Sum Rules for Magnetic Moments from Commutation Relations between Current Densities*

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It is shown that the magnetic-moment sum rule suggested by Drell and Hearn can be derived from the commutation relations between charge densities. We also derive another exact sum rule for the isovector magnetic moment in terms of cross sections:

$$\frac{1}{4\pi^2\alpha}\int_0^\infty \{(2\sigma_{1/2}{}^V(\nu)-\sigma_{3/2}{}^V(\nu))_P-(2\sigma_{1/2}{}^V(\nu)-\sigma_{3/2}(\nu))_A\}d\nu=\frac{\mu_T{}^V}{2m},$$

where $\sigma_{1/2}{}^V(\nu)_P$ $(\sigma_{1/2}{}^V(\nu)_A)$ and $\sigma_{3/2}{}^V(\nu)_P$ $(\sigma_{3/2}{}^V(\nu)_A)$ are the total cross sections for the absorption of a circularly polarized isovector photon by a proton polarized with its spin parallel (antiparallel) to the photon spin in the $I=\frac{1}{2}$ and $I=\frac{3}{2}$ channels, respectively. This sum rule is derived from the commutation relation between space and time components of vector current densities.

THE purpose of this article is to notice that the magnetic-moment sum rule suggested by Drell and Hearn[1] is also a consequence of commutation relations between vector charge densities, and to propose another sum rule for the isovector magnetic moment in terms of experimentally measurable quantities. The latter sum rule is derived from the commutation relation between space and time components of vector current densities.

The equal-time commutation relation between charge densities

$$[V_0{}^i(\mathbf{x}_1),V_0{}^j(\mathbf{x}_2)]=if_{ijk}V_0{}^k(\mathbf{x}_1)\delta(\mathbf{x}_1-\mathbf{x}_2)\,, \qquad (1)$$

or its Fourier transform,

$$[V_0{}^i(\mathbf{q}_1),V_0{}^j(\mathbf{q}_2)]=if_{ijk}V_0{}^k(\mathbf{q}_1+\mathbf{q}_2)\,, \qquad (2)$$

gives various dynamical information on strongly interacting particles.[2]

One obtains, for instance, by applying the Fubini-Furlan-Adler method,[3] the sum rule derived by Cabibbo and Radicati and several other people[4]:

$$\left(\frac{\mu_A{}^V}{2m}\right)^2+\frac{1}{2\pi^2\alpha}\int_0^\infty\frac{(2\sigma_{1/2}{}^V(\nu)-\sigma_{3/2}{}^V(\nu))}{\nu}d\nu=\tfrac{1}{3}\langle r^2\rangle_{F_1}{}^V\,, \qquad (3)$$

where $\langle r^2\rangle_{F_1}{}^V$ is the isovector Dirac charge radius, $\mu_A{}^V$ is the isovector anomalous magnetic moment, α is 1/137, and $\sigma_{1/2}{}^V(\nu)$ and $\sigma_{3/2}{}^V(\nu)$ are the total cross sections for the absorption of the isovector photon with the laboratory energy ν in the $I=\frac{1}{2}$ and $I=\frac{3}{2}$ channels, respectively. It is likely that the sum rule Eq. (3) is satisfied fairly well if one includes the contributions from the electric dipole transition near the threshold and the second higher resonance as well as the (3-3) resonance.[5]

It is to be recalled, however, that this sum rule is the one resulting from the part antisymmetric with respect to isospin (unitary spin) in the commutation relations Eq. (2). Likewise, we can make use of the symmetric part of Eq. (2) to get another sum rule for the magnetic moment. Let us choose $i=j=\{3\}+\frac{1}{3}\sqrt{3}\{8\}$, namely the electromagnetic charge densities; take the matrix element of Eq. (2) between one-proton states with spin up, for instance, and differentiate both sides of Eq. (2) with respect to q_{1x} and q_{2y}, putting $\mathbf{q}_1=\mathbf{q}_2\to 0$ after the differentiation. Obviously, the right-hand side of Eq. (2) is purely antisymmetric with respect to isospin indices and therefore there is no contribution from this side.

In the limit of infinite momentum in the z direction, one picks up a term proportional to the square of the proton anomalous magnetic moment as the one-particle-state contribution. The continuum part can be expressed in terms of total cross sections, and thus we are led to the following sum rule:

$$\left(\frac{\mu_A{}^P}{m}\right)^2+\frac{1}{2\pi^2\alpha}\int_0^\infty\frac{[\sigma_A(\nu)-\sigma_P(\nu)]}{\nu}d\nu=0\,, \qquad (4)$$

where the σ_P (σ_A) is the total cross section for the absorption of a circularly polarized photon by a proton polarized with its spin parallel (antiparallel) to the photon spin. The sum rule Eq. (4) is identical to the one pointed out by Drell and Hearn,[1] which is a consequence of the unsubtracted dispersion relation for the spin-flip part of the forward Compton scattering amplitude with no arbitrary constant. It is, of course, not surprising that one can reproduce the sum rule by making use

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[2] M. Gell-Mann, Phys. Rev. **125**, 1067 (1962).
[3] S. Fubini and G. Furlan, Physics **1**, 229 (1965); S. L. Adler, Phys. Rev. Letters **14**, 1051 (1965); Phys. Rev. **140**, B736 (1965). See also W. I. Weisberger, Phys. Rev. Letters **14**, 1047 (1965); Phys. Rev. **143**, B1302 (1966).
[4] N. Cabibbo and L. A. Radicati, Phys. Letters **19**, 697 (1966); S. L. Adler, Phys. Rev. **143**, 1144 (1966); J. D. Bjorken (unpublished) and Phys. Rev. **148**, 1467 (1966); R. Dashen and M. Gell-Mann, in *Proceedings of the Third Coral Gables Conference on Symmetry Principles at High Energies* (W. H. Freeman and Company, San Francisco, California, 1966).
[5] F. J. Gilman and H. J. Schnitzer, Phys. Rev. **150**, 1362 (1966).

of a commutation relation between charge densities. It is gratifying, however, that such a simple derivation can be given within the framework of the current commutation relations. We remark in passing that the sum rule Eq. (4) is derived from Eq. (2) which is, in turn, based on the assumption that the electromagnetic current is of minimal type. Otherwise we would have a symmetric term in the right-hand side of Eq. (2) and this would give rise to an additional contribution to the sum rule. This point, together with nonminimal effects on other sum rules, will be discussed in a separate paper.

We can extend our arguments to the commutator between space and time components of vector current densities. Their Fourier transforms $V_\alpha{}^i(\mathbf{q}_1)$ and $V_0{}^j(\mathbf{q}_2)$ obey the commutation rules

$$[V_\alpha{}^i(\mathbf{q}_1),V_0{}^j(\mathbf{q}_2)]=if_{ijk}V_\alpha{}^k(\mathbf{q}_1+\mathbf{q}_2) + (\text{contributions from the Schwinger term}), \quad (5)$$

where $\alpha=x$, y, and z.

We choose $i=\{1\}+i\{2\}$, $j=\{1\}-i\{2\}$, and $\alpha=x$, take the matrix element of Eq. (5) between one-proton states with spin up and differentiate it with respect to q_{2y}. Following then the similar procedure by which Eqs. (3) and (4) are derived, one arrives at a sum rule

$$\frac{1}{4\pi^2\alpha}\int_0^\infty \{[2\sigma_{1/2}{}^V(\nu)-\sigma_{3/2}{}^V(\nu)]_P - [2\sigma_{1/2}{}^V(\nu)-\sigma_{3/2}{}^V(\nu)]_A\}d\nu=\frac{\mu_T{}^V}{2m}, \quad (6)$$

provided the integral of the left-hand side is convergent. Here, $\mu_T{}^V$ is the total vector magnetic moment and the cross sections are those with isovector photons. Since the commutation relations used here are those between "good" and "bad" components in the terminology of the current algebra at infinite momentum,[6] in deriving the above sum rule, we have to keep terms of the order of $1/E$ and equate their coefficients. There is no contribution from the one-particle state to the sum rule. The Schwinger term[7] also will not contribute since we are concerned here only with spin-dependent terms.[8] We will not discuss the convergence problem of the integral that stands on the left-hand side of Eq. (6). We only note that it is possible that this integral is in fact convergent as a whole, since the difference of the cross sections is taken twice with respect to spin and isospin $[2\sigma_{1/2}(\nu)-\sigma_{3/2}(\nu)\to 0$ at $\nu\to\infty$ by the Pomeranchuk theorem].

At present, the comparison of the sum rule with experiment is necessarily of qualitative nature. Contributions from the (3-3) and the second resonance are tentatively estimated as follows[9]:

$$-0.083/m_\pi+0.240/m_\pi=\mu_T{}^V/2m\,, \quad (7)$$

which leads to $\mu_T{}^V=2.1$. This is to be compared with the experimental value 4.7. Clearly, one expects an appreciable amount of contributions from higher resonances, most of which are $I=\frac{1}{2}$ states and presumably contribute as positive to the integral.[10] Thus, there is some chance that the sum rule is in fact satisfied. We should, however, wait for more accurate experiments covering higher energy regions in order to test this sum rule.

The significance of the sum rule given here will be that it is derived from the commutation relations between "good" and "bad" components. Although several sum rules derived from "good" components have achieved great success, little is known about the sum rules derived from "bad" components and their convergence problem. For instance, Okubo[11] obtained a sum rule from the commutator between "bad" components, which involves a less convergent integral than Eq. (6) and it is not clear whether such an integral is finite or not. It is hoped that the experimental test of the sum rule Eq. (6) will shed more light on the structure of the current algebra.

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[6] The concepts of the good and the bad components are found, for example, in R. Dashen and M. Gell-Mann, Phys. Rev. Letters **17**, 340 (1966), and Ref. 3; S. Fubini, G. Segré, and J. D. Walecka, Ann. Phys. (N.Y.) (to be published).

[7] J. Schwinger, Phys. Rev. Letters **3**, 296 (1959); Phys. Rev. **130**, 406 (1963).

[8] This is the reason why we differentiate Eq. (5) with respect to q_{2y}. Were we to differentiate Eq. (5) with respect to q_{2x}, then we would pick up a possible contribution from the Schwinger term, arriving at a sum rule which looks manifestly divergent.

[9] For the single-pion production amplitudes, we have adopted the result cited in Ref. 1. It is known that the isovector photon is dominant over the isoscalar photon at the first and the second resonances. The branching ratio is taken from L. D. Roper *et al.*, Phys. Rev. **138**, B190 (1965); B. H. Bransden *et al.*, Phys. Letters **11**, 339 (1964); P. Auvil *et al.*, *ibid.* **12**, 76 (1964). See also A. H. Rosenfeld *et al.*, Rev. Mod. Phys. **37**, 633 (1965).

[10] Two different orbital angular momentum states are allowed at each resonance with definite spin and parity. The smaller orbital angular momentum is allowed more in the parallel spin states than in the antiparallel states. Consideration of phase volume, therefore, favors the parallel spin states over the antiparallel states. A crude estimate of the contribution from the third resonance (1688 MeV, $\frac{5}{2}^+$) along this line gives $0.043/m_\pi$ to the left-hand side of Eq. (7), which leads to $\mu_T{}^V=2.7$.

[11] S. Okubo, Ann. Phys. (N.Y.) (to be published); Phys. Letters **20**, 195 (1966).