constant corrections contribute to $\partial V/\partial E$. In the following we shall use the language of this model wherever it does not risk confusion.[7]

The short-range effects are demonstrably difficult to estimate; to do so reliably, one needs to know not only the "perturbation" $\partial V/\partial E$, but also the wave function at small distances. There have been several papers on this problem,[8–12] particularly in the pion-nucleon system; not only the magnitude, but even the sign of $\langle\partial V/\partial E\rangle$ still seem to be controversial. Perhaps this is not surprising, since no calculation as yet has managed simultaneously both to produce the nucleon as a pion-nucleon bound state and to account in any realistic manner for the ($T=\frac{1}{2}$, $P_{1/2}$) phase shift.[13,14] By contrast, the driving term in the numerator of (1.2) is easy to estimate if $\delta V(r)$ varies slowly compared to ψ; we shall see that in fact it is not very far from $\delta V(0)$.

A further distinction to keep in mind is that between, as we shall call them, sign reversals of the driving and the feedback type. To see what is implied, take for reference the classical expectation, based on the Coulomb energy, that in an isomultiplet the charged member is heavier than the neutral. By sign reversal we mean that in some multiplets the opposite is observed. For example, thinking of the proton as an *elementary* particle with spin and Pauli moment, it is conceivable *a priori* that the Coulomb repulsion could be overcompensated by the attraction between the parallel current loops resulting from the rotation of a more or less homogeneous charge distribution.[15] If this were to happen we should call it a sign reversal of the driving type. Whether it does happen depends on the spatial distribution of charge and magnetization; in fact the observed form factors show that the Coulomb energy easily dominates and there is no such reversal.[16,17]

In principle it is also possible that $\langle\partial V/\partial E\rangle$ is positive and exceeds unity; if so, then (1.2) shows that δE and $\langle\delta V\rangle$ have opposite signs, contrary to expectations based on ordinary Schrödinger theory. We should call this a sign-reversal of the (short-range) feedback type; long-range feedback can of course help or hinder it. In various contexts, Pagels and Fried and Truong,[5] and Goldberg,[11] have suggested that this mechanism does or might operate. In our view, feedback-type reversal is implausible for elementary particles. Feedback would disappear in the logically not impossible limit in which the strong couplings are switched off; as their strength increases from zero, the ratio $\delta E/\langle\delta V\rangle$ could change sign, in this way, only by becoming infinite at some stage, which seems repugnant on physical grounds. On the other hand, in the bootstrap model the self-consistency requirement makes it nonsense to vary the coupling strengths, and this argument against short-range feedback-type reversal does not apply. The differences between the two types are further illustrated just below Eq. (4.14).

Apart from the attempt to disentangle the general issues discussed above, our aim is to explore in a qualitative but physically reasonable manner those implications of the bootstrap model for the baryon octet which do not depend on short-range feedback. By "physically reasonable" we mean that the signs and orders of magnitude of specific effects should be correctly and clearly linked to specific causes. We ignore the short-range effects simply because we know of no convincing way to include them. Our excuse for proceeding at all is that even this oversimplified situation has sometimes been mishandled in more sophisticated calculations. Hence it might be useful to have a model, if only a partial one, whose mathematics are transparent enough for the physics to show, so that more realistic and therefore much more complicated calculations can be charted by comparison. Correspondingly we shall try to be careful in stating the physics; this must be our apology for the relatively large number of words.

For the reasons stated we take seriously only the $SU(3)$-symmetric case, incorporating the baryon octet

[7] Even on the elementary-particle picture, $\partial V/\partial E$ has an analog in the charge-dependent corrections to the strong self-energies (Refs. 4,5).

[8] R. F. Dashen and S. C. Frautschi, Phys. Rev. **137**, B1318 (1965); **137**, B1331 (1965).

[9] G. L. Shaw and D. Y. Wong, Phys. Rev. **147**, 1028 (1966).

[10] R. F. Sawyer, Phys. Rev. **142**, 991 (1966).

[11] H. Goldberg, Cornell reports, 1966 (unpublished).

[12] J. E. Paton, Nuovo Cimento **43**, A100 (1966).

[13] C. Lovelace, Proc. Roy. Soc. (London) **A289**, 547 (1966).

[14] C. Peyrou, in *Proceedings of the Oxford International Conference on Elementary Particles, 1965* (Rutherford High Energy Laboratory, Harwell, England, 1966).

[15] R. P. Feynman and G. Speisman, Phys. Rev. **94**, 500 (1954); K. Huang, *ibid.* **101**, 1173 (1956).

[16] M. Cini, E. Ferrari, and R. Gatto, Phys. Rev. Letters **2**, 7 (1954).

[17] In Cottingham's expressions (Ref. 1) these terms make up the "elastic contributions."

B and the pseudoscalar octet P which would seem to be a minimum set. Moreover we accept the meson masses from experiment and include them with the driving terms; thus at best we deal with only one-half of the full problem of the EM mass shifts.[18]

The dynamical model is discussed in Sec. 2, where we pretend for simplicity that only pions and nucleons exist, reducing $SU(3)$ to the isotopic $SU(2)$ symmetry. One and the same elementary method yields both the magnetic moments and the EM mass splittings; in fact, the moments guide us in further specifying the model by rejecting certain relativistic complications that one might have expected *a priori*. The driving terms, single-photon exchange between baryon and meson, are discussed in detail in Appendix A. The $SU(2)$ model is easily stretched to include any number of N^*'s as well as the N; Sec. 3 and Appendix B consider this extension in detail. All the $SU(2)$ results are equally unsatisfactory in yielding the wrong sign of δN; this is expected, and merely underlines the need to include strange particles. Section 4 gives the results of the $SU(3)$ calculations, and Sec. 5 contains comments and conclusions.

2. THE MODEL

For simplicity we ignore strange particles in this section and consider only pions and nucleons.

Even granted that the nucleon is a pion-nucleon bound state, there remains the problem of using this idea to calculate the EM properties. The original attempt to do so[6,19] used the N/D approach via the partial-wave amplitude A, where the nucleon is identified with a pole of $A=N/D$ due to a zero of D, this pole being then shifted by EM effects. In principle, the driving term of such a calculation is the true change δA of A which results from switching on photon exchange between N and π. This method seems too laborious to be correctly implemented in practice, not primarily because of difficulties in ascertaining the unperturbed D function, but because it is not even qualitatively adequate[12,20,21] to replace δA by the single-photon-exchange amplitude itself. The trouble is aggravated by the difficulty of dealing correctly with the infrared-divergent expressions which enter at intermediate stages of the calculation.[20]

But the essential physics of the bootstrap hypothesis (apart from short-range feedback), can be incorporated into a simple model which suffers from none of the disadvantages of the more ambitious N/D approach; in particular it does not need to specify details of the strong interactions. What adds spice to the problem is that the nucleon is itself one of its own constituents; thus, one is led to exploit certain obvious self-consistency requirements.

We adopt the static (no-recoil) approximation for the nucleons; (later, when mesons as heavy as the K and the η enter, we shall re-examine its adequacy). Thus, the bound state contains a (physical) pion orbiting around a stationary (physical) nucleon in a $P_{1/2}$ state; we symbolize such a system by $|N\pi\rangle$. Since by definition its properties are those of the nucleon, we write boldly

$$\begin{aligned}|p\rangle&=-\sqrt{\tfrac{1}{3}}\,|p\pi^0\rangle+\sqrt{\tfrac{2}{3}}\,|n\pi^+\rangle,\\ |n\rangle&=-\sqrt{\tfrac{2}{3}}\,|p\pi^-\rangle+\sqrt{\tfrac{1}{3}}\,|n\pi^0\rangle.\end{aligned}\tag{2.1}$$

Within the framework of the usual vector spaces of quantum mechanics such an equation is, of course, meaningless. The meaning that tentatively we do attach to it derives from the bootstrap hypothesis: The matrix elements of observables taken between two states both appearing on the left are to equal the matrix elements between the corresponding states, both appearing on the right. No meaning is attached to "mixed" matrix elements in which the bra, say, is taken from the left and the ket from the right. As we shall see below, in practice the procedure for exploiting (2.1) is quite clear.

The isotopic Clebsch-Gordan coefficients on the right of (2.1) are dictated for members of an isodoublet. Thus, everything about the vectors on the right is determined except the radial pion wave function, which depends specifically on the strong binding forces.

The radial uncertainty entails another, whether in this context the pion should be treated as relativistic. Though (2.1) clearly implies that the pion's binding energy equals its rest mass, this does not bear directly on its speed; by the uncertainty principle the latter can be low if the interaction region is wide enough. The procedure obvious at first sight might be to subject the pion to a Klein-Gordon equation with the unknown strong potential treated either as a Lorentz scalar or as the time-component of a four-vector; this was the viewpoint taken in Ref. 20. But in fact the customary method uses a dispersion relation in the total energy, and its potential-theory analog is in some important respects closer to an ordinary Schrödinger equation, especially with respect to the treatment of perturbations. To see this, let ω be the pion energy, and note that although from the viewpoint of the left-hand cut the value $\omega=0$ is a distinguished one, as it is in the Klein-Gordon equation, from the viewpoint of the possible location of the bound-state pole it is not. In other words an N/D calculation with appropriate forces can equally easily yield a bound state whose mass m_b is less than the nucleon mass m_N, or greater. This

[18] For this we make no apology: We can just about conceive of a reasonable dynamical calculation in the near future which might yield the baryons as baryon-meson bound states, but not at all of one yielding the mesons as baryon-antibaryon systems.

[19] R. F. Dashen, Phys. Rev. **135**, B1196 (1964).

[20] G. Barton, Phys. Rev. **146**, 1149 (1966). The inadequacy, in general, of the first approximation in the Dashen-Frautschi method (independently of other, infrared-divergent, difficulties) shows up in Eq. (2.9) of this paper and in the subsequent discussion.

[21] Y. S. Kim, Phys. Rev. **142**, 1150 (1966); Y. S. Kim and K. V. Vasavada, this issue, Phys. Rev. **150**, 1236 (1966).

property is shared in a natural way by the Schrödinger but not as readily by the Klein-Gordon equation. There may be genuine physical difficulties that arise if $m_b < m_N$, but they are disregarded by the usual N/D equation, and we ignore them now because we want to make our model as similar as possible to the usual N/D procedure. Moreover, the Klein-Gordon equation involves unavoidable and in this context probably unphysical ambiguities as to the magnetic effects[20] (both moments and energies). For all these reasons we use a nonrelativistic model, and shall merely note where its predictions deviate from relativistic expectations. In every case, comparison with experiment favors the nonrelativistic alternative.

We are now in a position to exploit the model, and start with the magnetic moments. The moment operator is

$$\boldsymbol{\mu}(\mathrm{op}) = \tfrac{1}{2}(\mu_S + \tau_0\mu_V)\boldsymbol{\sigma} + (e/2m_\pi)T_0\mathbf{L}, \tag{2.2}$$

where μ_S and μ_V, by definition, are the total isoscalar and isovector moments which are to be calculated (the observed values are also quoted):

$$\mu_S = (\mu_p + \mu_n) = 0.88\frac{e}{2m_N}; \quad \mu_V = (\mu_p - \mu_n) = 4.70\frac{e}{2m_N}; \tag{2.3}$$

$\boldsymbol{\tau}/2$ and $\mathbf{T}$ are the isotopic spin operators for nucleon and pion, $\boldsymbol{\sigma}$ the Pauli matrix for the nucleon spin, and $\mathbf{L}$ the orbital angular momentum. (Note explicitly that exchange moments are being ignored.) We use natural units, $\hbar = 1 = c$.

First we take the expectation value of $\mu_0(\mathrm{op})$ for a spin-up nucleon as represented by the vectors on the left of (2.1), which yields μ_p and μ_n by definition; then with the state vectors on the right, using standard vector addition for the $P_{1/2}$ state. Finally we equate the results, finding

$$\begin{aligned} \mu_p &= \{\tfrac{1}{3}[-\tfrac{1}{3}\mu_p] + \tfrac{2}{3}[-\tfrac{1}{3}\mu_n + \tfrac{1}{3}(e/m_\pi)]\}, \\ \mu_n &= \{\tfrac{2}{3}[-\tfrac{1}{3}\mu_p - \tfrac{1}{3}(e/m_\pi)] + \tfrac{1}{3}[-\tfrac{1}{3}\mu_n]\}. \end{aligned} \tag{2.4}$$

These are just two simultaneous linear equations for μ_p and μ_n, the inhomogeneous terms being proportional the pion's orbital g factor. The solutions are

$$\mu_S = 0, \quad \mu_V = e/2m_\pi = 6.7(e/2m_N), \tag{2.5}$$

to be compared with (2.3).

Note the following points: (i) The nucleon mass disappears completely from the equations once we can neglect m_π/m_N; the natural unit for expressing the total moment is $e/2m_\pi$. (ii) The unwelcome results $\mu_S = 0$ reflects the pure isovector nature of the pion current. (iii) If the motion were treated by the Klein-Gordon equation, the orbital g values would be reduced.[22,23] A similar result is known in the relativistic corrections to the Zeeman effect. The measured value clearly cannot tolerate any great reduction.

To obtain the EM mass splittings, we take the expectation value of the EM perturbation to second order in e. Using the vectors on the left of (2.1) we obtain, by definition, the EM selfmasses δp and δn. Using the vectors on the right, we get the following types of contribution: (i) from δp and δn; (ii) from the pion EM self-energies $\delta\pi^+ = \delta\pi^-$ and $\delta\pi^0$; and (iii) from the direct EM forces between nucleon and pion.

In common with all other workers, we assume that single-photon exchange dominates the EM force (iii). The details are considered in Appendix A; they depend on the form factors. The important difference between the first approximation here and in the N/D perturbation method is that we take photon exchange to determine only the perturbing *potential*, while in the N/D method an attempt was made to substitute it for the change δA of the *amplitude* itself.

The direct EM forces are of two kinds. The Coulomb (electrostatic) interaction acts only between the p and π^- in the first component of the neutron; being attractive it makes a negative contribution $-C<0$ which is defined by

$$\langle p\pi^- |\, (\text{electrostatic potential})\, | p\pi^- \rangle = -C. \tag{2.6}$$

The potential here would be $-\alpha/r$ if the form factors were unity.

There is also a hyperfine-structure (hfs)-type coupling[23] between the magnetic field of the orbiting charged pions and the moments of the nucleons, better thought of as a current-current coupling. Since in a $P_{1/2}$ state spin and orbital angular momentum are antiparallel, the associated current loops are parallel, so that the magnetic force is attractive both in the $|p\pi^-\rangle$ component of the neutron and in the $|n\pi^+\rangle$ component of the proton. We define, with $M>0$,

$$\begin{aligned} \langle p\pi^- |\, (\text{hfs potential})\, | p\pi^- \rangle &= -e\mu_p M/m_\pi, \\ \langle n\pi^+ |\, (\text{hfs potential})\, | n\pi^+ \rangle &= +e\mu_n M/m_\pi. \end{aligned} \tag{2.7}$$

If the form factors were unity, the hfs potential would be $(\mu e/m_\pi)(\boldsymbol{\sigma}\cdot\mathbf{L})r^{-3}$.

Equating the expectation values of the order e^2 perturbation, we find

$$\delta p = \{\tfrac{1}{3}[\delta p + \delta\pi^0] + \tfrac{2}{3}[\delta n + \delta\pi^+ + e\mu_n M/m_\pi]\}, \tag{2.8}$$

$$\delta n = \{\tfrac{2}{3}[\delta p + \delta\pi^+ - C - e\mu_p M/m_\pi] + \tfrac{1}{3}[\delta n + \delta\pi^0]\}. \tag{2.9}$$

We are interested only in the difference $\delta N = (\delta n - \delta p) = (n-p)$, whose observed value is $+1.3$ MeV. Subtracting (2.8) from (2.9) we get

$$\delta N = \{-\tfrac{1}{3}\delta N - \tfrac{2}{3}C - \tfrac{2}{3}e(\mu_p + \mu_n)M/m_\pi\}, \tag{2.10}$$

$$\delta N = \{-\tfrac{1}{2}C - \tfrac{1}{2}e\mu_S M/m_\pi\}. \tag{2.11}$$

[22] This result is well known in the relativistic corrections to the Zeeman effect (Ref. 23). It is commented on in Ref. 20.

[23] See for instance H. A. Bethe and E. E. Salpeter, *Quantum Mechanics of One- and Two-Electron Atoms* (Springer-Verlag, Berlin, 1957).

The right-hand side of (2.11) is unavoidably negative; this merely shows that, as anticipated, the $SU(2)$ symmetry is too simple. Of course one should ask whether it could be salvaged by including, besides N, other $SU(2)$ particles[24] such as the N^*'s. What one has in mind is to calculate both the N and the N^* splittings, not to infer the one from the observed value of the other. This extension is considered in Sec. 3, where we find that in the absence of short-range feedback-type reversal not even the extended $SU(2)$-symmetric model can reproduce the observed sign of δN.

The driving terms C and M are considered in Appendix A, where we find that the magnetic contributions are roughly a 20% effect, essentially because μ_S is fairly small. With a realistic attitude to the level of accuracy of the model we neglect this, and adopt the value

$$C \approx 1.9 \text{ MeV}, \tag{2.12}$$

which, by way of compensation, is a slight overestimate. In the $SU(3)$-symmetric case we shall need to reconsider the magnetic effects.

The cancellation of the boson mass difference from (2.10) is peculiar to $SU(2)$; it occurs because $\delta\pi = (\pi^+ - \pi^0)$ is a second-rank isotensor while δN is an isovector. Similarly, the reason why only μ_S occurs in δN is that the pion current is pure isovector; hence, only that part of photon exchange violates charge independence in which the photon couples to the isoscalar component of the nucleon current.[19]

The way in which magnetic and Coulomb energies combine illustrates vividly the differences between the elementary-particle and the bound-state approach. In the former the two energies have opposite signs; in the latter they have the same sign. In both cases the Coulomb energy tends to make δN negative, on the elementary particle picture through the electrostatic *repulsion* between the different parts of the *proton*, and on the bound-state picture through the *attraction* between the p and π^- constituents of the *neutron*. By contrast, on the elementary-particle picture the magnetic energy tends to make the *proton lighter* through the attraction between its predominantly parallel current loops[15]; but on the bound-state picture it makes the *neutron lighter* through the attraction between the parallel currents resulting from the antiparallel rotations of its two oppositely charged constituents p and π^-.

3. DIGRESSION ON THE EXTENDED $SU(2)$-SYMMETRIC MODEL

3.1 Introduction

It is a commonplace of πN bootstrap attempts that N and N^* exchange forces are of comparable importance; this suggests that N and N^* may be of comparable importance also in the direct channel. In other words, one should not neglect the possibility that N is a bound state just as much of N^* and π as it is of N and π; the fact that the $T=\frac{1}{2}$ partial wave in $N\pi$ scattering is very strongly inelastic[13,14] lends some support to this hypothesis.[25] Correspondingly the N^* may contain $|N^*\pi\rangle$ as well as $|N\pi\rangle$. In our model we represent such a situation by writing

$$|N\rangle = \alpha |N\pi(N)\rangle + (1-\alpha^2)^{1/2} |N^*\pi(N)\rangle, \tag{3.1}$$

$$|N^*\rangle = (1-\beta^2)^{1/2} |N\pi(N^*)\rangle + \beta |N^*\pi(N^*)\rangle. \tag{3.2}$$

Here, $|N\pi(N)\rangle$, for instance, represents the states (2.1); the labels in brackets show the quantum numbers (including isospin) to which the constituents are coupled. The coefficients α and β depend on the dynamics; from our viewpoint they are at this stage adjustable parameters.

All components are P states, but differ in their radial wave functions; for instance in (3.1) and (3.2) the first component of (3.2) is a continuum state, and all others are bound. Moreover one knows from analyses of the phase shifts[13,26] that the strong forces in the N and N^* channels are quite different. Therefore the Coulomb expectation values C will be different. But because the electrostatic potential has long range, and is likely to be smooth even at small distances (see Appendix A), we expect all the C's to be fairly close. Since we are looking only for qualitative conclusions we shall adopt a common value of C for all components. (For instance, we take $-2C$ as the Coulomb energy of the $N^{*++}\pi^-$ component.) Certainly they can differ neither in sign nor in order of magnitude; and the consequences of taking them equal are so clearcut that only incredibly drastic changes in the C's could upset them.

3.2 Magnetic Moments

Using the methods of Sec. 2 the moments are got from (3.1) and (3.2) by a straightforward but laborious calculation[27] making repeated use of the Wigner-Eckart theorem. The isoscalar moments again vanish, and one obtains simultaneous equations for the three isovector quantities, namely the N and N^* isovector moments μ_V and ν, and the $N \to N^*$ transition moment λ. These are defined by (2.3) and by

$$\langle N^*, t_3, s_3 = \tfrac{3}{2} | \mu_0(\text{op}) | N^*, t_3, s_3 = \tfrac{3}{2} \rangle = \tfrac{3}{2} \nu t_3, \tag{3.3}$$

$$\langle N^*, t_3, s_3 | \mu_0(\text{op}) | N, t_3, s_3 \rangle = \lambda, \tag{3.4}$$

where t_3 and s_3 are the third components of isospin and

[24] D. Morgan (private communication).

[25] E. J. Squires (private communication).

[26] J. Hamilton, in *Proceedings of the Seminar on Unified Field Theories* (Max-Planck-Institut, Munich, 1965). See also A. Donnachie and J. Hamilton, Ann. Phys. (N. Y.) **31**, 410 (1965); and Phys. Rev. **138**, B678 (1965).

[27] The results have been obtained independently by M. R. Wallace, University of Sussex, M.Sc. thesis, 1966 (unpublished).

TABLE I. The coefficients in Eq. (3.6).

$i \backslash$	A_{ij} $j=1$	2	3	B_i
1	$9-\alpha^2$	$-16\alpha(1-\alpha^2)^{1/2}$	$-(25/3)(1-\alpha^2)$	$2+6\alpha^2$
2	$-4\alpha(1-\beta^2)^{1/2}$	$9-\frac{1}{2}(1-\alpha^2)^{1/2}(1-\beta^2)^{1/2}-5\alpha\beta$	$-(10/3)\beta(1-\alpha^2)^{1/2}$	$2\alpha(1-\beta^2)^{1/2}+5\beta(1-\alpha^2)^{1/2}$
3	$-(1-\beta^2)$	$-(8/5)\beta(1-\beta^2)^{1/2}$	$3-(121/75)\beta^2$	$2-(42/25)\beta^2$

spin. Writing

$$\mu_V=\kappa_1,\quad \lambda=\kappa_2,\quad \nu=\kappa_3, \tag{3.5}$$

the equations are

$$A_{ij}\kappa_j=(e/2m_\pi)B_i; \tag{3.6}$$

the dummy-suffix convention is used. The coefficients A_{ij} and B_j are given in Table I. Experimentally, ν is unknown and[28]

$$\lambda_{\exp}=1.18\mu_p. \tag{3.7}$$

A numerical exploration of the solutions of (3.6) with a liberal variation of α and β reveals that λ and μ_V do not, simultaneously, assume their experimental values. But of course the most serious failure is the prediction $\mu_S=0$.

3.3 Mass Splittings

Since N^* has $T=\frac{3}{2}$, its EM splitting contains an isotensor component which is affected by $\delta\pi$. The isotensor splittings are easily disentangled from the isovector ones since the former depend only on the magnitude but not on the sign of t_3. We concern ourselves only with the vector splittings, i.e., with two independent quantities. By applying our method to (3.1) and (3.2) one easily finds

$$\delta N=N^{*0}-N^{*+}=\tfrac{1}{3}(N^{*-}-N^{*++})=-C/2, \tag{3.8}$$

independently of α and β. In the absence of strange particles the result (3.8) depends only on the assumptions that short-range feedback is negligible and that all the C's are equal; it remains true even when one introduces any number of N^* resonances with arbitrary values of the coefficients of the type α and β. Appendix B proves this general statement. Though the constancy of the C's is admittedly only an approximation, we think this result is acceptable evidence for the following statement:

If strange particles are excluded from the calculation, the observed sign of δN can be reproduced by theory only if there is a sign-reversal due to short-range feedback.

[28] It has been argued that λ should be compared not to μ_V, but to the value of the isovector nucleon magnetic form factor at a momentum-transfer value corresponding to that actually needed in the $N\to N^*$ photoproduction. Then the experimental ratio of λ to μ_V is even greater, and the inability of the theory to reproduce is more striking [R. H. Dalitz and D. G. Sutherland, Phys. Rev. **146**, 1180 (1966)].

4. THE $SU(3)$-SYMMETRIC MODEL

4.1 Introduction

Formally, the basic equation (2.1) is generalized[29] to $SU(3)$ simply by coupling the baryons B and the pseudoscalars P to an octet $|B\rangle$. This may be a mixture of the symmetric (D-type) and antisymmetric (F-type) states; hence, we introduce a parameter α which we treat as adjustable and define so that the $|B\rangle$'s are proportional to

$$(1+\alpha)|BP\rangle_D+(3/\sqrt{5})(1-\alpha)|BP\rangle_F, \tag{4.1}$$

where $|BP\rangle_{D,F}$ are normalized to unity and the factor $3/\sqrt{5}$ is inserted for convenience in the numerical work. For instance,[30]

$$|p\rangle=N(\alpha)\left\{\left[\frac{1}{\sqrt{2}}|p\pi^0\rangle-|n\pi^+\rangle\right]+\frac{(1-2\alpha)}{\sqrt{6}}|p\eta\rangle -\frac{(2-\alpha)}{\sqrt{6}}|\Lambda K^+\rangle+\left[\frac{\alpha}{\sqrt{2}}|\Sigma^0K^+\rangle-\alpha|\Sigma^+K^0\rangle\right]\right\}, \tag{4.2}$$

$$|n\rangle=N(\alpha)\left\{\left[|p\pi^-\rangle-\frac{1}{\sqrt{2}}|n\pi^0\rangle\right]+\frac{(1-2\alpha)}{\sqrt{6}}|n\eta\rangle -\frac{(2-\alpha)}{\sqrt{6}}|\Lambda K^0\rangle+\left[-\frac{\alpha}{\sqrt{2}}|\Sigma^0K^0\rangle+\alpha|\Sigma^-K^+\rangle\right]\right\}, \tag{4.3}$$

where $N(\alpha)$ is a normalization constant,

$$N(\alpha)=[3/(7-4\alpha+7\alpha^2)]^{1/2}. \tag{4.4}$$

But in view of the observed $SU(3)$-violating "medium-strong" (MS) mass differences within B and P, one must consider carefully to what extent this generalization is valid.

As regards the magnetic moments, $SU(3)$ violation cannot be neglected even if the wave functions on the right of (4.2) and (4.3), etc., are undisturbed by MS effects. The inhomogeneous terms in the equations for the moments involve the meson orbital g values $e/2m_\pi$ and $e/2m_K$, whose ratio differs by the factor $m_K/m_\pi=3.54$ from its $SU(3)$ value of 1. The isovector moments draw contributions both from π's and K's, and

[29] See for instance M. Gell-Mann and Y. Ne'eman, *The Eightfold Way* (W. A. Benjamin, Inc., New York, 1964).

[30] We use the phase conventions, and the tables, of P. McNamee and F. Chilton, Rev. Mod. Phys. **36**, 1005 (1964).

a priori any $SU(3)$ relations between them are worthless. The isoscalar moments depend on $1/2m_K$ alone since only the K current has an isoscalar component; thus, the $SU(3)$ relations involving them remain valid, but all isoscalar moments will turn out very much smaller than the isovector ones, and too small compared to experiment. This is a serious dilemma for the theory; an analogous problem is bound to occur, for the same reason $m_K \gg m_\pi$, on any picture of the dynamics, and we have no solution to offer. A second complication ensues because K and η are not light enough compared to the baryons to neglect the influence of baryon motion on the orbital g values. We have repeated all the moment calculations, allowing for this by replacing $e_P/2m_P$ with[23]

$$\frac{e_P m_B}{2m_P(m_B+m_P)}+\frac{e_B m_P}{2m_B(m_B+m_P)}, \tag{4.5}$$

where e_P, e_B are the charges and m_P, m_B the masses of the meson and the baryon in each component. It turns out that the change has no appreciable effect on the calculated moments; this can be seen from Figs. 1 and 2. In particular the isoscalar moments remain much too small.

Next, the MS mass differences could change the values of the coefficients on the right of (4.2), etc., or distort the radial wave functions differently in the several components, or both. We propose to ignore these complications, for two reasons. Empirically, the validity of the Coleman-Glashow rule[3] suggests either that the effects of $SU(3)$ impurities, separately, are very small, or that they conspire to cancel by some fantastic coincidence. Theoretically, the inner region contains most of the wave function because of the centrifugal barrier, and it is shielded by the barrier from the effects of the MS splittings. Dalitz has shown that even for the decuplet the shielding is very effective[31]; in the deeper lying octet it must be more so.

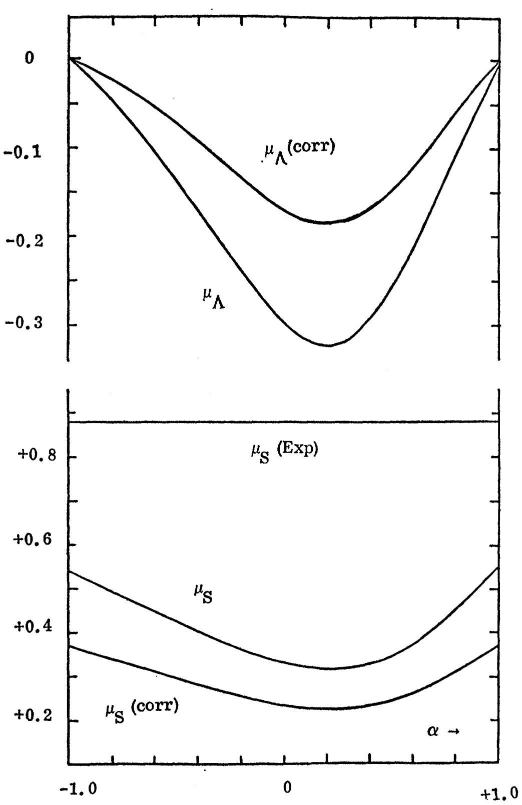

FIG. 1. The isoscalar moments in nucleon magnetons. The "corrected" curves result from the g value of Eq. (4.5).

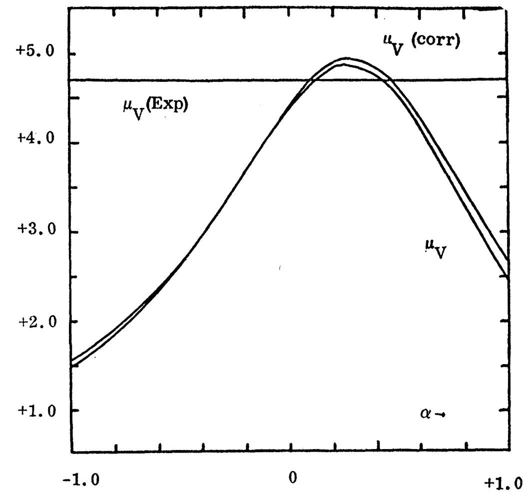

FIG. 2. The nucleon isovector moment in nucleon magnetons. The "corrected" curve results from the g value of Eq. (4.5).

As regards the EM splittings, we shall therefore retain a common value of C for all components; but we must reconsider whether the magnetic energies can still be neglected. In $SU(2)$ symmetry this was just possible, but only because the observed value of μ_S is fairly small. Now the contributions to $\langle \delta V_M \rangle$ of the component states containing an orbiting K meson carry a factor m_K^{-1} instead of m_π^{-1}, whence we need not worry about them. Moreover that part of the magnetic isovector splitting which involves m_π^{-1} depends only on the isoscalar moments, and, experimentally, μ_Λ as well as μ_S seems fairly small. This leaves only the isotensor Σ splitting, whose magnetic part combines the (large) isovector moments with an orbiting π, and is not negligible. We shall correct for this; it turns out eventually that the correction is quantitively but not qualitatively important. (See Fig. 4.)

4.2 Magnetic Moments

Proceeding exactly as in Sec. 2, we obtain two sets of simultaneous equations, one set for the isoscalar and

[31] R. H. Dalitz, Proc. Roy. Soc. (London) **A288**, 183 (1965). By contrast, Hamilton (Ref. 26) has stressed the difficulties of reconciling the validity of pure $SU(3)$ predictions with the dynamics, when the MS mass differences are taken into account.

TABLE II. The coefficients in Eq. (4.11). $L(\alpha)=(7-4\alpha+7\alpha^2)$.

	G_{ij}				H_i	J_i
$i \backslash$	$j=1$	2	3	4		
1	$L+\frac{1}{6}(1-2\alpha)^2-\frac{1}{2}$	α^2	$-\frac{1}{3}\alpha(2-\alpha)$	0	$\frac{1}{6}(2-\alpha)^2-\frac{1}{2}\alpha^2$	2
2	α^2	$L+\frac{1}{2}(1-\alpha)^2+\frac{1}{6}(1+\alpha)^2$	$-\frac{1}{3}(1-\alpha^2)$	1	$(1+\alpha^2)$	$(1-\alpha)^2+\frac{1}{3}(1+\alpha)^2$
3	$-\frac{1}{3}\alpha(2-\alpha)$	$-\frac{1}{3}(1-\alpha^2)$	$\frac{1}{3}L$	$-\frac{1}{3}(1-2\alpha)$	$\frac{1}{3}(1-\alpha^2)$	$\frac{2}{3}(1-\alpha^2)$
4	0	1	$-\frac{1}{3}(1-2\alpha)$	$L+\frac{1}{6}(\alpha-2)^2-\frac{1}{2}\alpha^2$	$-\frac{1}{2}+\frac{1}{6}(1-2\alpha)^2$	$2\alpha^2$

another for the isovector moments; the latter include the $\Sigma^0\Lambda$ transition moment $\mu(\Sigma\Lambda)$. As explained in Sec. 4.1, the $SU(3)$ relations among the isoscalar moments remain valid, and we use them to eliminate two of four. To do this, recall the standard $SU(3)$ result for the Pauli moments,[3,32] identified as such by primes:

$$\begin{aligned}
&\mu'(\Sigma^+)=\mu'(p), &&\mu'(\Lambda)=\mu'(n)/2,\\
&\mu'(\Sigma^0)=-\mu'(n)/2, &&\mu'(\Xi^0)=\mu'(n),\\
&\mu'(\Sigma^-)=-[\mu'(p)+\mu'(n)], &&\mu'(\Xi^-)=-[\mu'(p)+\mu'(n)],\\
&&&\mu'(\Sigma\Lambda)=-\sqrt{3}\mu'(n)/2.
\end{aligned}\tag{4.6}$$

Neglecting the relatively small effect of the MS baryon splittings, (4.6) leads to the following isoscalar relations between the total moments:

$$[\mu(\Sigma^+)+\mu(\Sigma^0)+\mu(\Sigma^-)]=-3\mu(\Lambda),\tag{4.7}$$

$$[\mu(\Xi^0)+\mu(\Xi^-)]=-[\mu(p)+\mu(n)]+2\mu(\Lambda).\tag{4.8}$$

With the definition (2.3) of μ_S, we are finally led to

$$\begin{bmatrix}(26-14\alpha+23\alpha^2) & 4(1-\alpha-2\alpha^2)\\ (1-\alpha^2) & 2(7-6\alpha+8\alpha^2)\end{bmatrix}\begin{bmatrix}\mu_S\\ \mu(\Lambda)\end{bmatrix}=\frac{e}{m_K}\begin{bmatrix}(2-2\alpha+5\alpha^2)\\ -(1-\alpha^2)\end{bmatrix}.\tag{4.9}$$

The solutions are plotted against α in Fig. 1. Experimentally, $\mu(\Lambda)=\{0; -0.6; -1.39; -1.5\}e/2m_N$, depending on one's choice of experiment.[33] The discrepancy with the data, due to the large K mass, is evident.

In the isovector equations no simplifications like (4.7) and (4.8) would be expected to remain valid for unequal π and K masses. Defining

$$\begin{aligned}
&[\mu(p)-\mu(n)]=\mu_V=\mu_1, &&2\sqrt{3}\mu(\Sigma\Lambda)=\mu_3,\\
&[\mu(\Sigma^+)-\mu(\Sigma^-)]=\mu_2, &&[\mu(\Xi^0)-\mu(\Xi^-)]=\mu_4,
\end{aligned}\tag{4.10}$$

one finds the equations

$$G_{ij}(\alpha)\mu_j=H_i(\alpha)/m_K+J_i(\alpha)/m_\pi;\tag{4.11}$$

the coefficients G_{ij}, H_i, J_i are given in Table II, and μ_V plotted against α in Fig. 2. (To our surprise, we found, after completing the calculations, that the rule $\mu_1-\mu_3-\mu_4=0$ does survive in spite of the π-K mass difference.)

The figures also show the result got with the orbital g value (4.5). The features anticipated in Sec. 4.1 are evident.

4.3 EM Mass Splittings

The method of Sec. 2 again applies straightforwardly. In order to keep in closer touch with the measured numbers, we prefer this method to decomposing the mass splitting into its $SU(3)$-irreducible tensor components.[8] We obtain simultaneous linear equations for the four isovector quantities defined by

$$\begin{aligned}
n-p&=\delta N=\delta_1=1.29,\\
\Sigma^--\Sigma^+&=\delta\Sigma=7.90\pm0.09,\\
\Xi^--\Xi^0&=\delta\Xi=\delta_2=6.5\pm1.0,\\
2\sqrt{3}\delta(\Sigma\Lambda)&=\delta_3,
\end{aligned}\tag{4.12}$$

where the numbers are the measured values in MeV. The equations satisfy, identically, the Coleman-Glashow rule[3] $\delta\Sigma=(\delta N+\delta\Xi)$, which we use to eliminate $\delta\Sigma$. [If magnetic energies had had to be included, this would no longer be possible in view of the $SU(3)$-violating isovector moments.] The driving terms contain C, for which we use the value 1.9 MeV (see Appendix A), and the two isovector boson masses $\delta K=(K^0-K^+)=3.9$ MeV and the $\eta\pi^0$ transition mass $\delta(\eta\pi)$. For the latter we adopt the $SU(3)$ relation

$$\delta(\eta\pi)=-(\delta K+\delta\pi)/\sqrt{3}\approx-5\text{ MeV}.\tag{4.13}$$

This step, taken faute de mieux, is somewhat controversial because of the MS η-π mass difference; $\delta(\eta\pi)$ depends on the value of the (necessarily common) mass for which it is evaluated,[34] and it is not really clear how this value should be chosen.

The resulting equations are

$$W_{ij}(\alpha)\delta_j=X_i(\alpha)C+Y_i(\alpha)\delta K+Z_i(\alpha)\delta(\eta\pi).\tag{4.14}$$

The coefficients W_{ij}, X_i, Y_i, and Z_i are given in Table III. The solutions for δN, $\delta\Sigma$, and $\delta\Xi$ are plotted against α in Fig. 3. We also show the result for δN when the boson driving terms are neglected (i.e., Y and Z set equal to zero), in order to illustrate the remarks made

[32] N. Cabibbo and R Gatto, Nuovo Cimento 21, 872 (1961).

[33] R. L. Cool *et al.* [Phys. Rev. 127, 2223 (1962)] find $\mu(\Lambda)=-1.5\pm0.5$; W. Kernan *et al.* [Phys. Rev. 129, 870 (1963)] find 0.0 ± 0.6; S. Anderson and F. S. Crawford [Phys. Rev. Letters 13, 167 (1964)] find -1.39 ± 0.72; P. Rosselet [Helv. Phys. Acta 38, 691 (1965)] finds -0.60 ± 0.34; all in units of $e/2m_N$.

[34] B. Barrett and G. Barton, Phys. Rev. 133, B466 (1964).

TABLE III. The coefficients in Eq. (4.14).

i	W_{ij} $j=1$	2	3	X_i	Y_i	Z_i
1	$(8-2\alpha+2\alpha^2)$	$-3\alpha^2$	$-\alpha(2-\alpha)$	$-3(1+\alpha^2)$	$(2-2\alpha-\alpha^2)$	$2\sqrt{3}(-1+2\alpha)$
2	-3	$(2-2\alpha+8\alpha^2)$	$-(1-2\alpha)$	$3(1+\alpha^2)$	$(1+2\alpha-2\alpha^2)$	$2\sqrt{3}\alpha(-2+\alpha)$
3	$(1+2\alpha-2\alpha^2)$	$(2-2\alpha-\alpha^2)$	$-\frac{1}{3}(7-4\alpha+7\alpha^2)$	$(1-4\alpha+\alpha^2)$	$(1-4\alpha+\alpha^2)$	0

in Sec. 5 below about a possible relativistic suppression of such effects.

The difference between possible feedback-type and driving-type sign reversals shows clearly in the structure of (4.14). Feedback-type reversal would result if $\det(W)$ were negative; driving-type reversal depends on the coefficients X_i, Y_i, and Z_i. With the long-range feedback used here, $\det(W)$ is positive; naturally we cannot rule out the possibility that with short-range feedback included it might change sign.

For the solitary isotensor splitting

$$\delta\Sigma^T=(\Sigma^+ + \Sigma^- - 2\Sigma^0)=1.8\pm0.1 \tag{4.15}$$

we obtain the formula

$$\delta\Sigma^T=\frac{[3C(3-4\alpha+3\alpha^2)-2\delta\pi(1-4\alpha+\alpha^2)]}{8(1-\alpha+\alpha^2)} \tag{4.16}$$

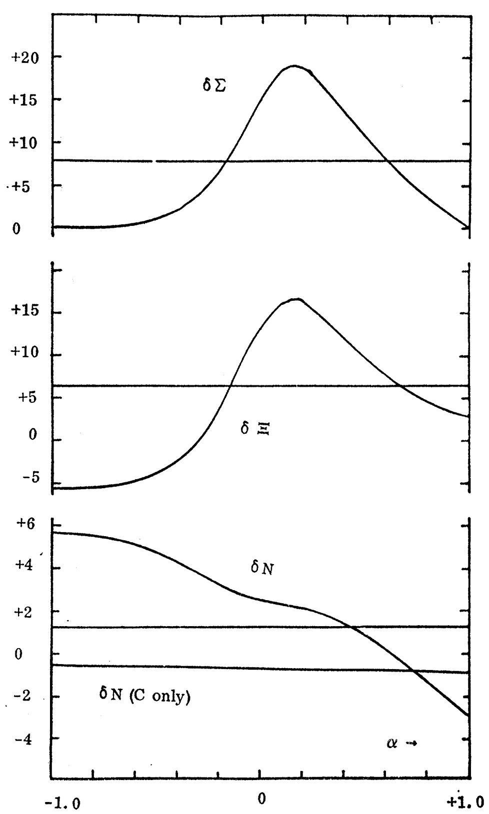

FIG. 3. The isovector splittings in MeV. The lower curve for δN results from neglecting the second and third terms on the right of Eq. (4.14). Horizontal lines show experimental values.

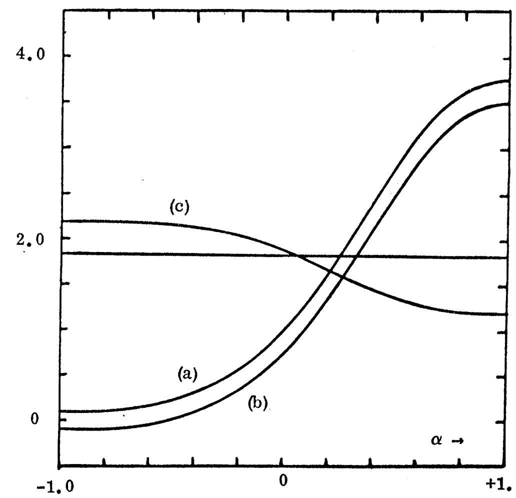

FIG. 4. $\delta\Sigma^T$ in MeV. (a): from Eq. (4.16) with $C=1.9$ MeV; (b): from Eq. (4.17) with $C=1.37$ MeV and $e\mu_pM/m_\pi=0.87$ MeV; (c): as in (b), but dropping $\delta\pi$. Horizontal line shows the experimental value.

if magnetic energies are neglected. With $C=1.9$ MeV, (4.16) is plotted in Fig. 4. Including the magnetic contributions, we find

$$\delta\Sigma^T=\Big\{3C(3-4\alpha+3\alpha^2)+3\frac{eM}{m_\pi}(\mu_{\Sigma^+}-\mu_{\Sigma^-})(1-\alpha)^2 + 3\frac{eM}{m_K}[\alpha^2(\mu_p-\mu_n)+(\mu_{\Xi^0}-\mu_{\Xi^-})] - 2\delta\pi(1-4\alpha+\alpha^2)\Big\}\{8(1-\alpha+\alpha^2)\}^{-1}. \tag{4.17}$$

Here we use our calculated values for the isovector moments, and, as explained in Appendix A, $C=\delta V_C(0)\times\langle f\rangle/f(0)=1.37$ MeV, and $e\mu_pM/m_\pi=\delta V_M(0)\langle g\rangle/g(0)=0.865$ MeV. The result is plotted in Fig. 4. For comparison we have also plotted the result which follows when $\delta\pi$ is dropped from (4.17).

5. CONCLUSIONS

In assessing the consequences of the model one must bear in mind that we are looking only for qualitative indications, and that for the reasons stated we are resigned to not explaining the isoscalar moments. Thus, we look not for a unique value, but only for a neighborhood, of α, where the calculated values approach the observed ones. Then the preferred range of values for the mixing parameter α is roughly $0<\alpha<1$, which is compatible with suggestions from other sources.[35] Within this range agreement with the data is reasonable. To what extent this supports the model, and the approximations made within it, must be a matter of taste.[1,2] As in the Introduction, we would claim only that the signs and orders of magnitude of the calculated results do follow straightforwardly from the underlying physics of the model, and that more sophisticated calculations based on similar physical assumptions must lead, if they are carried out correctly, to similar conclusions.

In the preferred range of α, we note that the effects of the two kinds of driving terms on the large splittings $\delta\Sigma$ and $\delta\Xi$ reinforce each other, while they tend to cancel in δN. Therefore the historically crucial fact that δN is actually positive in *sign* is almost accidental; what is significant is that it is considerably smaller in *magnitude* than $\delta\Sigma$ and $\delta\Xi$.

Also, it is the case that if the boson EM splittings are dropped from the driving terms, then one cannot reproduce the observed sign of δN. To illustrate this, we have plotted in Fig. 3 the solution of (4.14) with C alone in the driving terms, i.e., with $Y=0=Z$. (Since all isovector splittings are then simply proportional to C, uncertainties about the magnitude of C do not affect their signs. The other two could at least have their observed signs even without the boson driving terms.) This remark is important because the boson contributions might indeed be suppressed if the problem was essentially relativistic; a similar suppression can operate in certain dispersion-theory calculations because of kinematic factors. We conclude that if for any reason the boson EM splittings do not enter a calculation, then even with strange particles one cannot hope to reproduce all the observed signs except by invoking some very delicately balanced sign reversals due to short-range feedback. To accept such an explanation one would need firm faith in one's methods for dealing with details of the strong interactions. Again, it would seem to be a matter of taste whether, at the present time, one prefers to look for such an explanation; or whether one accepts, as we have done for the purpose of this discussion, the possibility that the problem is not essentially relativistic; or whether one abandons altogether the attempt to use the bootstrap model for explaining the baryon mass splittings.

ACKNOWLEDGMENTS

We welcome this opportunity to thank Dr. H. Goldberg and Dr. H. R. Pagels for valuable correspondence and for preprints of their papers; Dr. N. Cottingham, Dr. N. Dombey, and Professor E. J. Squires for stimulating conversations; Dr. L. M. Delves and Professor G. N. Ward for their mathematical advice on eigenvalue equations of the type (1.1); D. Newell for taking charge of the programming; and the Science Research Council for a grant to the second-named author.

APPENDIX A

How to begin with field theory or a dispersion representation and to end up with a potential usable in a Schrödinger equation, is a very delicate problem. But luckily there exists a consensus on what to do when the interaction in question is (i) weak and (ii) mediated by the exchange of a single particle other than one of the interacting particles themselves. Then one defines the equivalent potential δV by taking this single particle exchange diagram, which is essentially a function of the momentum transfer alone, and equating it to the first Born approximation to scattering by δV, i.e., to the Fourier transform of δV. One must face the problem that both particles are on their mass shells in the scattering amplitude, but not when one deals with the effects of δV on a bound state. In electrodynamics there is also the related problem of chosing a gauge for the exchanged photon. The correct choice is the Landau gauge; it leads to the potential in the Breit equation which successfully reproduces, for instance, the fine structure of positronium.[23] In our case δV is a pion-nucleon (eventually a P-B) potential due to photon exchange; in the nonrelativistic limit for the nucleons the effective differences between the gauges disappears. The resultant δV, using the Feynman gauge, has already been considered elsewhere.[6,20] Naturally it involves the pion and nucleon form factors, i.e., the finite spatial extension of their charge distributions.[36] For simplicity we adopt the same form factor, $F(K^2)$, for both, and indeed for all the B's and P's, making no distinction between Dirac and Pauli, or charge and magnetic form factors; these latter differences become unimportant for very heavy nucleons. In the spirit of our model, we treat the pions as nonrelativistic.

With these simplifications, consider as our reference system the state $|p\pi^-\rangle$. The electrostatic potential between p and π^- becomes[20]

$$\delta V_C = -\alpha f(r)\,, \tag{A1}$$

$$f(r) = (2/\pi)\int_0^\infty dK\, F^2(K^2)\sin(Kr)/Kr\,. \tag{A2}$$

The magnetic (hfs-type) potential is

$$\begin{aligned} \delta V_M &= -(2\mu_p e/m_\pi)(\mathbf{s}\cdot\mathbf{L})g(r)\,,\\ g(r) &= r^{-1}df/dr\,. \end{aligned} \tag{A3}$$

[35] Our definition of α differs from that of others. Most authors favor an F/D ratio close to $\frac{2}{3}$, which corresponds to $\alpha\approx 0.336$. See for instance J. C. Helder and J. J. de Swart, Phys. Letters **21**, 109 (1966).

[36] The connection between form factors and the spatial charge distribution is discussed, for instance, in G. Barton, *Introduction to Dispersion Techniques in Field Theory* (W. A. Benjamin, Inc., New York, 1965).

(These δV's differ slightly from those defined in Ref. 20.) Here, μ_p is the total magnetic moment of the proton, $\mu_p = 2.79e/2m_N$, $\mathbf{s} = \boldsymbol{\sigma}/2$ its spin, and $\mathbf{L}$ the orbital angular momentum. In the $P_{1/2}$ state, $(\mathbf{s}\cdot\mathbf{L}) = -1$, whence

$$\delta V_M = (2\mu_p e/m_\pi) g(r)\,; \tag{A4}$$

since $g(r)$ is negative, δV_M is attractive.

To estimate the expectation values $\langle \delta V_C \rangle$ and $\langle \delta V_M \rangle$ one should know something about the radial wavefunctions. This is particularly needful if $F(K^2)$ remains finite at infinity, for then δV_C and δV_M have components behaving like r^{-1} and r^{-3}, respectively, at small distances. But fortunately the dependence on the radial wavefunctions becomes less sensitive if we take advantage of the observed fact that $F(\infty)$ is very small and probably zero,[37] and if we are careful to construct the best possible approximation to the potentials at small distances by using an expression for $F(K^2)$ which is adequate up to large values of K^2. Such a parametrization seems to be

$$F(K^2) = a^4/(a^2+K^2)^2\,, \qquad a \approx 6m_\pi = 840\ \text{MeV}. \tag{A5}$$

Defining

$$x = ar\,, \tag{A6}$$

we find

$$f(r) = (a/48x)\{48 - e^{-x}[48 + 33x + 9x^2 + x^3]\}\,, \qquad f(0) = 5a/16\,; \tag{A7}$$

$$g(r) = -(a^3/48x^3)\{48 - e^{-x}[48 + 48x + 24x^2 + 7x^3 + x^4]\}\,, \qquad g(0) = -a^3/48\,. \tag{A8}$$

For a P state with its wave function well concentrated by the centrifugal barrier,[20] an appealing first approximation might be to adopt $\langle \delta V_C \rangle \approx \delta V_C(0)$, $\langle \delta V_M \rangle \approx \delta V_M(0)$; then (A5)–(A8) give

$$\delta V_C(0) \approx -1.9\ \text{MeV}, \quad \delta V_M(0) = -1.9\ \text{MeV}. \tag{A9}$$

To get a rough idea of the effects on the expectation values of the different ranges of f and g, we have calculated numerically the expectation values with wavefunctions for the $1p$ state in an infinitely deep square well of radius $1/m_\pi$; while this is larger than any realistic range for the πN force, the latter is not of course infinitely deep. Then one finds

$$\frac{\langle f(r) \rangle}{f(0)} = 0.72, \quad \frac{\langle g(r) \rangle}{g(0)} = 0.456\,. \tag{A10}$$

Using (A9) and (A10), the ratio of the magnetic to the Coulomb term, (both attractive), on the right of Eq. (2.11) becomes

$$\frac{\mu_S}{\mu_p} \frac{\langle g(r) \rangle}{\langle f(r) \rangle} \frac{\delta V_M(0)}{\delta V_C(0)} \approx 0.2\,. \tag{A11}$$

As explained in the text, the value of $\langle \delta V_C \rangle$ used in the isovector calculations is taken from (A9), being a slight overestimate in part-compensation for neglecting the magnetic energies. But when correcting the calculation of $\delta \Sigma^T$ for magnetic energies, we have used $\langle \delta V_C \rangle = \delta V_C(0) \langle f \rangle / f(0)$ and $\langle \delta V_M \rangle = \delta V_M(0) \langle g \rangle / g(0)$.

APPENDIX B

We prove the statement following Eq. (3.8). Denote by $|k\pi(i)\rangle$ the normalized state containing a pion and a (pion-nucleon) isobar $|k\rangle$ coupled together to give the quantum numbers of the isobar $|i\rangle$. In the spirit of Sec. 2, we write, for each i,

$$|i\rangle = \textstyle\sum_k \alpha_{ik} |k\pi(i)\rangle\,, \tag{B1}$$

$$\textstyle\sum_k |\alpha_{ik}|^2 = 1\,, \tag{B2}$$

where the α_{ik} are numerical coefficients that depend on the dynamics. Note that they need not, of course, form a unitary matrix; but in the special case of two isobars $|i\rangle$ and $|j\rangle$ with identical quantum numbers we do have

$$\textstyle\sum_k \alpha_{ik}^* \alpha_{jk} = \delta_{ij} \quad [\text{when } J^{PT}(i) = J^{PT}(j)]. \tag{B3}$$

On our model, the operator for the driving term is simply

$$T_0^\pi \tfrac{1}{2}(1 + T_0^B) C\,,$$

when $\mathbf{T}^\pi$ and $\mathbf{T}^B$ are the isospin operators for pions and baryons. We are interested only in the isovector component,

$$T_0^\pi C/2\,, \tag{B4}$$

and drop the rest. Let the operator for the isovector mass splitting be δ, and define its reduced matrix elements as usual[38] by

$$\langle i,\nu | \delta | j,\nu \rangle = \langle i || \delta || j \rangle C(t_j, 1, t_i | \nu, 0, \nu)\,, \tag{B5}$$

where t_i is the isospin of $|i\rangle$ and ν its third component. Naturally, $\langle i || \delta || j \rangle$ can be nonzero only if $|i\rangle$ and $|j\rangle$ have the same spin and parity. We shall need the orthonormality condition

$$\textstyle\sum_\mu C(t_k, 1, t_i | \mu, \nu-\mu, \nu) C(t_k, 1, t_j | \mu, \nu-\mu, \nu) = \delta_{t_i, t_j}\,, \tag{B6}$$

and the formula

$$C(t, 1, t | \nu, 0, \nu) = \frac{\nu}{[t(t+1)]^{1/2}}\,. \tag{B7}$$

The crucial first step is to note that the driving term, (B4), is electrostatic, and cannot change the nature of the isobar. Therefore, it is diagonal in the following

[37] For experimental information about the form factors, see, for instance, F. M. Pipkin, in *Proceedings of the Oxford International Conference on Elementary Particles* (Rutherford High Energy Laboratory, Harwell, England, 1966).

[38] M. E. Rose, *Elementary Theory of Angular Momentum* (John Wiley & Sons, Inc., New York, 1957).

sense:

$$\begin{aligned}\langle k\pi(i),\nu|T_0^\pi|l\pi(j),\nu\rangle &= \sum_\mu C(t_k,1,t_i|\mu,\nu-\mu,\nu)C(t_l,1,t_j|\mu,\nu-\mu,\nu) \\ &\quad\times\langle k,\mu|l,\mu\rangle\langle\pi,\nu-\mu|T_0^\pi|\pi,\nu-\mu\rangle \\ &= \delta_{kl}\sum_\mu C(t_k,1,t_i|\mu,\nu-\mu,\nu)C(t_l,1,t_j|\mu,\nu-\mu,\nu)(\nu-\mu).\end{aligned} \tag{B8}$$

Taking expectation values as in Secs. 2 and 3, and noting (B5), (B7), and (B8), we find at once

$$\begin{aligned}\langle i||\delta||j\rangle C(t_j,1,t_i|\nu,0,\nu) &= \sum_{k,l}\alpha_{ik}^*\alpha_{jl} \\ &\times\sum_\mu C(t_k,1,t_i|\mu,\nu-\mu,\nu)C(t_l,1,t_j|\mu,\nu-\mu,\nu) \\ &\quad\times\{\langle k||\delta||l\rangle C(t_l,1,t_k|\mu,0,\mu)+\tfrac{1}{2}C\delta_{kl}(\nu-\mu)\}.\end{aligned} \tag{B9}$$

On the right, the term containing $(C/2)\delta_{kl}\nu$ simplifies by virtue of (B6) and (B3), and one gets

$$\begin{aligned}\langle i||\delta||j\rangle C(t_j,1,t_i|\nu,0,\nu) &= \tfrac{1}{2}C\nu\delta_{ij} \\ &+\sum_{k,l}\alpha_{ik}^*\alpha_{jl}\sum_\mu C(t_k,1,t_i|\mu,\nu-\mu,\nu)C(t_l,1,t_j|\mu,\nu-\mu,\nu) \\ &\quad\times\{\langle k||\delta||l\rangle C(t_l,1,t_k|\mu,0,\mu)-\tfrac{1}{2}C\mu\delta_{kl}\}.\end{aligned} \tag{B10}$$

This is a set of simultaneous inhomogeneous linear equations for the $\langle i||\delta||j\rangle$ whose solution is unique. By inspection, the solution is given by

$$\langle i||\delta||j\rangle C(t_j,1,t_i|\nu,0,\nu)=\tfrac{1}{2}C\nu\delta_{ij},\quad\text{(for all } i,j). \tag{B11}$$

To check, notice that by virtue of (B11), the first term on the right of (B10) equals the left-hand side, and the second term on the right vanishes because the contents of the curly brackets vanish; here again one relies on (B7).

Evaluating the mass difference between substates of $|i\rangle$ with equal and opposite values of ν (to which the tensor splitting does not contribute), we find from (B5), (B7), and (B11),

$$[\langle i,\nu|\delta|i,\nu\rangle-\langle i,-\nu|\delta|i,-\nu\rangle]=C\nu,\quad\text{Q.E.D.}, \tag{B12}$$

independently of i, and independently of the dynamical coefficients α_{ij}. The results (2.11) (with $M=0$) and (3.8) are special cases of (B12).

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Generalized Nonet Representation of $SU(3)\otimes SU(3)$ and Its Applications

S. K. Bose*

International Atomic Energy Agency, International Centre for Theoretical Physics, Trieste, Italy

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A class of representations of the nonchiral $SU(3)\otimes SU(3)$ is worked out. These consist of a sequence of self-conjugate representations of $SU(3)$, starting always with a singlet and with each $SU(3)$ representation occurring once. An analog of the Gell-Mann–Okubo mass formula, valid for these representations of $SU(3)\otimes SU(3)$, is obtained. When applied to the lowest nontrivial representation, this formula correctly explains ω-ϕ mixing, thus providing a justification of Okubo's ansatz. Possible use of the next higher representation is indicated. From the same construction, the corresponding unitary irreducible representations of $SL(3,C)$ and $T_8\times SU(3)$ are simultaneously obtained.

I. INTRODUCTION

In this paper we describe the explicit construction of a class of irreducible representations of the nonchiral $SU(3)\otimes SU(3)$ and discuss their possible experimental relevance.[1] Let G_i and F_i denote the infinitesimal generators of the two commuting $SU(3)$'s. We now define a third $SU(3)$ whose infinitesimal generators are G_i+F_i. The special representations we have in mind are those in which this last $SU(3)$ is diagonal and which consist of a finite sequence of self-conjugate representations of this $SU(3)$, starting always with a singlet and with each representation occurring once. These representations are characterized by a single parameter which can take up odd integral values, and which is essentially a measure of the dimensionality of the representation.

Our construction also yields the corresponding representations of the noncompact $SL(3,C)$. In this case, the diagonal $SU(3)$ can be identified with the maximum compact subgroup. The single parameter that labels the irreducible representations can now take up real, odd-integral values or purely imaginary values. In the former case we get finite-dimensional nonunitary representations (unitary trick). In the latter case we get infinite-dimensional unitary representations. For the sake of completeness we also describe a similar representation of $T_8\times SU(3)$—the semidirect product of $SU(3)$ with eight mutually commuting translations [see Eq. (38)].

* On leave of absence from Center for Advanced Studies in Theoretical Physics and Astrophysics, University of Delhi, Delhi-7, India.

[1] J. Schwinger, Phys. Rev. Letters 12, 237 (1964); A. Salam and J. C. Ward, Phys. Rev. 136, B763 (1964).