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Classical Theory of Magnetic Monopoles*

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A classical system of n electric and n^* magnetic point charges is considered. The field equations (Maxwell-Lorentz equations, suitably generalized) and the particle equations are obtained by postulating duality invariance and coherence with the theory of only electric point charges. The particle equations together with the solutions of the field equations yield the (generalized) Lorentz-Dirac equations including radiation reaction. The question is then raised whether this system of equations can be derived from an action principle, as is the case for only electric or only magnetic charges. It is shown that the particle equations can be derived only from a nonlocal action integral. If an electric and a magnetic point charge are allowed to meet (crossing of world lines), they must do so with equal velocity (in magnitude and direction) at the instant of crossing. An action integral from which the field equations can be derived is not difficult to obtain, but it is proven that no action integral exists from which both the particle equations and the field equations can be derived. Nevertheless, there exist a local symmetric energy tensor and a corresponding angular momentum tensor which yield ten conservation laws when the field and particle equations hold.

1. INTRODUCTION

THE ingenious suggestion by Dirac[1] that the existence of a magnetic monopole would lead to a quantization of electric charge gave rise to a considerable literature on the subject.[2] After recent failure to detect such a particle this literature has turned partially negative, trying to cast doubt on the existence of such a theory. A notable exception is the recent work by Schwinger.[3] The latter's work makes the present failure to observe the monopole less serious, since his quantization leads to a quantization[4] $\alpha^*\alpha=n^2$ instead of Dirac's $n^2/4$. Here $\alpha=e^2/4\pi$, $\alpha^*=e^{*2}/4\pi$, and e and e^* are the electric and magnetic fundamental charges. If the classical electron radius and the classical monopole radius are about equal, the ratio of the monopole mass m^* to the electron mass m is found to be

$$m^*/m\sim\alpha^*/\alpha=(n/\alpha)^2 \tag{1.1}$$

according to Schwinger's quantization. This is four times larger than the ratio in Dirac's quantization, yielding $m^*\sim 9.6$ BeV for $n=1$. The present experimental limit is only a fraction of this.

All the negative papers[5–7] are studies in quantum field theory or S-matrix theory and are consequently subject to the same criticism as these theories themselves at the present state of the art. It is therefore not unreasonable to raise this question on the classical level where the situation is fully understood and where there are no divergences and no renormalizations for point-particle electrodynamics.[8]

It is our philosophical belief that quantum electrodynamics has a demonstrable classical limit (never proven) and that consequently the nonexistence of a classical theory of magnetic point charges would imply the nonexistence of the corresponding quantum electrodynamics. Conversely, the existence of a classical theory does not necessarily ensure a corresponding quantized theory.

Specifically, we wish to consider a system of n electric point charges e_k ($k=1, \cdots, n$) and n^* magnetic point charges e_l^* ($l=1, \cdots, n^*$) in interaction with each other by means of the electromagnetic field they produce and subject to incident radiation F_{in} as well as possibly an externally controlled field F_{ext}. This theory is to be Lorentz-invariant and, as long as electric and magnetic charges do not occur on the same particle, also P-, C-, and T- invariant.

The construction of the theory proceeds in two steps. The first is the establishment of the basic field equations and particle equations. This part is easy, especially if one postulates a certain invariance [duality invariance, Eqs. (2.5) and (2.6)] and coherence to the classical theory of point charges, in order to resolve various ambiguities of definitions and sign choices. By "coherence" we mean that the theory should reduce to the usual theory in the limit of all $e^*=0$; it also means that the field of a single positive magnetic charge approximates that of the positive pole of a magnetic dipole. This is done in Secs. 2 and 3.

The second step in the theory construction consists in ensuing its Lorentz-invariance properties by providing the infinitesimal generators of that group and the associated conservation laws. This can be done most easily by exhibiting a Lorentz-invariant action integral whose Euler-Lagrange equations are the fundamental equations obtained in step one. Noether's theorem will then provide the generators and the conservation laws. This is studied in Secs. 4 and 5.

The result of our investigation is that within the framework which we have set ourselves no action

* Work supported in part by the National Science Foundation.
[1] P. A. M. Dirac, Proc. Roy. Soc. (London) A133, 60 (1931); Phys. Rev. 74, 817 (1948).
[2] A summary up to 1959 can be found in H. Bradner and W. M. Isbell, Phys. Rev. 114, 603 (1959).
[3] J. S. Schwinger, Phys. Rev. 144, 1087 (1966).
[4] We use Heaviside-Lorentz units with $\hbar=c=1$ and a signature +2 of Minkowski space.
[5] D. Zwanziger, Phys. Rev. 137, B647 (1965).
[6] S. Weinberg, Phys. Rev. 138, B988 (1965).
[7] C. R. Hagen, Phys. Rev. 140, B804 (1965).
[8] F. Rohrlich, *Classical Charged Particles* (Addison-Wesley Publishing Company, Inc., Reading, Massachusetts, 1965).

integral exists that would provide both the field equations and the equations of motion. This conclusion casts severe doubt on the consistency of the theory of magnetic charges with the Lorentz group. However, this problem is resolved in Sec. 6, where the Lorentz invariance of the theory is proven.

2. THE FIELD EQUATIONS

The Maxwell-Lorentz equations specify the fields **E** and **H** produced by the electric source-current density **j** and electric source-charge density ρ. They consist of the inhomogeneous equations[4]

$$\nabla\cdot\mathbf{E}=\rho, \tag{2.1}$$

$$\mathrm{P}\times\mathbf{H}-\dot{\mathbf{E}}=\mathbf{j}, \tag{2.2}$$

as well as the homogeneous equations

$$\nabla\cdot\mathbf{H}=0 \quad\text{and}\quad \nabla\times\mathbf{E}+\dot{\mathbf{H}}=0.$$

These latter equations will no longer hold if magnetic charge densities ρ^* and current densities $\mathbf{j}^*$ are present.

We now assume that *positive magnetic charges are sources of* ***H*** *while negative ones are sinks.* This assumption permits one to construct a magnetic dipole as a combination of a positive and a negative monopole so that it would be indistinguishable from that produced by a suitable electric current. This determines the signs on the right side of the following equations:

$$\nabla\cdot\mathbf{H}=\rho^*, \tag{2.3}$$

$$\nabla\times\mathbf{E}+\dot{\mathbf{H}}=-\mathbf{J}^*. \tag{2.4}$$

The system (2.1) to (2.4) is invariant under either one of the following two substitutions (duality invariance):

$$\begin{array}{ll}\text{(A)} & \mathbf{E}\to-\mathbf{H}\\ & \mathbf{H}\to\mathbf{E}\\ & e\to-e^*\\ & e^*\to e\end{array} \tag{2.5}$$

$$\begin{array}{ll}\text{(B)} & \mathbf{E}\to\mathbf{H}\\ & \mathbf{H}\to-\mathbf{E}\\ & e\to e^*\\ & e^*\to -e\end{array} \tag{2.6}$$

Here we anticipate the point-charge assumption which implies that **j** and ρ contain a factor e while $\mathbf{j}^*$ and ρ^* contain a factor e^*.

In order to write these equations in manifestly covariant form we define $H_i=\epsilon_{ijk}F^{jk}$, $E_i=F^{0i}$, $j^\mu=(\rho,\mathbf{j})$, $\mathbf{j}_*{}^\mu=(\rho^*,\mathbf{j}^*)$ and find

$$\partial_\mu F^{\mu\nu}=-j^\nu, \tag{2.7}$$

$$\partial_\mu F_*{}^{\mu\nu}=+j_*{}^\nu. \tag{2.8}$$

Here $F_*{}^{\mu\nu}$ is just the dual of $F^{\mu\nu}$:

$$F_*{}^{\mu\nu}=F_D{}^{\mu\nu}, \tag{2.9}$$

$$F_D{}^{\mu\nu}\equiv\tfrac{1}{2}\epsilon^{\mu\nu\alpha\beta}F_{\alpha\beta}. \tag{2.10}$$

The four-dimensional Levi-Civita symbol is defined to vanish unless all indices differ, $\epsilon_{0123}=1=-\epsilon^{0123}$, and is completely antisymmetric.

The invariance characterized by (2.5) and by (2.6) can now be expressed by

$$\begin{array}{ll}\text{(A)} & F\to F_D\\ & F_D\to-F\\ & j\to-j^*\\ & j^*\to j\end{array} \tag{2.5'}$$

$$\begin{array}{ll}\text{(B)} & F\to-F_D\\ & F_D\to F\\ & j\to j^*\\ & j^*\to -j\end{array} \tag{2.6'}$$

We note that for both substitutions

$$jF\leftrightarrow-j^*F_D. \tag{2.11}$$

Since we no longer have homogeneous field equations, potentials can no longer be introduced by expressing F as the curl (in four-space). This is well known but does not prevent one from introducing potentials altogether, as was pointed out by Cabbibo and Ferrari.[9] One can define two four-vectors A^μ and B^μ by

$$F_{\mu\nu}=A_{\mu\nu}-B_{\mu\nu}{}^D, \tag{2.12}$$

so that

$$F_{\mu\nu}{}^*=B_{\mu\nu}+A_{\mu\nu}{}^D. \tag{2.12'}$$

Here

$$A_{\mu\nu}\equiv\partial_\mu A_\nu-\partial_\nu A_\mu,\quad B_{\mu\nu}\equiv\partial_\mu B_\nu-\partial_\nu B_\mu. \tag{2.13}$$

If one restricts these potentials by the Lorentz condition

$$\partial_\mu A^\mu=0,\quad \partial_\mu B^\mu=0, \tag{2.14}$$

the field equations (2.7) and (2.8) reduce to

$$\partial^\nu A_{\nu\mu}=\Box A_\mu=-j_\mu, \tag{2.15}$$

$$\partial^\nu B_{\nu\mu}=\Box B_\mu=+j_\mu{}^*. \tag{2.16}$$

In this way we can speak of the field F as consisting of two parts which superpose linearly:

$$F=F_A+F_B,\quad F_A{}^{\mu\nu}\equiv A^{\mu\nu},\quad F_B{}^{\mu\nu}\equiv-B_D{}^{\mu\nu}. \tag{2.17}$$

The A field has its source in the electric current j, the B field in the magnetic current j^*. This will make the following consideration a great deal easier to carry through than would otherwise be the case.

3. THE PARTICLE EQUATIONS

The Lorentz-force equation for an electrically charged particle in an *external* field is

$$m\dot{\mathbf{v}}=e(\mathbf{E}+\mathbf{v}\times\mathbf{H}). \tag{3.1}$$

In order to obtain the analogous equation for a magnetic charge we assume *that the substitutions (A) and (B), Eqs. (2.5) and (2.6), which leave the field equations invariant also leave the particle equations invariant.*

[9] N. Cabbibo and E. Ferrari, Nuovo Cimento **23**, 1147 (1962).

Only the special substitution (2.11) is actually necessary for Eq. (3.1). Thus, one finds

$$m^*\dot{\mathbf{u}}=e^*(\mathbf{H}-\mathbf{u}\times\mathbf{E}). \quad (3.2)$$

The manifestly covariant form of these equations is (the dot now refers to differentiation with respect to proper time)

$$m\dot{v}^\mu=eF^{\mu\nu}v_\nu, \quad (3.3)$$

$$m^*\dot{u}^\mu=-e^*F_*{}^{\mu\nu}u_\nu, \quad (3.4)$$

where F^* is the dual according to (2.9).

The connection between the particle equations and the field equations is obtained when the charges e and e^* act as sources of a field. For this purpose one needs the associated currents of these point charges,

$$j^\mu(x)=e\int\delta(x-z)v^\mu dt, \quad (3.5)$$

$$j_*{}^\mu(x)=e^*\int\delta(x-y)u^\mu dt. \quad (3.6)$$

The world lines of e and e^* are here denoted by $z^\mu(t)$ and $y^\mu(t)$, respectively.

In order that the particle equations (3.3) and (3.4) be generally valid, the force on the right side of these equations must be caused by the external field as well as the field due to the other particles in the system (mutual e-e and e-e^* interactions). It must also contain the particles' self-interaction, which gives rise to the radiation reaction force. There is no static self-interaction because the masses m and m^* are by definition the physical rest masses of the particles and the theory will be so constructed that such an interaction cannot arise.[8]

Let us therefore now envision a system consisting of a finite number of electric and magnetic point charges e_k and e_l^* and the electromagnetic fields produced by them, all under the influence of an incident radiation field F_{in} (given at $t=-\infty$) and an externally controlled field F_{ext}. This system is an open system because of the presence of F_{ext}. When $F_{\text{ext}}=0$ the system is closed.

In this general case of many particles, the field F at a point on the world line of a particle of charge e consists of five different parts:

$$F=F_{\text{ext}}+F_{\text{in}}+F_{\text{self}}+F_{A,\text{ret}}{}^{\text{other}}+F_{B,\text{ret}}. \quad (3.7)$$

The external (controlled) and the incident (radiation) field are both assumed as given without knowledge of their sources. F_{self} is the field produced by e, but only the radiation reaction part, i.e., the free field

$$F_-\equiv\tfrac{1}{2}(F_{\text{ret}}-F_{\text{adv}}) \quad (3.8)$$

will contribute to the particle equation.[8] $F_{A,\text{ret}}{}^{\text{other}}$ is the (retarded) field of all other electric charges in the system; F_B is the (retarded) field of all magnetic charges in the system. The interrelation between the particle equations and the field equations (2.15) and (2.16) must be completed by the specification of the point currents

$$j^\mu(x)=\sum_{k=1}^{n} j_k{}^\mu(x), \qquad j_k{}^\mu(x)=e_k\int\delta(x-z_k)v_k{}^\mu d\tau; \quad (3.9)$$

$$j_*{}^\mu(x)=\sum_{k=1}^{n^*} j_k{}^{*\mu}(x), \quad j_k{}^{*\mu}(x)=e_k{}^*\int\delta(x-y_k)u_k{}^\mu d\tau. \quad (3.10)$$

In analogy to (3.7) the field F_* at a point on the world line of a magnetic point charge e^* can be written as the sum of the following five parts:

$$F_*=F_{\text{ext}}{}^D+F_{\text{in}}{}^D+F_{\text{self}}{}^D+F_{A,\text{ret}}{}^D+F_{B,\text{ret}}{}^{D,\text{other}}, \quad (3.11)$$

with obvious meanings analogous to (3.5). We note that the total field (3.11) is *not* the dual of (3.7) but differs by self-field contributions and by the mutual interaction between electric and magnetic charges. In particular, e acts on e^* by means of $F_A{}^D$ produced by e, while e^* acts on e by means of F_B produced by e^*.

The generalization of the Lorentz-Dirac equation to a system of n electric and n^* magnetic point charges is now easy to write down. We introduce the notation

$$\bar{F}\equiv F_{\text{in}}+\tfrac{1}{2}(F_{\text{ret}}{}^s-F_{\text{adv}}{}^s), \quad (3.12)$$

where the superscript s indicates "self-field," and find

$$m_k\dot{v}_k{}^\mu=e_kF_{\text{eff},k}{}^{\mu\nu}v_\nu{}^k, \quad (3.13)$$

$$F_{\text{eff},k}=F_{\text{ext}}+\bar{F}_{(k)}{}^{(1)}+\sum_{l\neq k}^{n}F_{(l)}{}^{A,\text{ret}}+\sum_{l=1}^{n^*}F_{(l)}{}^{B,\text{ret}}, \quad (3.14)$$

$$m_k{}^*\dot{u}_k{}^\mu=-e_k{}^*F_{\text{eff},k}{}^{\mu\nu*}u_\nu{}^k, \quad (3.15)$$

$$F_{\text{eff},k}{}^*=F_{\text{ext}}{}^*+F_{(k)}{}^{(2)}+\sum_{l\neq k}^{n^*}F_{(l)D}{}^{B,\text{ret}}+\sum_{l=1}^{n}F_{(l)D}{}^{A,\text{ret}}. \quad (3.16)$$

The subscript in parenthesis indicates the source particle of the field.

The physical system of n electric and n^* magnetic point charges is now completely specified by the differential equations (3.13) to (3.16) for the particle, (2.15) and (2.16) for the fields, by the current–world-line relations (3.9) and (3.10), and by suitable asymptotic conditions which specify the state of the system in the distant past and/or future.

The above equations are best exemplified by the particular case of a system consisting of only one charge e and one charge e^*. This will be of special interest to us since it is the simplest system which contains all the relevant complications introduced by having both electric and magnetic point charges present,

$$m\dot{v}^{\mu}=e[F_{\text{ext}}{}^{\mu\nu}+\bar{F}_{(1)}{}^{\mu\nu}-B_{D,\text{ret}}{}^{\mu\nu}]v_{\nu}, \tag{3.17}$$

$$m^{*}\dot{u}^{\mu}=-e^{*}[F_{\text{ext }D}{}^{\mu\nu}+\bar{F}_{(2)}{}^{\mu\nu}+A_{D,\text{ret}}{}^{\mu\nu}]u_{\nu}. \tag{3.18}$$

Here we used (2.17). We note that

$$\bar{F}_{(1)}{}^{\mu\nu}=F_{\text{in}}{}^{\mu\nu}+\tfrac{1}{2}(A_{\text{ret}}{}^{\mu\nu}-A_{\text{adv}}{}^{\mu\nu}), \tag{3.19}$$

$$\bar{F}_{(2)}{}^{\mu\nu}=F_{\text{in},D}{}^{\mu\nu}+\tfrac{1}{2}(B_{\text{ret}}{}^{\mu\nu}-B_{\text{adv}}{}^{\mu\nu}). \tag{3.20}$$

The index s is here unnecessary, since there is only one field A and only one field B.

The characteristic features of the Lorentz-Dirac equation become explicit when the Liénard-Wiechert solutions of (2.15) and (2.16) with the sources (3.5) and (3.6) are substituted:

$$e\bar{F}_{(1)}{}^{\mu\nu}v_{\nu}=eF_{\text{in}}{}^{\mu\nu}v_{\nu}+(e^{2}/6\pi)(\ddot{v}^{\mu}-\dot{v}_{\alpha}\dot{v}^{\alpha}v^{\mu}), \tag{3.21}$$

$$-e^{*}\bar{F}_{(2)}{}^{\mu\nu}u_{\nu}=-e^{*}F_{\text{in},D}{}^{\mu\nu}u_{\nu}+(e_{*}{}^{\nu}/6\pi)(\ddot{u}^{\mu}-\dot{u}_{\alpha}\dot{u}^{\alpha}u^{\mu}). \tag{3.22}$$

Note that the radiation reaction terms in these two equations have the *same* sign (as must be the case physically) because the minus sign on the right of (3.18) and the minus sign by which (2.7) and (2.8) differ compensate each other.

We must now recall that these particle equations are really *not* the equations of motion since they contain higher derivatives than the second in the particle positions. The equations of motion are obtained from the generalized Lorentz-Dirac equations (3.17) through (3.22) by taking into account the asymptotic postulate of the theory which specifies free particles in the limit $|t|\to\infty$. One then obtains integro-differential equations as in the case of electric charges only.[10]

4. CONSTRUCTION OF A VARIATIONAL PRINCIPLE FOR THE PARTICLE EQUATIONS

The question now arises whether the fundamental equations of the theory, viz., the field equations and the particle equations connected by the current equations can be derived from a variational principle. To this end it is again sufficient to consider the two-particle system consisting of a change e of mass m and a charge e^* of mass m^*, electric and magnetic, respectively. Also, the presence of an external field is quite irrelevant to this question and we shall therefore assume $F_{\text{ext}}=0$ in this section, i.e., we assume that we are dealing with a closed system.

In these considerations we shall of course be guided by the variational principle for a system of electric point charges only,[11] since the present formulation must reduce to it when the magnetic charges are removed.

Since F_{in} is a free field it can certainly be derived as the curl of a potential, $A_{\text{in}}{}^{\mu}$. The equations

$$\begin{aligned} m\dot{v}^{\mu}&=e\bar{F}_{(1)}{}^{\mu\nu}v_{\nu},\\ m^{*}\dot{u}^{\mu}&=-e^{*}\bar{F}_{(2)}{}^{\mu\nu}u_{\nu}, \end{aligned} \tag{4.1}$$

which are (3.17) and (3.18) with $F_{\text{ext}}=0$ and with omission of the mutual interaction, can then easily be obtained from the following variational principle:

$$I_{p}=-m\int(-z'^{\mu}z_{\mu}')^{1/2}d\lambda-m^{*}\int(-y'^{\mu}y_{\mu}')^{1/2}d\lambda+e\int z_{\mu}'\bar{A}^{\mu}d\lambda-e^{*}\int y_{\mu}'\bar{B}^{\mu}d\lambda. \tag{4.2}$$

Here the prime indicates differentiation with respect to λ, and $\bar{A}^{\mu}$ and $\bar{B}^{\mu}$ are the potentials of $\bar{F}_{(1)}$ and $\bar{F}_{(2)}$:

$$\begin{aligned} \bar{F}_{(1)}{}^{\mu\nu}&=\partial^{\mu}\bar{A}^{\nu}-\partial^{\nu}\bar{A}^{\mu},\\ \bar{F}_{(2)}{}^{\mu\nu}&=\partial^{\mu}\bar{B}^{\nu}-\partial^{\nu}\bar{B}^{\mu}. \end{aligned} \tag{4.3}$$

Eventually, by means of the asymptotic conditions these will have to be shown to be of the form

$$\begin{aligned} \bar{A}^{\mu}&=A_{\text{in}}{}^{\mu}+\tfrac{1}{2}(A_{\text{ret}}{}^{\mu}-A_{\text{adv}}{}^{\mu})\\ \bar{B}^{\mu}&=B_{\text{in}}{}^{\mu}+\tfrac{1}{2}(B_{\text{ret}}{}^{\mu}-B_{\text{adv}}{}^{\mu}), \end{aligned} \tag{4.4}$$

where

$$\partial^{\mu}A_{\text{in}}{}^{\nu}-\partial^{\nu}A_{\text{in}}{}^{\mu}=F_{\text{in}}{}^{\mu\nu},\quad \partial^{\mu}B_{\text{in}}{}^{\nu}-\partial^{\nu}B_{\text{in}}{}^{\mu}=F_{\text{in }D}{}^{\mu\nu}. \tag{4.5}$$

The next problem is to add suitable terms to I_p [Eq. (4.2)] so that (4.1) is extended to (3.17) and (3.18) with $F_{\text{ext}}=0$. Such a term for the charge e, say, must necessarily be of the form

$$e\int_{-\infty}^{\infty}z_{\mu}'V^{\mu}d\lambda, \tag{4.6}$$

since only the vector z_{μ}' is available for the interaction of e with other fields or particles, if one does not want higher than second-order equations. But (4.6) yields a term

$$e(\partial^{\mu}V^{\nu}-\partial^{\nu}V^{\mu})v_{\nu}$$

which cannot be identified with $eB_{D}{}^{\mu\nu}v_{\nu}$, because $B_{D}{}^{\mu\nu}$ cannot be written in the form of a curl. Indeed, if it could,

$$\partial^{\lambda}B_{D}{}^{\mu\nu}+\partial^{\mu}B_{D}{}^{\nu\lambda}+\partial^{\nu}B_{D}{}^{\lambda\mu}\equiv-\partial^{\alpha}B_{\alpha\kappa}\epsilon^{\kappa\lambda\mu\nu} \tag{4.7}$$

would have to vanish identically which contradicts (2.16) whenever $j_{\mu}{}^{*}\neq0$.

This argument shows that a *local* Lagrangian does not exist. However, one can proceed by means of the following nonlocal ansatz. We define the four-vector

$$B_{D}{}^{\mu}(x)\equiv\int_{-\infty}^{0}B_{D}{}^{\alpha\beta}(\zeta)\partial_{x}{}^{\mu}\zeta_{\beta}\partial\zeta_{\alpha}d\xi,\quad(\partial\equiv\partial/\partial\xi) \tag{4.8}$$

[10] F. Rohrlich, Ann. Phys. (N.Y.) 13, 399 (1961); see also Ref. 8.
[11] F. Rohrlich, Phys. Rev. Letters 12, 375 (1964); see also Ref. 8.

where the four-vector $\zeta(x,\xi)$ depends on the field point x as well as on the parameter ξ. The latter is characterized by

$$\zeta^\mu(x,0)=x^\mu,\qquad \lim_{\xi\to-\infty}\zeta^\mu(x,\xi)=\text{space-like infinity.} \tag{4.9}$$

This parametrization was used by DeWitt[12] and is also implied in the work of Mandelstam.[13]

The path from x to space-like infinity which is traversed by ζ as ξ varies from 0 to $-\infty$ is defined to avoid all singularities e^*. Since $B_D{}^\mu$ is not singular at the electric charges, their presence on the path is irrelevant.

By differentiation of (4.8) one finds

$$\partial^\mu B_D{}^\nu-\partial^\nu B_D{}^\mu=B_D{}^{\mu\nu}+\int_{-\infty}^0 d\xi\,\partial\zeta_\alpha\partial_x{}^\nu\zeta_\beta\partial_x{}^\mu\zeta_\gamma \times(\partial^\gamma B_D{}^{\alpha\beta}+\partial^\alpha B_D{}^{\beta\gamma}+\partial^\beta B_D{}^{\gamma\alpha}).$$

The parenthesis inside the integral can be expressed in terms of j^* by means of (4.7) and (2.16). Therefore,

$$\partial^\mu B_D{}^\nu-\partial^\nu B_D{}^\mu =B_D{}^{\mu\nu}+\int_{-\infty}^0 d\xi\,\partial\zeta_\alpha\partial{}^\nu\zeta_\beta\partial_x{}^\mu\zeta_\gamma\epsilon^{\alpha\beta\gamma\sigma}j_\sigma{}^*(\zeta). \tag{4.10}$$

But this integral vanishes because the integration path by definition never meets a magnetic charge,[14]

$$\partial^\mu B_D{}^\nu-\partial^\nu B_D{}^\mu=B_D{}^{\mu\nu}. \tag{4.11}$$

This is exactly what one needs for the action integral. We can assume it to contain a term

$$-e\int z_\mu{}'d\lambda\, B_D{}^\mu(z), \tag{4.12}$$

and obtain a term $-eB_D{}^{\mu\nu}v_\nu$ on the right side of the particle equation for e. Similarly, a term of the form

$$-e^*\int y_\mu{}'d\lambda\, A_D{}^\mu(y), \tag{4.13}$$

where A_D is defined by

$$A_D{}^\mu(x)=\int_{-\infty}^0 A_D{}^{\alpha\beta}(\zeta)\partial_x{}^\mu\zeta_\beta\partial\zeta_\alpha d\xi, \tag{4.14}$$

will lead to

$$-e^*A_D{}^{\mu\nu}u_\nu. \tag{4.15}$$

[12] B. S. DeWitt, Phys. Rev. **125**, 2189 (1962).
[13] S. Mandelstam, Ann. Phys. (N.Y.) **19**, 1 (1962).
[14] The nonlocal expression (4.8) permits us therefore to specify exactly under what conditions (4.11) holds without contradicting the existence of a nonvanishing $j_\mu{}^*$ via (4.7) and (2.16): $B_D{}^{\mu\nu}$ is never to be used at points x where $j_\mu{}^*\neq0$

The path in (4.14) is not permitted to meet electric point charges.

The result of these considerations is that the nonlocal, path-dependent action integral

$$I=-m\int(-z'^\mu z_\mu{}')^{1/2}d\lambda-m^*\int(-y'^\mu y_\mu{}')^{1/2}d\lambda +e\int z_\mu{}'d\lambda(\bar{A}^\mu-B_{D,\text{ret}}{}^\mu) -e^*\int y_\mu{}'d\lambda(\bar{B}^\mu+A_{D,\text{ret}}{}^\mu) \tag{4.16}$$

will yield the local particle equations (3.17) and (3.18) of the two-particle system e-e^*.

The following difficulty now arises. It is conceivable that during the time development of the system the two charges e and e^* pass each other arbitrarily closely. As point particles, their world lines can intersect. But $B_D{}^\mu(z)$ is given by the path integral (4.8) which starts at $x=z$ where the charge e is located. If at some instant of time this is also the location of e^*, we violate the condition under which (4.11) was derived, viz., that there never be an e^* on this path.

The difficulty can be resolved as follows. If e^* is on the path of the integral (4.10) it will no longer vanish and will give an extra term to the right side of the particle equation (3.17). In the limit as e^* is met closer and closer to $\xi=0$ one finds for this contribution

$$\lim_{\xi_0\to0}-e\epsilon_\alpha\epsilon^{\alpha\nu\mu\sigma}\int_{\xi_0}^0 j_\sigma{}^*(\zeta)d\xi v_\nu, \tag{4.17}$$

where

$$\epsilon_\alpha\equiv\partial\zeta_\alpha|_{\xi=0}.$$

On expressing $j_\sigma{}^*$ in terms of u_σ one sees that (4.17) will vanish provided $\epsilon^{\alpha\mu\nu\sigma}u_\mu v_\sigma=0$ or, since u_μ and v_ν are unit vectors,

$$u_\mu=v_\mu \tag{4.18}$$

at the instant of coincidence of the two charges. Thus, we can still be assured of the correct particle equations even in the case of crossing world lines provided that at the instant of crossing the two velocity four-vectors of e and e^* are equal. Their world lines must therefore have a common tangent at the point of intersection. Physically, the two particles must both be at rest in some instantaneous inertial system at the instant of intersection.[15]

We conclude this section with the observation that the derivation of the particle equations from the variational principle (4.16) made use, and therefore presupposes, the field equations. These must therefore be derived from I before the particle equations.

[15] A condition equivalent to (4.18) was first derived by a different method by D. Rosenbaum (private communication). I am indebted to Dr. Rosenbaum for informing me of his work.

5. FAILURE OF A VARIATIONAL PRINCIPLE FOR THE COMPLETE THEORY

A variational principle for the full theory of electric and magnetic point charges requires a derivation of the field equations as well as the particle equations from an action integral. Since we have seen that the particle equations can be derived only by giving a very special nonlocal action integral, it remains to be proven that one can adjoin to this action integral suitable field terms which, upon variation of the fields (or potentials), will then yield the desired field equations. We shall now show that this is not possible.

The interaction terms in I, Eq. (4.16), can be written

$$\int j_\mu(x)d^4x[\bar{A}^\mu(x)-B_D{}^\mu(x)] - \int j_\mu{}^* d^4x[\bar{B}^\mu(x)+A_D{}^\mu(x)]. \quad (5.1)$$

The index "ret" can be omitted for the present purpose.

Given these terms in the action integral, we are forced to deal with four independent four-vector potentials (for the purpose of variation). This is (not surprisingly) twice as many as in the case of electric monopoles only. Of course, their curls combine to one single field observable by an electric charge,

$$F^{\mu\nu}=\partial^\mu\bar{A}^\nu-\partial^\nu\bar{A}^\mu-(\partial^\mu B_D{}^\nu-\partial^\nu B_D{}^\mu) \quad (5.2)$$

and an analogous one observable by a magnetic charge,

$$F_*{}^{\mu\nu}=\partial^\mu\bar{B}^\nu-\partial^\nu\bar{B}^\mu+(\partial^\mu\bar{A}_D{}^\nu-\partial^\nu\bar{A}_D{}^\mu). \quad (5.2')$$

If I is to contain no higher than first derivatives of these potentials, then relativistic invariance restricts the field term to integrals bilinear in the field or their duals constructed from them: $\bar{A}^{\mu\nu}$, $\bar{B}^{\mu\nu}$, $A_D{}^{\mu\nu}$, $B_D{}^{\mu\nu}$.

Let us recall the prototype of the variations to be carried out. A variation δG^μ on

$$\int j_\nu G^\nu d^4x-\frac{1}{2}\int G^{\mu\nu}H_{\mu\nu}d^4x, \quad (5.3)$$

where $G^{\mu\nu}$ is the curl of G^μ yields

$$j_\nu+\partial^\mu H_{\mu\nu}=0. \quad (5.4)$$

Consider now the field term containing $\bar{A}^{\mu\nu}$. The variation $\delta\bar{A}^\mu$ will yield a field equation in which j_μ, the factor of $\bar{A}^\mu$ in (5.1), will be the source of the field multiplying $\bar{A}^{\mu\nu}$ in the field term. This field is $A^{\mu\nu}$ according to (2.15). [Equivalently, we could take $F_{\mu\nu}$, following (2.7), but the argument is the same either way.] Therefore, we must have a field term

$$-\frac{1}{2}\int \bar{A}^{\mu\nu}A_{\mu\nu}d^4x. \quad (5.5)$$

$\delta\bar{A}^\mu$ will now yield the correct field equation, but we must also consider the variation δA^μ. Since (5.1) contains $A_D{}^\mu$ we use the identity

$$\bar{A}_D{}^{\mu\nu}A_{\mu\nu}{}^D=-\bar{A}^{\mu\nu}A_{\mu\nu} \quad (5.6)$$

and vary $A_D{}^\mu$. This yields the field equation, following (5.4),

$$-j_\nu{}^*-\partial^\nu\bar{A}_{\mu\nu}=0$$

which is an incorrect equation. To remedy this we need another term in I which contains $A_{\mu\nu}{}^D$ and whose source is $j_\nu{}^*$, viz. $B_{\mu\nu}$, so that we must have the two field terms

$$+\frac{1}{2}\int \bar{A}_D{}^{\mu\nu}A_{\mu\nu}{}^D-\frac{1}{2}\int B^{\mu\nu}A_{\mu\nu}{}^D. \quad (5.7)$$

Variation of $A_\mu{}^D$ now yields

$$-j_\nu{}^*-\partial^\mu(\bar{A}_{\mu\nu}-B_{\mu\nu})=0 \quad (5.8)$$

which is correct according to (2.16) provided we can eventually show that $\bar{A}_{\mu\nu}$ is a free field.

Next we must vary B^μ and, since $B_D{}^\mu$ occurs in (5.1), we write the second term in (5.7) as

$$-\frac{1}{2}\int B_D{}^{\mu\nu}A_{\mu\nu}. \quad (5.9)$$

Its variation yields

$$-j_\nu+\partial^\mu A_{\mu\nu}=0 \quad (5.10)$$

which has the wrong sign, contradicting (2.15). Therefore, the original assumption of having $\bar{A}^{\mu\nu}$ occur in the field term cannot be maintained.

Since an analogous argument can be made starting with $\bar{B}^{\mu\nu}$, we conclude that only $A_{\mu\nu}$ and $B_{\mu\nu}$ or their duals can occur. Now $B_{\mu\nu}{}^D$ must be multiplied by the field produced by j_μ, the factor of $B_\mu{}^D$ in (5.1), so that we must have

$$\frac{1}{2}\int A^{\mu\nu}B_{\mu\nu}{}^D d^4x \quad (5.11)$$

to obtain the correct equation under $\delta B_\mu{}^D$. The variation of $A_D{}^\mu$ in the equivalent term

$$\frac{1}{2}\int A_D{}^{\mu\nu}B_{\mu\nu}d^4x \quad (5.12)$$

will then yield

$$-j_\nu{}^*-\partial^\mu B_{\mu\nu}=0$$

which is again of incorrect sign, contradicting (2.16).

Thus, there is no consistent term that can be adjoined to the interaction term (5.1) to yield the correct field equations.

In order to avoid misunderstanding it should be emphasized that a variational principle can easily be constructed which is consistent and which yields the desired field equations. But such a principle cannot be amended to yield also the correct particle equations. In fact, the action integral

$$I_f=\int j_\mu(x)d^4x\,\bar{A}^\mu+\int j_\mu^*(x)d^4x\,\bar{B}^{\mu\nu}(x)-\frac{1}{2}\int\bar{F}^{\mu\nu}(x)F_{\mu\nu}(x)d^4x, \quad (5.13)$$

where

$$F=\bar{F}+F_+=A-B_D, \quad (5.14)$$

yields the inhomogeneous field equations (2.7) and (2.8) under the variations of $\bar{A}^\mu$ and $\bar{B}^\mu$, and it yields the homogeneous equations

$$\partial_\mu\bar{F}^{\mu\nu}=0 \quad\text{and}\quad \partial_\mu\bar{F}_D{}^{\mu\nu}=0 \quad (5.15)$$

under the variations of A^μ and B^μ.

6. THE CONSERVATION LAWS

The nonexistence of an action integral which gives the differential equations of the theory seems to cast doubt on the existence of the conservation laws associated with Lorentz invariance, and thus also on the Lorentz invariance of the theory. If the theory is to be invariant under the inhomogeneous Lorentz group then it must be possible to exhibit a symmetric energy tensor and an antisymmetric angular-momentum tensor both of which have vanishing divergence as a consequence of the field equations and the particle equations of the theory. This can indeed be done.

Consider the symmetric tensor

$$\Theta^{\mu\nu}=\Theta_p{}^{\mu\nu}+\bar{\Theta}_+{}^{\mu\nu}+\Theta_{A_DB}{}^{\mu\nu}, \quad (6.1)$$

where

$$\Theta_p{}^{\mu\nu}=-m\int\delta(x-z)v^\mu v^\nu d\tau-m^*\int\delta(x-y)u^\mu u^\nu d\tau, \quad (6.2)$$

$$\bar{\Theta}_+{}^{\mu\nu}=\bar{F}^{\mu\alpha}F_{\alpha+}{}^\nu+F_+{}^{\mu\alpha}\bar{F}_\alpha{}^\nu+\tfrac{1}{2}\eta^{\mu\nu}F_+{}^{\alpha\beta}\bar{F}_{\alpha\beta}, \quad (6.3)$$

$$\Theta_{A_DB}{}^{\mu\nu}=A_{+D}{}^{\mu\alpha}B_{\alpha+}{}^\nu+B_+{}^{\mu\alpha}A_{\alpha+D}{}^\nu+\tfrac{1}{2}\eta^{\mu\nu}A_{+D}{}^{\alpha\beta}B_{\alpha\beta}{}^+. \quad (6.4)$$

It is a matter of straightforward computation to prove that

$$\partial_\mu\Theta^{\mu\nu}=0 \quad (6.5)$$

as a consequence of the field equations (2.7), (2.8), (2.15), and (2.16) and of the particle equations (3.7) and (3.18) with $F_{\text{ext}}=0$. The latter can for this purpose be conveniently written in the form

$$m\dot{v}^\mu=e(\bar{F}^{\mu\nu}-B_{+D}{}^{\mu\nu})v_\nu \quad (6.6)$$

$$m^*\dot{u}^\mu=-e^*(\bar{F}_D{}^{\mu\nu}+A_{+D}{}^{\mu\nu})u_\nu, \quad (6.7)$$

where $\bar{F}$ is defined as in (3.12) by

$$\bar{F}=F_{\text{in}}+\tfrac{1}{2}(F_{\text{ret}}-F_{\text{adv}}) \quad (3.12')$$

and

$$A_+=\tfrac{1}{2}(A_{\text{ret}}+A_{\text{adv}}) \quad (6.8)$$

with B_+ being analogous. The total field is $F=\bar{F}+F_+$ and $F_+=A_+-B_+{}^D$ as defined in (5.14).

In order to prove (6.5) one notes first that

$$\partial_\mu\bar{\Theta}_+{}^{\mu\nu}=\bar{F}^{\nu\mu}j_\mu-\bar{F}_D{}^{\nu\mu}j_\mu{}^* \quad (6.9)$$

and that

$$\partial_\mu\Theta_{A_DB}{}^{\mu\nu}=-A_{+D}{}^{\nu\mu}j_\mu{}^*-B_{+D}{}^{\nu\mu}j_\mu \quad (6.10)$$

as a consequence of the field equations. Combining this with the divergence of $\Theta_p{}^{\mu\nu}$ and using (3.5) and (3.6) yields the desired result (6.5) provided the particle equations (6.6) and (6.7) are satisfied.

The energy tensor (6.1) is not symmetric under duality substitutions such as (2.5′) and (2.6′). However, it can be symmetrized in this respect as follows. One notes that

$$\bar{\Theta}_{+D}{}^{\mu\nu}\equiv\bar{F}_D{}^{\mu\alpha}F_{\alpha+D}{}^\nu+F_{+D}{}^{\mu\alpha}\bar{F}_{\alpha D}{}^\nu+\tfrac{1}{2}\eta^{\mu\nu}\bar{F}_D{}^{\alpha\beta}F_{\alpha\beta}{}^{+D} \quad (6.3')$$

satisfies the same divergence relation as $\bar{\Theta}_+{}^{\mu\nu}$, viz.,

$$\partial_\mu\bar{\Theta}_{+D}{}^{\mu\nu}=\partial_\mu\bar{\Theta}_+{}^{\mu\nu}. \quad (6.11)$$

Furthermore, the "dual" to (6.4),

$$\Theta_{AB_D}{}^{\mu\nu}\equiv A_+{}^{\mu\alpha}B_{\alpha+D}{}^\nu+B_{+D}{}^{\mu\alpha}A_{\alpha+}{}^\nu+\tfrac{1}{2}\eta^{\mu\nu}A_+{}^{\alpha\beta}B_{\alpha\beta}{}^{+D} \quad (6.4')$$

satisfies

$$\partial_\mu\Theta_{AB_D}{}^{\mu\nu}=-\partial_\mu\Theta_{A_DB}{}^{\mu\nu}. \quad (6.12)$$

Therefore, the tensor

$$\Theta^{\mu\nu}=\Theta_p{}^{\mu\nu}+\tfrac{1}{2}[\bar{\Theta}_+{}^{\mu\nu}+\bar{\Theta}_{+D}{}^{\mu\nu}]+\tfrac{1}{2}[\Theta_{A_DB}{}^{\mu\nu}-\Theta_{AB_D}{}^{\mu\nu}] \quad (6.13)$$

is also divergence free. But this tensor is symmetric under the duality substitution:

$$\begin{array}{llll}
\text{(A)} & A\to B, & A_D\to B_D & \\
& B\to -A, & B_D\to -A_D & \\
& e\to -e^*, & e^*\to e & \\
& v\leftrightarrow u, & m\leftrightarrow m^* & \\
\text{or} & & & (6.14)\\
\text{(B)} & A\to -B, & A_D\to -B_D & \\
& B\to A, & B_D\to A_D & \\
& e\to e^*, & e^*\to -e & \\
& v\leftrightarrow u, & m\leftrightarrow m^* &
\end{array}$$

which is a generalization of (2.5′) and (2.6′).

Having established the existence of a conserved symmetric energy tensor, it is trivial to give the conserved angular-momentum tensor:

$$J^{\mu\nu\alpha}\equiv x^\mu\Theta^{\nu\alpha}-x^\nu\Theta^{\mu\alpha} \quad (6.15)$$

satisfies

$$\partial_\alpha J^{\mu\nu\alpha}=0. \quad (6.16)$$

The ten conserved quantities are the four components of

$$P^{\mu}\equiv\int\Theta^{\mu\alpha}d^3\sigma_\alpha \tag{6.17}$$

and the six components of

$$J^{\mu\nu}\equiv\int J^{\mu\nu\alpha}d^3\sigma_\alpha; \tag{6.18}$$

the integrations being over a space-like surface. P^μ and $J^{\mu\nu}$ are a four-vector and an antisymmetric tensor, since the integrals are independent of σ.

7. DISCUSSION

In Secs. 2 and 3 the basic differential equations for any electrodynamic system consisting of a finite number of electric and magnetic point charges were given. These equations are formally Lorentz covariance but they do not suffice for the theory to be Lorentz invariant: It must be possible to ensure the ten conservation laws associated with Lorentz invariance. To this end it was attempted in Secs. 4 and 5 to construct an action integral which implies the equations of the theory and which would also give the conservation laws. This attempt failed and it was proven that an action integral which provides *all* the basic equations of the theory does not exist.[16]

An action integral that leads to the particle equations was constructed in Sec. 4. It is necessarily nonlocal. But it is completely general. While only the system e-e^* was considered explicitly, the action integral (4.16) can easily be extended to describe arbitrary numbers n and n^* of electric and magnetic point charges. The generalization proceeds as in the case of electrically charged particles[8] by a Fokker-type term for the interaction between the e_k and similarly between the $e_l{}^*$.

[16] Clearly, the relative sign of the integrals in (5.1) is crucial. If this sign is changed to conform with the relative signs in (5.13) a suitable field term *can* of course be found; but then the relative sign of the mass terms in (4.16) must also be changed in order to assure the correct particle equations. An action integral with mass terms of opposite signs however does not have the correct limit as the interaction vanishes and violates the positive definiteness of energy of a physically meaningful system.

A peculiar problem arises when two world lines belonging to one electric and one magnetic charge intersect. Whether this problem needs to be faced at all is not entirely clear. One can argue that if there are indeed initial conditions which lead to such intersections (which is not obvious), an infinitesimal modification will avoid it. Alternatively, one can point to the inapplicability of the classical theory for very small distances: One can endow each particle with very strong, very short-range repulsive forces so that particles never approach each other closer than the classical electron radius, say. But if world lines do intersect then, as was shown in Sec. 4, they must do so with equal velocity (in magnitude and direction) at the instant of intersection.

One can construct an action integral that leads to field equations of the theory, as was seen in Sec. 5. But there is no action integral that yields both the field equations and the particle equations. This means that the standard approach to conservation laws via Noether's theorem is not possible in this theory. While this does not rule out other means, the compatibility of the basic equations with the conservation laws corresponding to Lorentz invariance is certainly brought into question.

This question is resolved by giving the ten generators of the Lorentz group explicitly: A symmetric tensor $\Theta^{\mu\nu}$ and an antisymmetric tensor $J^{\mu\nu\alpha}=-J^{\nu\mu\alpha}$ are given in Sec. 6. These quantities are divergence free and local and yield conserved P^μ and $J^{\mu\nu}$.

Finally it should be noted that the present investigation is easily modified to the description of a system of particles each of which carries an electric as well as a magnetic charge. One must then give up invariance under space reversal. The action integral must again be nonlocal to permit the particle equations. The difficulties which prevent the existence of an action integral for all basic equations persist also for this physical system. But Lorentz invariance can again be proven by exhibiting the ten conservation laws explicitly.

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