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Saturation in Triplet Models of Hadrons

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Triplet models of hadrons are studied according to the criterion of saturation, namely, that the lowest-lying baryons contain exactly three triplets. Two main types of saturation are discussed: Pauli saturation, which depends on antisymmetrization of wave functions, and Coulomb saturation, which relies on the scheme of forces among the particles. The quark, quark-plus-singlet-core, two-triplet, three-triplet, and paraquark models are surveyed, and, using the saturation mechanisms discussed in the text, all of the models are made to satisfy the saturation criterion with the sole notable exception of the quark model, which fails.

1. INTRODUCTION

THE suggestion[1] that one or more fundamental unitary triplets can provide an explanation for the observed regularities of low-lying hadronic states received support from the $SU(6)$ classification[2,3] of these states. Since most of the successes[2–6] of this classification can be understood simply in terms of models in which the hadrons are composites of nonrelativistic triplets, it seems worthwhile to explore the consequences of the existence of triplets as real objects. In this direction, Thirring,[7] and Lipkin and Scheck[8] have found results, in agreement with experiment, which go beyond $SU(6)$ and appear to depend on the existence of real triplets.

There are three striking facts which provide criteria for such models: (1) a strong form of saturation whereby $N\equiv|n_t-n_{\bar{t}}|=0$, 3, only, for low-lying single-centered systems, where n_t ($n_{\bar{t}}$) is the number of triplet (antitriplet) particles[9]; (2) the lowest-lying baryons occur in the symmetric **56** of $SU(6)$ rather than in an antisymmetric $SU(6)$ configuration[10]; (3) the $SU(6)$ prediction of $-\frac{3}{2}$ for the ratio of the proton and neutron magnetic moments[6] is accurate to within 3%.

In the present article, we assume a nonrelativistic composite picture for the hadrons, and ignore the serious open dynamical problems connected with composite models of hadrons, such as the compatibility of strong binding with nonrelativistic triplet motion[11] and the coexistence of triplet and hadronic cloud structure. The question of saturation depends on the dominant $SU(6)$-invariant effects rather than on the weaker $SU(6)$-violating ones, and therefore we assume that the basic objects belong to low-dimensional irreducible representations of $SU(6)$, in particular to **6**, **6***, and **1**, and that the forces between the basic objects are $SU(6)$-invariant. There are only two different $SU(6)$-invariant forces between a pair of objects which are each in the **6**; these forces can be taken to be one, $V(21)$, which acts only between an $SU(6)$ symmetric pair in the **21**, and a second, $V(15)$, which acts only between an $SU(6)$ antisymmetric pair in the **15**. Similarly, for three-body forces between triples of objects, each in the **6**, there are four invariant forces, $V(56)$, $V(70)$, $V(70)'$, and $V(20)$, acting between triples in the associated representations, etc.

These forces are of exchange character, so that the theorem[12] that the nodeless wave function has the lowest energy is not relevant. To see this, consider a

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[1] M. Gell-Mann, Phys. Letters **8**, 214 (1964); G. Zweig, CERN Reports 8182/TH. 401 and 8419/TH. 412, 1964 (unpublished).

[2] F. Gürsey and L. A. Radicati, Phys. Rev. Letters **13**, 173 (1964).

[3] B. Sakita, Phys. Rev. **136**, B1756 (1964).

[4] A. Pais, Rev. Mod. Phys. **38**, 215 (1966); F. J. Dyson, *Symmetry Groups in Nuclear and Particle Physics* (W. A. Benjamin, Inc., New York, 1966); R. H. Dalitz, in *Proceedings of the Oxford International Conference on Elementary Particles, 1966* (Rutherford High Energy Laboratory, Harwell, England, 1966), and in *High Energy Physics*, edited by C. DeWitt and M. Jacob (Gordon and Breach, Science Publishers, Inc. New York, 1965), contain reviews, reprints, and bibliographies.

[5] F. Gürsey, A. Pais, and L. A. Radicati, Phys. Rev. Letters **13**, 299 (1964).

[6] M. A. B. Bég, B. W. Lee, and A. Pais, Phys. Rev. Letters **13**, 514 (1964); B. Sakita, *ibid.* **13**, 643 (1964).

[7] W. Thirring, Phys. Letters **16**, 335 (1965).

[8] H. J. Lipkin and F. Scheck, Phys. Rev. Letters **16**, 71 (1966).

[9] This saturation is similar to that in molecular physics, where fixed numbers of atoms combine to form molecules. This should be contrasted with the kind of saturation which occurs in solid or liquid bulk matter or in nuclear matter in which the binding energy per particle is roughly constant. No "quark matter" analogous to nuclear matter seems to exist.

[10] The antisymmetric $SU(6)$ states would be analogous to the ground states of the lightest nuclei. See discussion of type 1b.

[11] O. W. Greenberg, Phys. Rev. **147**, 1077 (1966), discusses this problem for S states in a potential.

[12] The relevant theorem [R. Courant and D. Hilbert, *Methods of Mathematical Physics* (Interscience Publishers, Inc., New York, 1953), p. 452] applies in the case of a self-adjoint Schrödinger equation with a local potential: If the bound-state wave functions are ordered according to increasing energy, then the nodes of the nth wave function divide configuration space into no more than n disjoint regions. In particular, the lowest wave function is nodeless, and therefore all the others, which must be orthogonal to it, have nodes. Note that it is not necessary that a wave function which divides space into n regions lie higher than one which divides space into fewer than n regions. In addition, the Pauli principle might exclude the nodeless solution. However, for the special case of the one-dimensional Sturm-Liouville problem, the nth eigenfunction divides the fundamental domain into precisely n parts by means of its nodal points (see p. 454).

TABLE I. Types of saturation.

Pauli saturation	
1. internal	a-absolute b-known quantum numbers
2. orbital	a-with a core b-without a core
3. accidental	
Coulomb saturation	
4. generalized charge	a-additive quantum number b-nonadditive quantum number
5. hard core	
6. accidental	

pair of particles interacting via a space exchange potential which is repulsive in even space states and attractive in odd ones. Here any bound states which occur will have at least one node.

2. TYPES OF SATURATION

Now we discuss some mechanisms of saturation. They fall into two categories which we call "Pauli saturation" and "Coulomb saturation." Pauli saturation is the mechanism which operates when a given set of states is filled by the maximum number of Fermi or para-Fermi particles which can enter. Coulomb saturation operates when the number at which saturation occurs is determined by the scheme of forces among the particles, rather than by the permutation symmetry of the state. Table I lists the types of saturation which we will describe. These types are not disjoint.

Type 1: Internal saturation occurs when states labeled by internal quantum numbers are filled. Type 1a: A direct way to achieve saturation at three is to introduce a new three-valued internal quantum number[13] so that there are three sets of triplets, or altogether, nine fundamental particles. If the triplets are fermions, then requiring an antisymmetric three-particle wave function in the new quantum number leads to a symmetric wave function in the other quantum numbers, and for a symmetric three-particle S state the **56** follows. For the known hadron states, this threefold degenerate quark model is qualitatively the same as the order-three para-Fermi quark model,[14] as can be seen from Green's ansatz. However, these two models differ in the number of different quarks: The paraquark model has only three. Type 1b: Filling of states labeled by known internal quantum numbers is the familiar type which operates in atomic and nuclear physics, for example, the two-electron $SU(2)$ singlet for the atom, or the four-nucleon $SU(4)$ singlet for the α particle. The analog of these for $SU(6)$ would be a six-particle singlet. Type 2a: Orbital saturation occurs at $2l+1$ for fermions in an orbital l shell around a core. Type 2b: Without a core, the l shell saturates at $2l+2$, since the wave function for n particles depends on only $n-1$ independent relative coordinates, and the coordinates of one of the particles may be used kinematically to play the role of a core. Simple examples are: for symmetric S waves, 1; for antisymmetric P waves, $\mathbf{r}_{12}$, $\mathbf{r}_{12}\times\mathbf{r}_{13}$, $\mathbf{r}_{12}\times\mathbf{r}_{13}\cdot\mathbf{r}_{14}$, for 2, 3, and 4 particles, respectively. We took particle 1 as the core particle; however, these last three functions are totally antisymmetric under all permutations of particles because of the linear dependence of the relative coordinates. Here, as well as later in this article, we omit a function of the scalar products which is symmetric under particle permutations and which decreases exponentially for large magnitude of any relative coordinate vector. Because of this factor, the pairs are not in pure relative l waves, so our analysis applies only to the dominant terms. Note that $\mathbf{r}_{12}\times\mathbf{r}_{23}\cdot\mathbf{r}_{31}$ vanishes identically, so that there is no $L=0$ antisymmetric state of three particles in the same P shell. Type 3: Pauli-accidental saturation occurs when the antisymmetric wave functions are formed with particles in different principal quantum states. For example, n fermions can be put into S states with principal quantum numbers $1S$, $2S$, $\cdots$, nS.[15] Some potentials might lead to saturation in such a state, but we call this case accidental because detailed study of the relevant bound-state equation is needed to decide whether saturation occurs.

Type 4: In Coulomb saturation by a generalized charge, there are forces depending on discrete quantum numbers whose eigenvalues characterize, *a priori*, the lowest lying compound systems. Type 4a: The additive case includes Coulomb forces: a particle with charge Ze will not bind strongly more than Z particles with charge $-e$, regardless of their statistics. For some two-triplet models of baryons,[16] "charm" replaces charge and the zero-charm states are the saturated systems. Type 4b: A nonadditive quantum number can also characterize saturation, for example, the Casimir operator C_2 in a three-triplet model.[13] Type 5: Simple geometry shows that finite-range hard-core potentials lead to saturation. For example, if a particle of type A attracts particles of type B via a finite-range potential and the B particles repel each other with a hard-core potential, then there is an upper limit to the number of B particles which can be bound to A. Type 6: We include in the Coulomb-

[13] Y. Nambu, in *Preludes in Theoretical Physics*, edited by A. De-Shalit, H. Feshbach, and L. Van Hove (North-Holland Publishing Company, Amsterdam, 1966), pp. 133–142; M. Y. Han and Y. Nambu, Phys. Rev. **139**, B1006 (1965); N. N. Bogolyubov *et al.*, Dubna Reports D1968, D2015, D2141, 1965 (unpublished), cited in A. Pais, Ref. 4.

[14] O. W. Greenberg, Phys. Rev. Letters **13**, 598 (1964).

[15] It can be shown, provided the potential is regular at the origin, that baryons constructed in this way in the quark model have a wave function at the origin which is of degree 6, $(\mathbf{r}_{12}^2-\mathbf{r}_{23}^2)\times(\mathbf{r}_{23}^2-\mathbf{r}_{31}^2)(\mathbf{r}_{31}^2-\mathbf{r}_{12}^2)$, or greater.

[16] Y. Nambu, in *Symmetry Principles at High Energy*, edited by B. Kurşunoğlu, A. Perlmutter, and I. Sakmar (W. H. Freeman and Company, San Francisco, 1965), pp. 274–283; Y. Nambu, Ref. 13; H. Bacry, J. Nuyts, and L. Van Hove, Phys. Letters **9**, 279 (1964); **12**, 285 (1965); Nuovo Cimento **35**, 510 (1965); L. Van Hove, Progr. Theoret. Phys. (Kyoto) Suppl., p. 14 (1965).

accidental category those systems with forces which lead to saturation, but for which the number at which saturation occurs cannot be found *a priori*. An example of this category is obtained if in the discussion of type 5 above we replace the AB and BB potentials by attractive and repulsive Yukawa potentials, respectively. The Pauli and Coulomb mechanisms can combine to produce saturation; for example, the neutrality of atoms is due to Coulomb saturation, while chemical inertness of the (neutral) rare gases is due to Pauli saturation.

We remind the reader that the problem of saturation is removed if no triplets are present and the fundamental symmetry of particles is $SU(3)/Z_3$. However, it is not clear how to account for the successes of $SU(6)$ with this group.

3. SURVEY OF MODELS

We will now apply simple symmetry and energy agruments to a number of models and see if they saturate via one of the mechanisms described above.

A. Quark Model

The quark model[1,17,18] is the most economical, and the easiest to study. Among the three quark states, the symmetric **56** representation will be lowest if the two-body potential is attractive in the **21** state and repulsive in the **15**. The Pauli principle requires an antisymmetric space wave function. The lowest available orbital shell is the P shell which saturates with four particles rather than three, and which does not have an antisymmetric $L=0$ three-particle wave function. (See discussion of type 2b above.) There is an antisymmetric $L=1$ three-particle P-shell wave function, but $L=1$ will upset the proton/neutron magnetic-moment ratio, and the $L=1$ coupled to the $S=\frac{1}{2}, \frac{3}{2}$ of the **56** no longer produces a true **56**.[19] Accidental saturation could occur as under the discussion of type 3 above.

From the standpoint of energy, consider the phenomenological mass formula

$$M(N)=NM_q+\tfrac{1}{2}N(N-1)U, \quad 1\leq N\leq 4,$$

where M_q is the quark mass, and U includes the effect of two-body potentials as well as kinetic-energy changes due to binding. From $M(3)<M(1)$, we find $U<-\frac{2}{3}M_q$, and $M(4)<0$, so saturation occurs at 4 rather than 3.[20] Thus both symmetry and energy arguments indicate that the quark model fails.[21]

Further restrictions are obtained by applying the saturation requirement to the mesons which in the quark model are $q\bar{q}$ bound states in the **35** representation. For example, the $qq\bar{q}$ system in the **120** representation has three attractive pairs since both $q\bar{q}$ pairs are in the **35** and the qq pair is in the symmetric **21**. Such a state would lie too low unless the potentials are chosen properly: for example, a repulsive $qq\bar{q}$ three-body force, or a qq hard core which prevents the $q\bar{q}$ force from forming two bonds.

B. Quark-plus-Singlet-Core Model

The model with one triplet and an $SU(6)$ singlet core[22,23] avoids some difficulties of the quark model. There is an antisymmetric three-particle $L=0$ state, whose wave function is $\mathbf{r}_1\times\mathbf{r}_2\cdot\mathbf{r}_3$ (where $\mathbf{r}_j$ is the distance of particle j to the core), which saturates the P shell. (See discussion of type 2a above.) One can make this state lie lowest and make more complicated single-centered systems unbound if the forces between quarks (q) and cores (c) are: attractive in the P wave for $c-q$, attractive in **21** and repulsive in **15** for $q-q$, and repulsive for $c-c$. Thus it allows saturation.[24]

C. Two-Triplet Model

The two-triplet model[13,16] can be made to saturate using type 4a. Let the triplets[25] be t_1 and t_2 with masses M_1 and M_2, and charms C_1 and C_2. Coulomb-like forces among them give the mass formula

$$M\{\alpha_i\}=\sum_i M_{\alpha_i}+\sum_{i>j} C_{\alpha_i}C_{\alpha_j}V=\sum_i m_{\alpha_i}+\tfrac{1}{2}C^2V, \quad \alpha_i=1,2,$$

where the effective mass $m_\alpha=M_\alpha-C_\alpha{}^2V$ and the total charm $C=\sum_i C_{\alpha_i}$. If $m_\alpha\ll V$ then "neutral" systems with $C=0$ will lie lowest, and for these systems

[17] Y. Nambu in Ref. 16; T. S. Kuo and L. M. Radicati, Phys. Rev. **139**, B746 (1965); L. A. Radicati, Cargèse Lectures, 1965 (to be published); G. Morpurgo, Physics **2**, 95 (1965); R. H. Dalitz in Ref. 4; and A. N. Mitra, Phys. Rev. **142**, 1119 (1966).

[18] K. Kinoshita and Y. Kinoshita, Progr. Theoret. Phys. (Kyoto) **35**, 330 (1966) have also studied saturation in the quark model and have found similar conclusions.

[19] These objects apply to the model of A. N. Mitra, Ref. 18.

[20] This conclusion no longer holds if one uses the formula

$$M(N)=NM_q+\tfrac{1}{2}N(N-1)U_2+\tfrac{1}{6}N(N-1)(N-2)U_3, \quad 1\leq N\leq 4,$$

which may crudely account for the effects of three-body potentials. Now saturation can occur at three, provided $U_2<-\frac{5}{6}M_q$ and $U_3>\frac{1}{2}M_q$; i.e., attractive two-body forces to bind three quarks into baryons and repulsive three-body forces to effect saturation at three. Kuo and Radicati, Ref. 17, introduce instead three-body forces, attractive in the **56**, which bind three quarks into baryons but lead to four-quark states that are much more tightly bound. In particular, their V_3, Eq. (14), p. B748, has the value $6V$ ($V<0$) for the symmetric and antisymmetric three-quark states and $24V$ for the corresponding four-quark states. Thus their model does not saturate at three.

[21] We are not convinced by the arguments for saturation given in G. Morpurgo, Phys. Letters **20**, 684 (1966).

[22] F. Gürsey, T. D. Lee, and M. Nauenburg, Phys. Rev. **135**, B467 (1964), Appendix IV, model II. Only a neutral core and fractionally charged quarks will give the observed magnetic-moment ratio, as shown by M. A. B. Bég and A. Pais, Phys. Rev. **137**, B1514 (1965).

[23] P. G. O. Freund and B. W. Lee, Phys. Rev. Letters **13**, 592 (1964). The singlet core in this model is the internally saturated 6-particle S-shell antisymmetric $SU(6)$ singlet. It is arbitrary to assume, as in this model, that the next shell should saturate via orbital (P-shell) saturation, and have a totally symmetric $SU(6)$ wave function.

[24] However the core-core interaction would upset the results of Ref. 8.

[25] Here we follow Nambu, Ref. 13. In the model of Van Hove, Ref. 16, $\bar{\theta}\sim t_1$, $T\sim t_2$, and supercharge$\sim-$charm.

$M=\sum_i m_{\alpha_i}$. We expect more complicated systems to be multiple-centered, in analogy with collections of neutral atoms. The assignment $C_1=2$, $C_2=-1$ leads to $(t_1t_2t_2)$ for the lowest baryons. Let $\mathbf{r}_1$ and $\mathbf{r}_2$ be the distances of the t_2's from the t_1. For the **56**, the Pauli principle requires antisymmetry in $\mathbf{r}_1$ and $\mathbf{r}_2$. However, there is no antisymmetric S state with both t_2's in the same orbital shell relative to t_1. The lowest degree antisymmetric wave function with the t_2's in the same orbital shell is $\mathbf{r}_1\times\mathbf{r}_2$, which has $L=1$. However, the antisymmetrized $1S$, $2S$ wave function $\mathbf{r}_1^2-\mathbf{r}_2^2$ has the same degree, and if the forces are attractive (repulsive) for a pair in the **21** (**15**) the **56** can be the lowest three-particle state. An analogous situation occurs in orthohelium for which the electron spin state is symmetric: the lowest state is the $1S$, $2S$ with $1S$, $2P$ nearby, and the $2P$, $2P$ state, analogous to $\mathbf{r}_1\times\mathbf{r}_2$ above, lies much higher. Thus this model can saturate properly.[26]

D. Paraquark and Three-Triplet Models

The order three-paraquark[14] and three-triplet[13] models yield the same baryon states; in particular, both models require a symmetric three-particle joint orbital and $SU(6)$ wave function (see discussion of type 1a above). Paraquarks must have fractional charge, and among the three-paraquark states only the symmetric one is an effective fermion. If the Green components of the paraquark model are taken to be independent fields and are Klein transformed, then, as far as the known hadrons are concerned, the paraquark model can be considered a special case of the three-triplet model. Using the (three-valued) $SU(3)''$ degree of freedom, one can make pairs of particles in the antisymmetric $SU(3)''$ state attract and pairs in the symmetric one repel,[13] so that saturation will occur at three and the orbital-$SU(6)$ wave function will be symmetric.[27] By proper choice of such forces, more complicated single-centered systems can be made unbound.

The states produced by low orbital excitation in the paraquark model have been tabulated[14]; these same states are also relevant for orbital excitation in the three-Fermi-triplet model.[28]

In the model of Han and Nambu,[13] an octet of gauge fields couples to the $SU(3)''$ generators $\frac{1}{2}\lambda_\mu''$ of the triplets and leads to the mass formula

$$M(N)=NM_t+\tfrac{1}{4}g^2\sum_{\mu=1}^{8}\sum_{\substack{i,j=1\\ i<j}}^{N}\lambda_\mu''^{(i)}\lambda_\mu''^{(j)}$$
$$=Nm_t+\tfrac{1}{2}g^2C_2,$$

where $m_t=M_t-\frac{1}{2}g^2C_{20}$ $(C_{20}=\frac{4}{3})$ is the effective mass, and

$$C_2=\tfrac{1}{4}\sum_{\mu=1}^{8}\Big[\sum_{n=1}^{N}\lambda_\mu''^{(n)}\Big]^2$$

is the quadratic Casimir operator for $SU(3)''$. If $m_t\ll g^2$, the lowest-lying states will be those for which $C_2=0$ [i.e., the singlets of $SU(3)''$, all of which have $SU(3)$ triality 0], which here play the role of "neutral" systems. Note, however, that C_2 is not an additive quantum number. Thus this model makes essential use of saturation mechanism 4b.

In conclusion, although we have shown how several models can be made to satisfy the saturation criteria, it remains a challenge to show how, if at all, these criteria can be satisfied in the quark model.[29]

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[26] The proton/neutron magnetic-moment ratio is not determined uniquely here; however, the experimental value can be gotten with a reasonable value of the free parameter. See Van Hove, Ref. 16.

[27] This can be done directly in the paraquark model.

[28] Dalitz, Ref. 4, suggested that a universal short-range repulsion between baryons might result from overlapping wave functions in the paraquark model. In view of the fact that the Pauli principle alone does not suffice for saturation in nuclei, it is not clear that overlapping wave functions, without suitable dynamics, will give such a repulsion. Proper choice of forces in the models which saturate, i.e., core-core repulsion, t_2-t_2 repulsion, or repulsion in $SU(3)''$ symmetric pairs, will give such a repulsion.

[29] S. Meshkov and H. Lipkin have independently studied saturation in triplet models [S. Meshkov (private communication)].