Gauge Properties of the Minkowski Space*

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The dilatations and the 4-parameter special conformal group are interpreted as geometrical gauge transformations of the Minkowski space and new arguments are given as to why the conventional interpretation of the special conformal group as a set of transformations connecting systems with constant relative accelerations cannot be true. The dilatations and the special conformal group are considered to be approximate symmetries in particle physics in the sense that they become "good" groups at very high energies, but are broken for low energies when the rest masses are important. These properties are illustrated in the case of classical point particles by the partially conserved quantities associated with the new symmetries. One of the interesting features is that the transformation by reciprocal radii appears as a new discrete approximate space-time symmetry. This is probably of interest for the problem of P and CP invariance. The geometrical interpretation is analyzed in terms of homogeneous coordinates and the physical significance of these coordinates is discussed. This analysis leads in a straightforward way to the group $O(2,4)$, which is isomorphic to the full 15-parameter conformal group in Minkowski space, including the full Poincaré group. A conformal-invariant generalization of the usual notion of Einstein causality is given and analyzed. The relationship between tensors and spinors of the group $O(2,4)$ and the usual space-time spinors and tensors is discussed, employing results obtained by Dirac. Finally the field equations for electrodynamics and the pseudoscalar coupling are written down in terms of the new spinors and tensors. These equations do not contain a bare-mass term. The masses are considered to be consequences of the interaction.

I. INTRODUCTION

BROKEN and approximate symmetries have been playing a very stimulating and interesting part in particle physics. The unitary groups in particular appear to be useful in classifying elementary particles and in describing some of the patterns which characterize the interactions of these particles. The group SU_2 seems to provide a good description of the isospin; the higher symmetry SU_3—the eightfold way[1]—and its breaking has given a number of insights into the physics of particles and something similar may be said about even higher symmetries.[2]

A characteristic feature of these higher symmetries is that one starts by neglecting the mass differences of a certain class of particles. The higher the energies of the particles involved the better is this approximation, for at higher energies the mass differences, or even the masses themselves, become negligible. Thus these symmetries seem to be good symmetries at very high energies, but they have to be broken, as soon as mass differences become interesting at lower energies.[3]

In all these symmetry considerations the breaking affects the so-called internal symmetries: The Poincaré group is always considered a good symmetry group, with the restriction that space-reflection and CP invariance may be violated in certain types of interactions.

These last restrictions show that approximate or broken space-time symmetries exist, too, and one may, therefore, ask whether there are additional approximate space-time symmetries, which may be badly broken for low energies, but which are good symmetries for particle physics in the very high energy region.

It has been proposed in several papers[4–8] that the dilatations and the special conformal group—see Eqs. (19)-(21) below—are symmetries of this kind. We have not yet discussed the usefulness of these groups for the classification of particle multiplets,[9] but a comparison with experiments of the consequences of these higher symmetries for scattering amplitudes at high energies is very encouraging[8] and a more detailed analysis of the physical meaning of these new groups would seem to be worthwile.

Such an analysis has been given for the dilatations.[6,7] This group maps a certain length into another one which differs from the first one by a factor independent of the position in space-time. In the following sections we shall interpret the special conformal group similarly, namely as a geometrical gauge group the elements of which map a given length into another one which differs from the first one by a space- and time-dependent factor.[10]

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[1] M. Gell-Mann and Y. Ne'eman, *The Eightfold Way* (W. A. Benjamin, Inc., New York, 1964).

[2] Freeman J. Dyson, *Symmetry Groups in Nuclear and Particle Physics* (W. A. Benjamin, Inc., New York, 1966).

[3] The viewpoint we are adopting here has been discussed persuasively by M. Gell-Mann and F. Zachariasen [Phys. Rev. **123**, 1065 (1961), Introduction] and more recently by R. P. Feynman, in *Symmetries in Elementary Particle Physics*, edited by A. Zichichi (Academic Press Inc., New York, 1965).

[4] H. A. Kastrup, Phys. Letters **3**, 78 (1962). The inequality following Eq. (10) of this paper should read $\varepsilon \geqslant 0$ (see Ref. 6).

[5] H. A. Kastrup, Phys. Letters **4**, 56 (1963).

[6] H. A. Kastrup, Nucl. Phys. **58**, 561 (1964).

[7] H. A. Kastrup, Phys. Rev. **142**, 1060 (1966).

[8] H. A. Kastrup, Phys. Rev. **147**, 1130 (1966).

[9] See however D. Bohm, M. Flato, D. Sternheimer, and J. P. Vigier, Nuovo Cimento **38**, 1941 (1965). This paper contains references to related work. The covering group $SU(2,2)$ of the conformal group appears also as a subgroup of the group $\tilde{U}(12)$, etc., discussed by Salam *et al.* [Proc. Roy. Soc. (London) **A284**, 146 (1965)] and by others [see Ref. 2 and *Proceedings of the Second Coral Gables Conference on Symmetry Principles at High Energies*, edited by B. Kurşunoğlu (W. H. Freeman and Company, San Francisco, 1965)].

[10] H. A. Kastrup, Ann. Phys. (Leipzig) **9**, 388 (1962); this paper contains many references on the conformal group.

We have already discussed earlier[10,11] why the conventional interpretation of the special conformal group as a system of transformations which connect frames of references with constant relative acceleration cannot be the right one. We shall give another simple argument in Sec. V which shows that such an interpretation would imply that velocities of particles could be larger than the velocity of light. As far as we know this is impossible.

The present paper is intended to clarify some of the elementary physical properties of the dilatations and particularly those of the special conformal group. Since almost nothing is known about the conserved or almost conserved quantities associated with these groups, we discuss some of their simple properties in Secs. II–IV. This can be done without referring to the space-time interpretation of the special conformal group, which will be given in Sec. V.

In order to discuss the space-time properties of the special conformal group we employ the notion of homogeneous coordinates.[10] This notion is very convenient and has also a simple physical interpretation. The main idea is that one can characterize space-time positions by coordinates without a dimension of length, whereas this is not possible for a length ds which is always characterized by two points. These considerations lead in a natural way to the group $O(2,4)$ which is isomorphic to the full 15-parametric conformal group including the full Poincaré group.

In this context we also discuss the notion of Einstein causality: Whereas one has to use the differential form $dx^i dx_i$ in x space, in order to define space-like and time-like distances in a conformal-invariant way, one can use a global quadratic form of the homogeneous coordinates, in order to distinguish between these two types of distances.

In the final sections we relate some spinors and tensors of the group $O(2,4)$ to the corresponding spinors and tensors which occur in the Dirac equation, in Maxwell's equations, etc. These relations, which are mainly attributable to Dirac,[12] are intended to serve the following purpose, which will be discussed in detail elsewhere: The interaction Lagrangians in quantum electrodynamics, in the pseudoscalar coupling and in weak interactions, mediated by a vector boson, are all invariant—at least formally—under the new groups discussed here.[4] The only terms in these (unrenormalized) theories which are not invariant are the kinetic bare mass terms. Since the physical meaning of these bare masses is unclear anyhow, one is inclined to discard them completely and consider the physical masses as consequences of the interactions.[13–17] As the dilatations and the special conformal group seem to imply continuous mass values[6,7] these symmetries have to be broken somehow in order to account for the observed discrete mass spectrum of particles. We do not know, for the time being, how to break these symmetries. One possibility was pointed out in Ref. 11; another one is the spontaneous breakdown[13,14,17] of these symmetries as a consequence of the ground state not being invariant under dilatations and special conformal transformations. The latter possibility seems to be supported by the comparison with experiments[8] of the high-energy predictions of these groups. In this case one has to take into account the soft-meson background which is certainly not invariant under dilatations.

We shall write down the field equations for interacting systems with spin $\frac{1}{2}$ and spin 0 or 1. These equations correspond to the usual equations of quantum electrodynamics and the pseudoscalar coupling without a bare mass term. They explicitly show the invariance of the interaction terms under the full conformal group. Their interpretation, quantization, and other properties will be discussed elsewhere.

II. CLASSICAL KINEMATICS

In order to illustrate some elementary properties of the new quantities associated with the dilatations and the special conformal group, we shall discuss the example of classical relativistic point particles, characterized by their momenta $\mathbf{p}$ and energy[18] $p_0=(\mathbf{p}^2+m^2)^{1/2}$. The motion of a free particle is described by the relation

$$\mathbf{x}(t)=(\mathbf{p}/p_0)t+\mathbf{a}\,, \tag{1}$$

where $\mathbf{a}$ is the position of the particle at time $t=0$. The constant angular momentum $\mathbf{m}$ of such a particle is given by

$$\mathbf{m}\equiv\mathbf{x}\times\mathbf{p}=\mathbf{a}\times\mathbf{p}\,, \tag{2}$$

and its Lorentz momentum $\mathbf{n}$ is

$$\mathbf{n}\equiv\mathbf{x}p_0-x_0\mathbf{p}=p_0\mathbf{a}\,. \tag{3}$$

The quantities in Eqs. (2) and (3) are associated with the homogeneous Lorentz group. The corresponding quantities of the dilatations and the special conformal group are respectively[4]:

$$s\equiv x^i p_i=p_0 t-\mathbf{p}\cdot\mathbf{x}\,,$$

and

$$h_i\equiv 2x_i x^j p_j - x^2 p_i\,,\quad i=0,1,2,3\,.$$

[11] H. A. Kastrup, Phys. Rev. **143**, 1021 (1966); contractions of the conformal group have also been considered by R. Prasad, Nuovo Cimento **38**, 1921 (1965).

[12] P. A. M. Dirac, Ann. Math. **37**, 429 (1936).

[13] H. P. Dürr, W. Heisenberg, H. Mitter, S. Schlieder, and K. Yamazaki, Z. Naturforsch. **14a**, 441 (1959).

[14] Y. Nambu and G. Jona-Lasinio, Phys. Rev. 122, 345 (1961); **124**, 965 (1962).

[15] K. Johnson, M. Baker, and R. Willey, Phys. Rev. **136**, B1111 (1964).

[16] R. Haag and Th. A. J. Maris, Phys. Rev. **132**, 2325 (1963).

[17] *Proceedings of the Seminar on Unified Theories of Elementary Particles*, edited by H. Rechenberg (Max-Planck-Institut für Physik und Astrophysik, München, 1965). Symmetry-breaking solutions of the conformal-invariant Thirring model have been discussed by H. Leutwyler, Helv. Phys. Acta **38**, 431 (1965).

[18] We use the units $c=1=\hbar$ and the metric $x^2=(x^0)^2-\mathbf{x}^2$.

If we insert the relation (1) into these quantities we get

$$s=(m^2/p_0)t-\mathbf{ap}\,, \tag{4}$$

$$h_0=(m^2/p_0)t^2+\mathbf{a}^2p_0\,, \tag{5}$$

$$\mathbf{h}=(m^2/p_0)[(\mathbf{p}/p_0)t^2+2\mathbf{a}t]+\mathbf{a}^2\mathbf{p}-2(\mathbf{a}\cdot\mathbf{p})\mathbf{a}\,. \tag{6}$$

The expressions (4)–(6) show that s and h_i are constants of the free motion only, if either $m=0$ or $p_0\rightarrow\infty$. In analogy to the language adopted in connection with the axial-vector current in weak interactions, where the situation is similar,[19–21] we call s and h_i, $i=0, 1, 2, 3$, partially conserved quantities.

In the following discussions we shall assume that we can neglect the time-dependent terms in Eqs. (4)–(6), i.e., we either deal with a massless particle or with a particle at very high energies.

Whereas the angular momentum essentially describes the perpendicular distance between the origin and the orbit of the free particle, the quantity s, the "central phase,"[4,6] gives the projection of the vector $\mathbf{a}$ onto $\mathbf{p}$. For $\mathbf{a}\neq 0$, we have $\mathbf{m}\neq 0$, but $s=0$, if $\mathbf{a}\perp\mathbf{p}$; and $\mathbf{m}=0$, but $s\neq 0$, if $\mathbf{a}\|\mathbf{p}$.

The Bessel-Hagen momenta[11] h_i are of second order in the position variables or in $\mathbf{a}$. It is easy to verify that $h_0{}^2=\mathbf{h}^2$. Furthermore, if φ is the angle between $\mathbf{a}$ and $\mathbf{p}$, then $\pi-\varphi$ is the angle between $\mathbf{a}$ and $\mathbf{h}$, i.e., if $\mathbf{a}$ is parallel to $\mathbf{p}$, then $\mathbf{h}$ is antiparallel to $\mathbf{a}$, etc.

The usual definition for the velocity of a particle is $\mathbf{v}_p=\mathbf{p}/p_0$. We can define a new h velocity by the expression $\mathbf{v}_h=\mathbf{h}/h_0$. Since the direction of $\mathbf{h}$ differs in general from that of $\mathbf{p}$, then $\mathbf{v}_h$ is in general different from $\mathbf{v}_p$. One asks, of course, what the physical meaning of this second velocity is. The quantum-mechanical discussion of Sec. V will show that in general $\mathbf{p}$ and $\mathbf{h}$ cannot be measured simultaneously and that they are complementary in very much the same way that the positions and momenta are complementary in quantum mechanics.[11] Thus, only one of the two velocities can have a sharp value.

It is interesting to eliminate $\mathbf{a}$ from the Eqs. (2)–(6) and express all quantities in terms of the vectors $\mathbf{p}$ and $\mathbf{h}$:

$$p_0=+(\mathbf{p}^2)^{1/2}\,,\quad h_0=+(\mathbf{h}^2)^{1/2}\,,$$

$$\mathbf{h}\times\mathbf{p}=2s\mathbf{m}\,,\quad p_0\mathbf{h}-h_0\mathbf{p}=2s\mathbf{n}\,,\quad h\cdot p=2s^2\,.$$

These relations bear a close resemblance to the Lie algebra of the conformal group,[11] the elements of which can be generated by the generators P_i and K_i of the translations and the special conformal transformations.

To get a feeling for the orders of magnitude of s and h_0 in particle physics let us assume that $\mathbf{a}$ is of the order of 1 fermi and p_0 of the order 10^3 F^{-1} (≈ 200 GeV). s is then of the order 10^3, which is the same as that of $\mathbf{m}$, and h_0 is of the order 10^3 F, a rather small length which becomes larger with increasing energy.

Next we wish to discuss some transformation properties of s and h_i. It is obvious from their definition that s is a scalar and h_i a 4-vector with respect to the orthochronous proper homogeneous Lorentz group. Under space translations $\mathbf{x}\rightarrow\mathbf{x}+\mathbf{y}$, s goes over into $s-\mathbf{p}\cdot\mathbf{y}$. This behavior is similar to that of $\mathbf{m}$: $\mathbf{m}\rightarrow\mathbf{m}+\mathbf{y}\times\mathbf{p}$. Under a time translation $t\rightarrow t+\tau$ we have $s\rightarrow s+p_0\tau$. The transformation properties of h_i under translations are not so simple. The reason is that the translations and the special conformal transformations combined do not form a group.[11]

From the usual properties of $\mathbf{p}$ and p_0 under space reflections $P(\mathbf{p}\rightarrow-\mathbf{p},\, p_0\rightarrow p_0)$ and time reversal $T(\mathbf{p}\rightarrow-\mathbf{p},\, p_0\rightarrow p_0)$ the following relations can be derived from the definitions of s and h_i:

$$P:\quad s\rightarrow s\,,\qquad \mathbf{h}\rightarrow-\mathbf{h}\,,\quad h_0\rightarrow h_0\,;$$

$$T:\quad s\rightarrow -s\,,\quad \mathbf{h}\rightarrow-\mathbf{h}\,,\quad h_0\rightarrow h_0\,.$$

As an additional and new discrete symmetry group we get the transformation R by reciprocal radii [see Eq. (21) below]. We shall discuss its geometrical meaning in detail in Sec. V. This group induces the following transformations[11]:

$$R:\qquad \mathbf{m}\rightarrow\mathbf{m}\,,\quad \mathbf{n}\rightarrow\mathbf{n}\,,\quad s\rightarrow -s\,,$$

$$\mathbf{p}\rightarrow\mathbf{h}\,,\quad p_0\rightarrow h_0\,,\quad \mathbf{h}\rightarrow\mathbf{p}\,,\quad h_0\rightarrow p_0\,.$$

All the above discussions show that the quantities s and h_i constitute an additional set of observables which characterize a particle. Since these new observables are only partially conserved, they might not be very useful for low energies and finite rest masses. But we have learned during the last years that partially conserved quantities (currents) can be of considerable physical interest at very high energies.[19–23] The same might be true for our case here. Furthermore, we have seen that $\mathbf{h}$ is a vector which can carry parity, and it might be of some significance for the problem of P violation in weak interactions, particularly in connection with the new discrete group R, for this transformation maps the piece of the full conformal group which contains the unity transformation into the same piece into which it is mapped by the space reflections.[10] This conjecture about a possible interesting application of the conformal group is emphasized by the fact that the weak interactions have a form (V-A coupling) which suggests that the masses of the particles are dynamically unimportant, for they are γ^5-invariant whereas the kinematical mass terms are not. In addition to this the masses of the leptons are small and very soon become negligible for higher energies.

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[19] M. L. Goldberger and S. B. Treiman, Phys. Rev. **111**, 354 (1958).
[20] M. Gell-Mann and M. Lévy, Nuovo Cimento **16**, 705 (1960).
[21] Y. Nambu, Phys. Rev. Letters **4**, 380 (1960).
[22] Stephen L. Adler, Phys. Rev. **140**, B736 (1965); **143**, 1144 (1966).
[23] J. D. Bjorken, Phys. Rev. Letters **16**, 408 (1966).

III. THE ELASTIC SCATTERING OF TWO CLASSICAL POINT PARTICLES AT VERY HIGH ENERGIES

As a nontrivial application of the new quantities discussed above we consider the elastic scattering of two classical point particles which have either the mass zero or have very high energies. It might also be possible to consider such a system as a crude model for the centers of gravity of two astrophysical systems which are far apart from each other for large negative times—the time scale is here, of course, quite different from that in atomic collisions—but which come close to each other around the time $t=0$, move with an average relative velocity which is almost equal to the velocity of light, and which are again far apart for very large positive t.

We do not specify the interactions of the two particles; we merely assume that they allow for the conservation of s and h_i at very high energies. It is not clear to me whether this is so for gravitational interactions,[24] but it seems to be the case for weak (mediated by a vector meson), electromagnetic, and strong interactions.[4] We further assume that the motion of the particles can be characterized asymptotically by an expression of the form (1). We observe the collision in the c.m. system of the two particles.[25]

For very large negative times we have

$$\mathbf{x}_{1,2}=\pm(\mathbf{p}/p_0)t+\mathbf{a}_{1,2} \tag{7}$$

for the position of the two particles. t is the time of the observer. By a suitable choice of the orgin we can make $\mathbf{a}_1$ vanish and put $\mathbf{a}_2=\mathbf{a}$. For very large positive times we have

$$\mathbf{x}_{1,2}=\pm(\mathbf{p}'/p_0)t+\mathbf{a}_{1,2}'. \tag{8}$$

The differences between $\mathbf{p}$ and $\mathbf{p}'$ and between $\mathbf{a}_i$ and $\mathbf{a}_i'$, $i=1$, 2, describe the asymptotic effects of the interaction between the particles around $t=0$. We ask now which conditions are imposed on the final-state parameters, if we assume not only the usual 10 conservation laws associated with the Poincaré group but also those which are associated with the dilatations and the special conformal group.

Conservation of the Lorentz momenta gives

$$\mathbf{n}_1+\mathbf{n}_2=\mathbf{a}=\mathbf{n}_1'+\mathbf{n}_2'=\mathbf{a}_1'+\mathbf{a}_2'. \tag{9}$$

[24] Conformal invariance within the framework of general relativity has been discussed by R. H. Dicke, Phys. Rev. **125**, 2163 (1962); see also R. H. Dicke, in *Relativity, Groups, and Topology, Les Houches Lectures 1963*, edited by DeWitt and DeWitt (Gordon and Breach, Science Publishers, Inc., New York, 1964). Our discussion is restricted to the Minkowski space and at the moment we do not know its consequences for general relativity.

[25] These assumptions are supposed to avoid possible conflicts with various "no interaction" theorems and to avoid interaction momenta which vanish in the c.m. system [H. Van Dam and E. P. Wigner, Phys. Rev. 142, 838 (1966); references to related papers can be found here]. Additional terms in Eq. (1) due to long-range interactions seem to cause no troubles anyway at very high energies (see Ref. 6). Our main interest, for the time being, is to illustrate by a simple model the implication of additional conservation laws, not whether this model can be justified for the various types of interaction.

From angular momentum conservation it follows that

$$-\mathbf{a}\times\mathbf{p}=-(\mathbf{a}_1'-\mathbf{a}_2')\times\mathbf{p}'. \tag{10}$$

This equation means that the impact parameter $a\sin(\mathbf{a},\mathbf{p})$ is an invariant of the collision and that the plane defined by the orbits of the two outgoing particles is parallel to the corresponding plane of the incoming particles. In the following discussion we shall assume that the two planes are identical.

We see that there is at least one nontrivial solution $\mathbf{p}\neq\mathbf{p}'$ satisfying all 15 conservation laws, namely $\mathbf{a}=\mathbf{a}_1'=\mathbf{a}_2'=0$. This occurs if the two particles have a delta-function-like interaction at time $t=0$, which results in a change of the direction of motion. In this case all the quantities $\mathbf{n}_i$, $\mathbf{m}_i$, s_i, $h_j(i)$, etc., $i=1$, 2, are zero (s-wave scattering). In the following, this case is excluded.

We further have

$$s_1+s_2=\mathbf{a}\cdot\mathbf{p}=s_1'+s_2'=(\mathbf{a}_2'-\mathbf{a}_1')\cdot\mathbf{p}'. \tag{11}$$

From Eqs. (9), (10), and (11) it follows that

$$\mathbf{a}_1'\cdot\mathbf{a}_2'=0. \tag{12}$$

If $\mathbf{a}_1'=0$, then one concludes that $\mathbf{p}=\mathbf{p}'$. This is a trivial case. For $\mathbf{a}_2'=0$ we get $\mathbf{p}=-\mathbf{p}'$. This is again trivial, since only the labeling of the particles is interchanged. But if $\mathbf{a}_1'$, $\mathbf{a}_2'\neq0$, then the Eq. (12) means that they are orthogonal and this is a nontrivial restraint which goes beyond those imposed by momentum, angular momentum and Lorentz momentum conservation. Since the quantity s is a measure for the projection of the parameters $\mathbf{a}_i'$ on the orbit of the particles, the condition (12) implies a constraint on the time delay of the particles in the interaction region. This can be seen more clearly in a quantum-mechanical description where the conservation of s imposes conditions on the derivative of the phase shift.[6] Here it follows from the properties $|\mathbf{a}_1'|$, $|\mathbf{a}_2'|\leqslant|\mathbf{a}|$ which are a consequence of Eqs. (9) and (12), but which cannot be inferred from Eq. (9) alone.

It is relatively easy to see that the conservation laws for the Bessel-Hagen momenta $h_j(i)$, $i=1$, 2, do not introduce new restrictions which go beyond those already obtained. We have

$$h_0(1)+h_0(2)=\mathbf{a}^2p_0=h_0'(1)+h_0'(2)=(\mathbf{a}_1'^2+\mathbf{a}_2'^2)p_0,$$

which is satisfied according to Eqs. (9) and (12). Taking the square of both sides of the equation

$$\begin{aligned}\mathbf{h}_1+\mathbf{h}_2&=-\mathbf{a}^2\mathbf{p}+2(\mathbf{a}\cdot\mathbf{p})\mathbf{a}=\mathbf{h}_1'+\mathbf{h}_2'\\&=(\mathbf{a}_1'^2-\mathbf{a}_2'^2)\mathbf{p}'-2(\mathbf{a}_1'\cdot\mathbf{p}')\mathbf{a}_1'+2(\mathbf{a}_2'\cdot\mathbf{p}')\mathbf{a}_2'\end{aligned} \tag{13}$$

and taking into account (9) and (12) shows that the moduli of the vectors $\mathbf{h}_1+\mathbf{h}_2$ and $\mathbf{h}_1'+\mathbf{h}_2'$ are equal without additional restrictions on the parameters. The same holds true for the components of these vectors along the direction $\mathbf{a}=\mathbf{a}_1'+\mathbf{a}_2'$, which can be seen from the scalar product between this vector and the two sides of Eq. (13). The equality of the corresponding cross product finally completes the proof that Eq. (13) is valid for all parameters which fulfill the conditions (9), (10), and (11).

IV. QUANTUM-MECHANICAL KINEMATICS

In order to get some insight into the simultaneous measurability of p_i, s, h_i, etc., we consider the Lie algebra of the conformal group. We denote the generators of the Poincaré group as usual by P_i and M_{ik} and the generators of the dilatations and the special conformal group by D and K_i, $i=0, 1, 2, 3$, respectively. We assume all these operators to be Hermitian with respect to a certain scalar product (ϕ_1,ϕ_2) of a given Hilbert space. The expectation value $\langle O\rangle$ of an operator O with respect to a state ϕ is defined by $\langle O\rangle=(\phi,O\phi)$ and the corresponding fluctuation (uncertainty) ΔO by $\langle(O-\langle O\rangle)^2\rangle^{1/2}$. If we have two operators O_1 and O_2, then the general uncertainty principle is[26]

$$(\Delta O_1)(\Delta O_2)\geqslant\tfrac{1}{2}|(\phi,[O_1,O_2]\phi)|\,. \tag{14}$$

Heisenberg's quantum-mechanical uncertainty principle for p and q is a special example of this relation.

From the Lie algebra of the conformal group[7] it follows that

$$(\Delta K_i)(\Delta P_i)\geqslant|(\phi,D\phi)|\,,\quad i=0,1,2,3. \tag{15}$$

We see that the product $(\Delta K_i)(\Delta P_i)$ depends on the state considered. Since D is an unbounded operator, the right-hand side of Eq. (15) can become arbitrarily large and therefore the quantities P_i and K_i in general cannot have sharp values at the same time. Furthermore, we have

$$(\Delta K_i)(\Delta P_k)\geqslant|(\phi,M_{ik}\phi)|\,,\quad i\neq k;\, i,k=0,1,2,3. \tag{16}$$

Since the operators M_{ik}, $i, k=1, 2, 3$, are bounded, the right-hand side of relation (16) is at least bounded for these indices, but the operators K_i and P_i again cannot be measured simultaneously, if the right-hand side does not happen to vanish for that particular state. Because of $RD=-DR$ the right-hand side of Eq. (15) vanishes, for instance, if ϕ is an eigenfunction[7] of R. We also see from the Lie algebra that we can, for instance, choose K_0 and the angular momentum as commuting set of observables instead of P_0, $\mathbf{M}^2$ and M_{12}. In that case P_0 is no longer sharp. It is worthwhile to illustrate this in more detail. We take the example of a particle with both spin and mass zero,[7] where the scalar product has the form

$$(\phi_1,\phi_2)=\int\frac{d^3p}{2p_0}\phi^*(p)\phi(p)\,,\quad p_0=(\mathbf{p}^2)^{1/2}. \tag{17}$$

For this example the elements of the Lie algebra are explicitly given in Ref. 7. If we put $\phi=\beta e^{i\alpha}$, $(\phi,\phi)=1$, where α and β are real functions of $\mathbf{p}$, then we have, for instance, for

$D=i(\sum_{j=1}^{3}p^j\partial_j+1)$:

$$\langle D\rangle=(\phi,D\phi)=-\int\frac{d^3p}{2p_0}\beta^2(p)p^j\partial_j\alpha(p)\,. \tag{18}$$

If $\beta(\mathbf{p})$ is different from zero only in a small neighborhood of $\mathbf{p}=\mathbf{p}^{(0)}$, the expression (18) becomes

$$\langle D\rangle=-\sum_{j=1}^{3}(p^j\partial_j\alpha)_{\mathbf{p}=\mathbf{p}^{(0)}}.$$

We see that the gradient of the phase α assumes the role of the constant $\mathbf{a}$ of Sec. II [Eq. (4)]. Similar expressions can be obtained for the expectation values of the other generators.

If we have the scattering of two particles, then the influence of the interactions shows itself in a shift of the phase α. The consequences of the asymptotic conservation of D for the phase shift at very high energies have been discussed in Ref. 6 and their comparison with experiments in Ref. 8.

The momentum wave packets $\phi(p)$ are transformed into Bessel-Hagen momentum wave packets $\psi(h)$ by the unitary and Hermitian transformation[7]

$$R(p,h)=\frac{1}{2\pi}J_0[(2p\cdot h)^{1/2}],$$

where J_0 is a Bessel function of order zero. We have the relations

$$\psi(h)=\int\frac{d^3p}{2p_0}R(p,h)\phi(p)\,,\quad \phi(p)=\int\frac{d^3h}{2h_0}R(p,h)\psi(h)\,.$$

In the limiting case of sharp momentum, $\phi(p)=2p_0\delta(\mathbf{p}-\mathbf{p}^{(0)})$, we get $\psi(h)=R(p^{(0)},h)$, i.e., $\psi(h)$ is a Bessel function. A state with a sharp momentum contains therefore all possible Bessel-Hagen momenta. This situation is very similar to that of the momenta p and the positions q in quantum mechanics, where we have a plane wave instead of a Bessel function as a unitary transformation connecting p and q wave packets. The close relationship between the Lie algebra of the conformal group and the algebra of quantum mechanics has been analyzed previously.[11] On the other hand, if we have a state with sharp Bessel-Hagen momentum, it contains, loosely speaking, arbitrary energies. Since the Bessel function $J_0(u)$ behaves like $(2/\pi u)^{1/2}\cos(u-\pi/4)$ for very large u, the high-energy tail of J_0 is perhaps not very important in the above transformation.

The above discussion shows the importance of the transformation R for the experimental determination

[26] A simple derivation can be found in Becker-Sauter, *Electromagnetic Fields and Interactions* (Blaisdell Publishing Company, New York, 1964), Vol. II, p. 118.

of Bessel-Hagen momenta: We can obtain their distribution $\psi(h)$ by measuring the momentum distribution $\phi(p)$ and then making the integral transformation with the kernel $R(p,h)$.

If we forget for a moment that the photon has a spin, we can prepare a wave packet with sharp Bessel-Hagen momentum $h^{(0)}$ if the momentum space amplitude is given by $J_0(ph^{(0)})$. Optics is probably one of the possible fields of application of the conformal group. Since there is no experimental experience with Bessel-Hagen momenta at all, proposals, discussions, and performances of such experiments are extremely desirable.

V. THE GEOMETRICAL INTERPRETATION OF THE SPECIAL CONFORMAL GROUP

The dilatations and the special conformal group are defined by the following relations

$$x^i \to \rho x^i; \quad \rho>0,\ i=0,1,2,3; \tag{19}$$

$$x^i \to RT(c)Rx^i = (x^i - c^i x^2)/\sigma(x), \quad \sigma(x) = 1-2c\cdot x + c^2x^2; \tag{20}$$

$$Rx^i = -x^i/x^2; \quad T(c)x^i = x^i + c^i. \tag{21}$$

The physical interpretation of the dilatations has been discussed to some extent in earlier papers.[6–8,27] Here we want to give an analysis of the special conformal group and the inversion by reciprocal radii R. We immediately encounter the following problem: The image Rx^i of all coordinates with the property $x^2=0$ is infinite, and one would like to know the physical meaning of this. Since $\sigma(x) = c^2(x^i - c^i/c^2)(x_i - c_i/c^2)$ the same holds true for the light cone $(x-c/c^2)^2=0$ in the case of the transformations (20). As c^i is an arbitrary parameter, we can map any quadruple x of coordinates into an infinite one.

The mathematical tools to treat this problem are the homogeneous coordinates. In order to keep their introduction as physical as possible we start with the simpler case of the dilatations. As usual we define the number ds which characterizes the distance between a point and other points in its neighborhood by the form

$$ds^2 = dx^i dx_i = (dx^0)^2 - (dx^1)^2 - (dx^2)^2 - (dx^3)^2. \tag{22}$$

In the following we always keep the coordinate system fixed and interpret the transformations as mappings, i.e., we adopt the so-called[28] "active" viewpoint.

The dilatations (19) mean that any length, characterized by the components dx^i and the number ds^2, is transformed parallel to itself into another length, characterized by ρdx^i and $\rho^2 ds^2$. For instance, a length of 10 mm is mapped onto another one which is 80 mm long. Equation (19) says that the dilatations also map the coordinates x^i, not only the lengths ds. The crucial point for the following discussion is that it is *not* necessary for the *coordinates* of a space-time point P to have a dimension of length. They can be pure numbers. Indeed, we can always characterize a *position* by four dimensionless numbers, for instance by (1,1,1,1), without referring to any unit of length, but we cannot characterize a *length*, which by definition involves *two* points, without giving the standard of length to which we refer.[10]

We introduce the dimensionless coordinates η^i, $i=0, 1, 2, 3$, which are supposed to characterize the position of a space-time point relative to a given coordinate system, in the following way: Let us assume we have a translation and Lorentz-invariant standard of length at every space-time point P, for instance the proper time interval of a clock, which can be translated into a standard of length by multiplication with the velocity of light. Let us characterize this standard by a number κ with the dimension of length -1, for instance 10 cm^{-1}. The relationship between the coordinates x^i and η^i is then defined by

$$x^i = \eta^i/\kappa, \quad i=0,1,2,3. \tag{23}$$

The quantity κ is supposed to be invariant under all transformations of the full Poincaré group. Under dilatations it transforms as follows: $\kappa \to \rho^{-1}\kappa$, for instance $10 \to 100\ \text{cm}^{-1}$. The coordinates η^i are invariant under dilatations, i.e., the positions of space-time points are considered invariant under this group.

Under translations and proper orthochronous Lorentz transformations the coordinates transform like this:

$$\eta^i \to \eta^i + a^i\kappa,$$
$$\eta^i \to \Lambda_k{}^i \eta^k, \quad i=0,1,2,3.$$

Since the Poincaré group leaves κ invariant, it does not distinguish between the coordinates x^i and η^i at all. By a suitable choice of the standard of length we can put $\kappa=1$. The difference appears if we consider dilatations. We have seen in Refs. 6 and 7 that the invariance under dilatations is not as general as the invariance under the Poincaré group, and that for each system considered, one has to find out whether there is such a symmetry or not. For macroscopic systems and very high-energy atomic physics this invariance seems to exist. In other words, we have to make sure that the possible standards of length, characterized by κ, form a continuous manifold, if we want to introduce the dilatations as a physical symmetry group. To summarize: We characterize a point P in space-time by the coordinates $(\eta^0,\eta^1,\eta^2,\eta^3;\kappa)$, the first four of which denote the position of the point and the last of which characterizes the standard of length associated with that point. For $\kappa\neq 0$ we can always use the old coordinates x^i, but for $\kappa=0$, which can occur as soon as we have the transformation by reciprocal radii (see below), this is no longer possible.

From Eq. (23) it follows that

$$dx^i = \kappa^{-1}d\eta^i - \kappa^{-2}\eta^i d\kappa. \tag{24}$$

[27] Th. A. J. Maris, Nuovo Cimento **30**, 378 (1963).
[28] A. S. Wightman, Nuovo Cimento Suppl. **14**, 81 (1959).

If κ does not vary from point to point, we have $d\kappa=0$. This is no longer true if we consider the transformation by reciprocal radii and the special conformal group. In order to discuss these groups we introduce the spurious variable λ, defined by $\lambda=\kappa x^2$ or

$$\eta^i\eta_i-\kappa\lambda=0\,. \tag{25}$$

We now write the transformation by reciprocal radii in the following form:

$$R:\quad \eta^i\to\eta^i,\ i=0,1,2,3;\quad \kappa\to-\lambda,\ \lambda\to-\kappa\,. \tag{26}$$

These relations show that the position of a space-time point is not affected at all by R, only the standard of length is changed, but generally in a different way for different space-time points.

Since $\lambda=0$ for $x^2=0$, the transformation (26) means for these points on the light cone that the new standard of length is vanishingly small. The number $ds'^2=(x^2)^{-2}ds^2$ which characterizes the image lengths, therefore goes to infinity at these points. I do not suggest that such a gauge is convenient from a practical point of view, but I want to emphasize that the Minkowski space has these geometrical gauge properties which may have very interesting consequences.[29]

According to Eq. (20) we can construct the elements of the special conformal group from the translations and the inversion R. This gives

$$\begin{aligned}&\eta^i\to\eta^i-c^i\lambda\,,\\&\kappa\to-2c_i\eta^i+\kappa+c^2\lambda\,,\\&\lambda\to\lambda\,.\end{aligned} \tag{27}$$

It is easy to verify the fact that all transformations discussed so far leave the quadratic form $\eta^i\eta_i-\kappa\lambda$ invariant. This is the well-known theorem[30] that the full conformal group of the Minkowski space, including the Poincaré group, is isomorphic to the group $O(2,4)$.

From Eqs. (23), (24), and (25) it follows that

$$ds^2=dx^idx_i=\kappa^{-2}(d\eta^id\eta_i-d\kappa d\lambda)\,. \tag{28}$$

This relation is extremely important, for it shows that we can express ds^2 in terms of the new coordinates and that we can define space-like and time-like distances by means of the sign of the quadratic form $d\eta^id\eta_i-d\kappa d\lambda$, where the differentials are subject to the condition [Eq. (25)] $2\eta^id\eta_i-\kappa d\lambda-\lambda d\kappa=0$. Since $d\eta^id\eta_i-d\kappa d\lambda$ is invariant under the full conformal group, time-like and space-like distances are invariant concepts for the conformal group, too. This is important to our understanding of the so-called Einstein causality, which states that space-like events cannot be causally dependant.

There seems to be a certain amount of confusion as to whether the special conformal group violates causality or not. Since this is an important point, a more detailed discussion may be appropriate. It was, as to my knowledge, first pointed out by Wess[31] that the sign of the global form $(x-y)^2$ is not invariant under the special conformal group, whereas the sign of the differential form dx^idx_i is invariant. The physical aspects of these properties were discussed by me in Ref. 10. I argued that the differential form is essential for physics, because the coordinate transformations induced by the special conformal group in x space are nonlinear and it follows from all we know about differential geometry and general relativity that we have to use the differential form for defining distances, if we are dealing with nonlinear coordinate transformations.

Later[32] Gamba and Luzzato, obviously unaware of the discussions by Wess and myself, argued that the conformal group is unphysical, because the sign of the global form $(x-y)^2$ is not invariant. But Zeeman's theorem, on which these authors base their arguments, assumes that distances are defined by a global quadratic form $(x-y)^2$. For the reasons given above, I think one has to use the differential quadratic form in x space, if one discusses the conformal group. By doing so Zeeman's theorem does not apply and by using the differential form it is quite possible to describe a causal ordering of events in x space even for the special conformal group.

Furthermore, we have seen that one has to introduce the coordinates η^i, κ, λ in order to get a consistent description of the properties of the conformal group, for otherwise there would be no one-to-one correspondence between points and their images. We have already mentioned that we can define space-like and time-like distances by means of the differential form $d\eta^id\eta_i-d\kappa d\lambda$. Since the transformations of the full conformal group are linear in η space, we can even use the sign of the global form

$$(\eta_2-\eta_1)^2-(\kappa_2-\kappa_1)(\lambda_2-\lambda_1)=-2\eta_1\cdot\eta_2+\kappa_1\lambda_2+\kappa_2\lambda_1$$

in order to define, when the two arbitrary points $P_1=(\eta_1,\kappa_1,\lambda_1)$ and $P_2=(\eta_2,\kappa_2,\lambda_2)$ are space-like or time-like relative to each other.

[29] The notion of gauge transformations has always been very stimulating for particle physics. H. Weyl [*Space-Time-Matter*, translation of the fourth German edition (Dover Publications, Inc., New York, 1922)] considered a very general class of geometrical gauge transformations in the framework of general relativity. He tried to associate these transformations with the gauge transformations in electrodynamics, but later considered this attempt a failure [Naturwiss. **19**, 49 (1931)]. Gauge groups similar to those in electrodynamics were introduced into particle physics by Yang and Mills [Phys. Rev. **96**, 191 (1954)] and utilized by J. J. Sakurai [Ann. Phys. (N.Y.) **11**, 1 (1960)], M. Gell-Mann [California Institute of Technology Laboratory Report CTSL-20, 1961 (unpublished), reprinted in Ref. 1], Y. Ne'eman [Nucl. Phys. **26**, 222 (1961)], and others. At the moment the relationship between these gauge transformations and those of the conformal group is not clear to us, but they do not seem to be independent (see Ref. 11).

[30] See, for instance, F. Klein, *Vorlesungen über die Entwicklung der Mathematik im 19. Jahrhundert* (Julius Springer-Verlag, Berlin, 1927), Vol. II, p. 77.

[31] J. E. Wess, Nuovo Cimento **18**, 1086 (1960).

[32] A. Gamba and G. Luzzato, Nuovo Cimento **33**, 1732 (1964); this note is based on an article by E. C. Zeeman, J. Math. Phys. **5**, 490 (1964).

The difference between the global quadratic forms in x space and η space can be seen from the relation

$$(x_1-x_2)^2=\frac{1}{\kappa_1\kappa_2}[(\eta_1-\eta_2)^2-(\kappa_1-\kappa_2)(\lambda_1-\lambda_2)].$$

The crucial point is that the sign of $\kappa_1\kappa_2$ can change under special conformal transformations and therefore only the sign of $\kappa_1\kappa_2(x_1-x_2)^2$ has a conformal-invariant meaning.

Our main conclusion from the above discussion is the following: The squared numerical value of the local distance between two points which are near to each other is given by $\kappa^{-2}(d\eta^i d\eta_i-d\kappa d\lambda)$, where the scale factor κ is in general a function of the point under consideration, as can be seen from Eq. (27). But the possible forms of κ as a function of the different positions is not arbitrary. Only those κ are admitted here which can be generated from a constant by the elements of the full conformal group. This is a severe restriction in comparison to Weyl's theory.[29]

The possibility of having different standards of length at different points is not unusual in physics: Physicists in Princeton may measure a certain length, for instance the wavelength of a spectral line, in inches, whereas physicists in Munich may measure it in meters. In order to compare their different numerical results, one has to know the ratio $\kappa(P_1)/\kappa(P_2)$ of their standards of length.

If we want to determine whether two points are space-like or time-like relative to each other, we can use the global quadratic form in η-space (but not in x space): Given any two points $P_1(\eta_1,\kappa_1,\lambda_1)$ and $P_2(\eta_2,\kappa_2,\lambda_2)$, we can infer in a conformal invariant way from the sign of $(\eta_2-\eta_1)^2-(\kappa_2-\kappa_1)(\lambda_2-\lambda_1)$ whether the two points are space-like or time-like relative to each other. This definition of space-like and time-like distances coincides with the conventional one, if we restrict ourselves to the Poincaré group and the dilatations, i.e., if κ is the same for all points.

I have already given arguments before,[10,11] as to why the conventional interpretation of the special conformal group as a system of transformations which connects physical systems with relative constant accelerations cannot be the right one. The origin of this interpretation can be seen from the first of the Eqs. (27), replacing λ by $\kappa^{-1}\eta^i\eta_i$. If we keep the transformed coordinates fixed, these equations impose a hyperbolic relationship on the original space and time coordinates. Since constant accelerations lead to so-called hyperbolic motions if one solves Lorentz's equation,[33] people have suggested the interpretation of the conformal group just mentioned.[34]

A close look at the transformation formulas reveals that this interpretation is not possible, for the alleged accelerated system could move with a velocity larger than the velocity of light. As far as we know, only phase velocities can have such a property, but not the velocities of particles. Let us assume that only $c^3\neq 0$ in Eqs. (27) and let us define $c^3\kappa^{-1}=b$. We then have

$$\begin{aligned}\eta^{0\prime}&=\eta^0,\\ \eta^{1\prime}&=\eta^1,\\ \eta^{2\prime}&=\eta^2,\\ \eta^{3\prime}&=\eta^3-b\eta^i\eta_i.\end{aligned}\tag{29}$$

We ask which velocity a fixed point in the primed system has with respect to the old one. Differentiation gives

$$0=v-b(2\eta^0-2v\eta^3),\tag{30}$$

where $v=\partial\eta^3/\partial\eta^0$. By substituting the relation (29) into Eq. (30) we get

$$v^2[1+4b^2(\eta^3)^2+4b\eta^3]=4b(\eta^3-\eta^{3\prime}+b\boldsymbol{\eta}^2).$$

We see that $v^2>1$, if $4b^2(\eta_1^2+\eta_2^2)-4b\eta^{3\prime}>1$. But this condition can be fulfilled quite easily for appropriate values of the different variables. The possibility $v^2>1$ clearly contradicts the interpretation of v as the (group-) velocity of a physical system. Thus, it is not possible to interpret the elements of the special conformal group as a class of transformations from a physical system at rest to physical systems which move under the influence of a certain force which imparts accelerations, for as far as we know, even the velocity of an accelerated system cannot exceed the velocity of light.

We have discussed in Ref. 11 that this type of velocity which appears in connection with the transformation by reciprocal radii and the special conformal group can be understood as a kind of *phase* velocity. In Ref. 11 it was also discussed that the *group* velocity associated with the eigenfunctions of the special conformal group describes linear motions and this velocity does not exceed the velocity of light. The group velocity of Ref. 11 is identical with the h velocity in Sec. II of the present paper.

We want to emphasize again that the discrete group R is extremely powerful. Together with the translations, it generates the special conformal group, the dilatations and the proper orthochronous Lorentz group.[7,11]

It is also important to know that the gauge transformations discussed here are the only ones of the Minkowski space. This follows from a theorem[35] first proved by Liouville for three dimensions and later generalized by Lie for arbitrary dimensions $n\geqslant 3$, which says that the full conformal group in n dimensions is the

[33] See, for instance, W. Pauli, *Theory of Relativity* (Pergamon Press, Ltd., London, 1958).

[34] A long list of papers can be found in Ref. 10; see also T. Fulton, R. Rohrlich, and L. Witten, Rev. Mod. Phys. **34**, 442 (1962); Nuovo Cimento **26**, 652 (1962).

[35] Liouville's proof is contained in his editorial note VI of G. Monge's book: *Application de l'Analyse à la Géométrie* (Bachelier, Paris, 1850). S. Lie and F. Engel, *Theorie der Transformationsgruppen* (B. G. Teubner Verlag, Leipzig, 1893), Vol. III, Chaps. 17 and 18. A proof of the theorem can also be found in H. Weyl's very interesting lectures: *Mathematische Analyse des Raumproblems* (Julius Springer-Verlag, Berlin, 1923).

largest group with the property leaving the quantity $\sum_{i=1}^{n} dx^i dx_i = 0$ of a Euclidean (or pseudo-Euclidean) space invariant.

VI. TRANSFORMATION PROPERTIES OF PHYSICAL QUANTITIES UNDER THE CONFORMAL GROUP

In the previous section we have discussed the geometrical interpretation of the dilatations and the special conformal group. We want to discuss now how physical quantities like field strengths, vector potentials, etc., transform under these new groups. All these quantities have a certain dimension of length and their numerical value is therefore a function of the standard of length which is employed. If we change this standard, then the numerical values of the physical quantities are changed according to their dimension of length. In the case of the transformation by reciprocal radii and the special conformal group, the field quantities are transformed nonlinearly[4] in the coordinates x^i, but since the full conformal group is isomorphic to the group $O(2,4)$, it is very convenient to relate the usual field quantities to the tensor and spinor representations of that group. This problem has been discussed mathematically by Dirac,[12] and we shall rely heavily on his results.

Our aim is to write down field equations of interacting fields which are manifestly covariant under the conformal group and which are closely related to the usual field equations in quantum electrodynamics, the pseudoscalar coupling, etc. One of the features of these new equations is that they do not contain any bare rest masses. We therefore have to assume that the rest masses are a consequence of the interaction. In the rest of this paper we shall provide the means which are needed in order to write down these new field equations. Their analysis will be given elsewhere.

In order to utilize the summation convention we define $\kappa=\eta^4-\eta^5$, $\lambda=\eta^4+\eta^5$ and let Latin indices run from 0 to 3 and Greek indices from 0 to 5. This means, for instance,

$$\eta^i\eta_i-\kappa\lambda=\eta^\mu\eta_\mu .$$

Let $\tilde{A}(x)$ be any invariant with respect to the Poincaré group. We assume that $\tilde{A}(x)$ transforms under dilatations as follows $\tilde{A}(x)\rightarrow\tilde{A}'(\rho x)=\rho^n\tilde{A}(x)$, i.e., $\tilde{A}(x)$ has the dimension of length n. If we define

$$A(\eta)\equiv A(\eta^0,\cdots,\eta^5)=\kappa^n\tilde{A}(\eta^0/\kappa,\cdots,\eta^3/\kappa), \quad (31)$$

then $A(\eta)$ is not only invariant under the Poincaré group but also invariant under dilatations. Furthermore, $A(\eta)$ is homogeneous in η of degree n, i.e.,

$$\eta^\mu\partial_\mu A(\eta)=nA(\eta), \quad \partial_\mu=\partial/\partial\eta^\mu. \quad (32)$$

A similar discussion can be given for vectors, tensors etc. and we conclude that we can confine ourselves to those field quantities in η space which are homogeneous in the coordinates η.

If $F(\eta)$ is any homogeneous function of the coordinates η, its physical domain of definition is confined to the hyper-light-cone $\eta^\mu\eta_\mu=0$. Thus we can add to $F(\eta)$ any function $\eta^2B(\eta)$ without altering the properties of $F(\eta)$ on the hyper cone. Dirac has investigated a number of consequences of this feature, and we refer to his paper for further details.

If the two quantities $\tilde{F}(x)$ and $F(\eta)$, where $F(\eta)$ is homogeneous of degree n, are related to each other by

$$\tilde{F}(x)=\kappa^{-n}F(\eta), \quad (33)$$

their derivatives obey the following equation

$$\tilde{\partial}_k\tilde{F}(x)\equiv\frac{\partial}{\partial x^k}\tilde{F}(x)=\kappa^{-n}\left[\kappa\partial_kF(\eta)+2\eta_k\frac{\partial}{\partial\lambda}F(\eta)\right]. \quad (34)$$

This relation has a simple meaning which can be seen from Eqs. (27): $\tilde{\partial}_k$ is the generator of the infinitesimal translations in the space of the functions $\tilde{F}(x)$, and $\kappa\partial_k+2\eta_k(\partial/\partial\lambda)$ is the generator of the translations in the space of the functions $F(\eta)$.

Because of Eq. (23) the relation (33) is defined only for $\kappa\neq0$. But we have seen before that κ can become zero for certain classes of points if we implement the transformation by reciprocal radii. Therefore it would be better to write $\tilde{F}(P)=\kappa^{-n}F(\eta)$, where P denotes $(\eta^1,\eta^2,\eta^3,\eta^4;\kappa)$. By so doing we extend the domain of definition of $\tilde{F}(x)$ and that should be kept in mind during the following discussions.

Important physical quantities in particle physics are the conserved currents. For these currents the following interesting lemma holds: If $j^\mu(\eta)$ is a vector with respect to the group $O(2,4)$ with the properties

$$\eta^\mu\partial_\mu j^\nu(\eta)=nj^\nu, \quad \nu=0,\cdots,5; \quad \eta_\mu j^\mu=0,$$

and if we define the 4-vector $\tilde{j}^k(x)$ by the equation

$$\tilde{j}^k(x)=\kappa^{-n}[j^k(\eta)-\eta^k/\kappa(j^4-j^5)],$$

then we have the following relation

$$\tilde{\partial}_k\tilde{j}^k(x)=\kappa^{-n+1}[\partial_\mu j^\mu(\eta)-\kappa^{-1}(n+3)(j^4-j^5)], \quad (35)$$

i.e., $\tilde{j}^k(x)$ is a conserved current in x-space if $j^\mu(\eta)$ is a conserved current in η-space and if $n=-3$. All the physically interesting currents seem to be of this type. Examples are given in the next sections.

By integrating $\tilde{j}^k(x)$ over the total three-dimensional space, we get the "charge" associated with the conserved current. Because of Eq. (23) we have in general

$$dx^1dx^2dx^3=\kappa^{-3}d\eta^1d\eta^2d\eta^3 -\kappa^{-4}(\eta^1d\kappa d\eta^2d\eta^3+\eta^2d\eta^1d\kappa d\eta^3+\eta^3d\eta^1d\eta^2d\kappa).$$

If we choose our gauge in such a way that κ is the same for all space-time points then we have $d\kappa=0$. This means

$$Q=\int dx^1dx^2dx^3\tilde{j}^0(x)$$

$$=\int d\eta^1 d\eta^2 d\eta^3[j^0(\eta)-(\eta^0/\kappa)(j^4-j^5)],$$

where Q is time-independent, if the current is conserved.

VII. THE FREE SCALAR FIELD

It is easy to verify that the relations

$$\tilde{A}(x)=\kappa A(\eta),\quad \eta^\mu\partial_\mu A(\eta)=-A(\eta) \tag{36}$$

imply the equation

$$\tilde{\partial}_k\tilde{\partial}^k\tilde{A}(x)=\kappa^3\partial^\mu\partial_\mu A(\eta); \tag{37}$$

i.e., if $\tilde{A}(x)$ is a solution of the massless Klein-Gordon equation, then $A(\eta)$ is a solution of $\partial^\mu\partial_\mu A(\eta)=0$ and vice versa. The current

$$j_\mu=(A^*\partial_\mu A-\partial_\mu A^* A)$$

is conserved and homogeneous of degree -3. It follows from the second of the Eqs. (36) that it fulfills the condition $\eta^\mu j_\mu=0$ and we have, therefore, exactly the situation encountered in our lemma of the last section.

Of particular interest are those solutions of the equation $\partial^\mu\partial_\mu A(\eta)=0$ which can be expanded in terms of the plane waves $e^{-ip\cdot x}=e^{-ip\eta/\kappa}$ or in terms of the corresponding functions $e^{ih\cdot x/x^2}=e^{ih\cdot\eta/\lambda}$ of the special conformal group. Both types of functions are homogeneous in the new coordinates, and they are eigenfunctions of the operators

$$P_j=i\left(\kappa\partial_j+2\eta_j\frac{\partial}{\partial\lambda}\right),\quad K_j=i\left(-\lambda\partial_j-2\eta_j\frac{\partial}{\partial\kappa}\right),$$

respectively. If we want expansions in terms of these functions to be solutions of the above free-field equation we have to require $p^2=0$, $h^2=0$.

Usually only expansions in terms of the plane waves are considered, but in the framework of the conformal group the operators K_i are on the same level as the P_i, and this leads to a number of very interesting problems if one considers field theories in the framework of the conformal group.[7,36]

VIII. MAXWELL'S EQUATIONS

The classical Maxwell fields have been discussed by Dirac and we confine ourselves to some definitions we shall need later on. The free-field equations for the vector potential are

$$\partial^\mu\partial_\mu A_\nu(\eta)=0,\quad \nu=0,\cdots,5;\quad \partial^\mu A_\mu(\eta)=0,$$

$$\eta^\mu\partial_\mu A_\nu(\eta)=-A_\nu(\eta),\quad \eta^\mu A_\mu(\eta)=0.$$

The usual vector potential $\tilde{A}^k(x)$ is given by

$$\tilde{A}_k(x)=\kappa A_k(\eta)-\eta_k(A^4-A^5).$$

With the definition $F_{\mu\nu}(\eta)=\partial_\mu A_\nu-\partial_\nu A_\mu$ we have for the physical field strengths $\tilde{F}_{ik}(P)$, $i, k=0, 1, 2, 3$,

$$\tilde{F}_{ik}(P)=\kappa^2F_{ik}(\eta)+\kappa(\eta_i\partial_k-\eta_k\partial_i)(A^4-A^5)+2\kappa\left(\eta_i\frac{\partial}{\partial\lambda}A_k-\eta_k\frac{\partial}{\partial\lambda}A_i\right).$$

In our new framework we have two types of gauge transformations for electrodynamics: those which are a consequence of the field strengths being the physically interesting quantities, not the potentials, and the geometric gauge transformations associated with the dilatations and the special conformal group. The latter affect the field strengths, too. We shall not investigate all these different gauge properties and their relationship to each other in this paper.

IX. THE SPIN-½ FIELD

The following discussion differs to some extent from Dirac's, who considers 4-component spinors in η-space. But if we want to include the reflections, we have to double the components.[37]

The Clifford algebra we are going to employ is generated by six elements β^μ defined by the relations

$$\beta^\mu\beta^\nu+\beta^\nu\beta^\mu=2g^{\mu\nu}, \tag{38}$$

where $g^{00}=-g^{11}=-g^{22}=-g^{33}=-g^{44}=g^{55}=1$; all other $g^{\mu\nu}$ vanish. Equations (38) are fulfilled by the explicit construction

$$\beta^k=\gamma^k\otimes\sigma_1,\quad k=0,1,2,3;$$
$$\beta^4=i\gamma^5\otimes\sigma_1,\quad \beta^5=1\sigma\otimes\sigma_2, \tag{39}$$

where the γ^k are the usual Dirac matrices, the σ_α, $\alpha=1,2,3$, are the Pauli matrices and $\gamma^5=i\gamma^0\gamma^1\gamma^2\gamma^3=-\gamma_5$.

With $\hat{M}_{\mu\nu}=i(\eta_\mu\partial_\nu-\eta_\nu\partial_\mu)$ we can write down the following free-field equation for the 8-component spinor $\chi(\eta)$:

$$\beta^\mu\beta^\nu\hat{M}_{\mu\nu}\chi(\eta)=4i\chi(\eta). \tag{40}$$

$\chi(\eta)$ is homogeneous of degree -2, i.e.,

$$\eta^\mu\partial_\mu\chi(\eta)=-2\chi(\eta). \tag{41}$$

The equation (40) is invariant under the transformations $S(\alpha)$ of the covering group $SU(2,2)$ of the con-

[36] H. A. Kastrup, Phys. Rev. 140, B183 (1965).

[37] The spinor representations of the general group $O(p,q)$, $p+q=n$, have been treated by R. Brauer and H. Weyl, Am. J. Math. 57, 425 (1935), and the special application of their work for the group $O(2,4)$ has been discussed in Ref. 10, where references to related papers can be found. W. A. Hepner's paper appeared in Nuovo Cimento 26, 351 (1962). See also R. J. Finkelstein, Nuovo Cimento 1, 1104 (1955). For a more modern analysis of spinors see M. Riesz, University of Maryland, The Institute for Fluid Dynamics and Applied Mathematics, Lecture Series No. 38, 1957–1958 (unpublished). I am indebted to Dr. D. Hestenes for drawing these lectures to my attention.

formal group. This can be inferred from the relation

$$S(\alpha)\beta^\mu S^{-1}(\alpha)=\alpha^\mu{}_\nu\beta^\nu,$$

where the real numbers $\alpha^\mu{}_\nu$ are the coefficients of an element of $O(2,4)$. The adjoint spinor $\tilde{\chi}$ is defined by

$$\tilde{\chi}\to\chi^\dagger A\,,\quad A=-i\beta^0\beta^5=\gamma^0\otimes\sigma_3\,,$$

and we have $\tilde{\chi}\to\tilde{\chi}S^{-1}$ if $\chi\to S\chi$.

Because of the relations (39) Eq. (40) decomposes into two equations for 4-component spinors χ_1 and χ_2, where

$$\chi=\begin{pmatrix}\chi_1\\ \chi_2\end{pmatrix}.$$

Using the relations $\hat{P}_j=\hat{M}_{j4}+\hat{M}_{j5}$, $\hat{K}_j=-\hat{M}_{j4}+\hat{M}_{j5}$, $\hat{D}=-\hat{M}_{45}$, we get

$$\gamma^k\gamma^l\hat{M}_{kl}\chi_1+i\gamma^k(1+\gamma^5)\hat{P}_k\chi_1+i\gamma^k(1-\gamma^5)\hat{K}_k\chi_1+2\gamma^5\hat{D}\chi_1=4i\chi_1\,,\quad(42)$$

$$\gamma^k\gamma^l\hat{M}_{kl}\chi_2-i\gamma^k(1-\gamma^5)\hat{P}_k\chi_2-i\gamma^k(1+\gamma^5)\hat{K}_k\chi_2-2\gamma^5\hat{D}\chi_2=4i\chi_2\,.\quad(43)$$

Each of these equations is invariant separately under the group $SU(2,2)$.

It is easy to verify the fact that the 2-component spinors

$$\psi_1(x)=\kappa\tfrac{1}{2}(1+\gamma^5)(\kappa+i\eta^k\gamma_k)\chi_1(\eta)\quad(44)$$

and

$$\psi_2(x)=\kappa\tfrac{1}{2}(1-\gamma^5)(\kappa-i\eta^k\gamma_k)\chi_2(\eta)\quad(45)$$

are solutions of the Dirac equation

$$\gamma^k\tilde{\partial}_k\psi_\alpha(x)=0\,,\quad\alpha=1,2\,.$$

A few remarks about reflections: for space reflections P we have[10]

$$S(P)=\omega_P\beta^0\beta^4\beta^5=i\omega_P\gamma^0\gamma^5\otimes\sigma_2\,.\quad(46)$$

ω_P is a phase factor. The factor σ_2 shows that χ_1 and χ_2 are interchanged. Because of the relations (44) and (45), $\psi_1(x)$ and $\psi_2(x)$ are transformed as usual.

The transformation $S(R)$ by reciprocal radii is represented by

$$S(R)=\omega_R\beta^0\beta^1\beta^2\beta^3\beta^5=-i\omega_R\gamma^5\otimes\sigma_2\,.\quad(47)$$

Again, χ_1 and χ_2 are interchanged by $S(R)$.

It is easy to see that the quantity $\tilde{\chi}\beta^7\chi$,

$$\beta^7=-i\beta^0\beta^1\beta^2\beta^3\beta^4\beta^5=1\otimes\sigma_3$$

changes its sign under $S(P)$ and $S(R)$, i.e., $\tilde{\chi}\beta^7\chi$ is a pseudoscalar. We also observe the close relationship between P and R. This is probably quite important for the problem of parity conservation, etc.

It follows from Eqs. (40) and (41) that the current

$$j^\mu(\eta)=\frac{i}{2}\eta_\nu\tilde{\chi}(\beta^\nu\beta^\mu-\beta^\mu\beta^\nu)\chi\quad(48)$$

is conserved, is homogeneous of degree -3, and obviously has the property $\eta_\mu j^\mu=0$.

We are at last able to write down nonlinear equations for spin-0, spin-$\frac{1}{2}$, and spin-1 fields. With the definitions of the preceding discussions we have

$$\beta^\mu\beta^\nu[\hat{M}_{\mu\nu}-q(\eta_\mu A_\nu-\eta_\nu A_\mu)]\chi(\eta)=4i\chi(\eta)\,,$$
$$\partial^\mu\partial_\mu A_\nu(\eta)=qj_\nu(\eta)\,,\quad\nu=0,\cdots,5\,,\quad\partial^\mu A_\mu(\eta)=0\,,\quad(49)$$
$$\eta^\mu A_\mu(\eta)=0\,,\quad\eta^\mu\partial_\mu A_\nu(\eta)=-A_\nu(\eta)\,,\quad\eta^\mu\partial_\mu\chi(\eta)=-2\chi(\eta)\,.$$

$j_\nu(\eta)$ has the same form as in Eq. (48). The system (49) is our new set of equations for quantum electrodynamics.

The pseudoscalar coupling is given by

$$\beta^\mu\beta^\nu\hat{M}_{\mu\nu}\chi(\eta)-4i\chi(\eta)=gA(\eta)\eta_\nu\beta^\nu\beta^7\chi(\eta)\,,\quad(50)$$
$$\partial^\mu\partial_\mu A(\eta)=-g\eta_\nu\tilde{\chi}\beta^\nu\beta^7\chi\,.$$

Other nonlinear field equations can be constructed accordingly.

In order to extract physical information out of Eqs. (49) and (50) we have to discuss the properties of the groundstate and the algebraic structure of the field operators. This will be done elsewhere.

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Note added in proof. After the present paper was submitted for publication, I learned about an article by F. Rohrlich [Ann. Phys. (N. Y.) **22**, 169 (1963)] in which he shows that there are hyperbolic motions which are not elements of the conformal group. The discussions in the present paper go beyond this and I maintain that the special conformal group has nothing to do with accelerated motions at all. I am indebted to Professor Rohrlich for calling his paper to my attention.