Frequency Shift of Radiation from Collapsing (or Expanding) Bodies

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(Received 23 May 1966)

The spectral shift of radiation from the surface of an expanding or collapsing spherically symmetric dust distribution is calculated. It is found that for the explosion case, there is a strong violet shift when the surface emitting radiation is near or inside the Schwarzschild radius.

I. INTRODUCTION

THE gravitational frequency shift, i.e., the shift observed by a body situated in a gravitational potential different from that of the source even though the two may be at relative rest, is a well-known phenomenon. However, in the case of an imploding or exploding massive body emitting radiation there would be an additional shift as the emitting surface is receding from or approaching the external observer. This effect would indeed be very considerable if the contraction or expansion were as rapid as occurs when the surface of a spherically symmetric body is near or inside the Schwarzschild radius. In a recent communication, Faulkner, Hoyle, and Narlikar[1] (hereinafter referred to as FHN) have sought to calculate the resultant frequency shifts. However, for the external empty space they have taken the line element given by Oppenheimer and Snyder[2] and have assumed the imploding (exploding) surface to be at rest in this coordinate system. As Nariai and Tomita[3] have pointed out, this is not correct because the Oppenheimer-Snyder exterior and interior metrics do not satisfy the conditions of fit. This corresponds to the fact that the coordinate frame of the exterior metric is not the co-moving system and hence the surface of the collapsing (or exploding) matter could not be stationary in this coordinate system.

The purpose of the present article is to make a correct evaluation of the frequency shift and for this purpose it would be convenient if one could write the exterior metric in the co-moving coordinates. This has indeed been attempted by Nariai and Tomita, but "except for various approximate forms" they found it "very difficult" to give an explicit solution. We shall here utilize a solution given some years earlier by Raychaudhuri,[4] of which Nariai and Tomita are apparently unaware. The Raychaudhuri exterior metric is of the desired type in that it is in the co-moving coordinate system, satisfies the condition of fit, and although it involves the Weierstrass elliptic function, the calculation of frequency shift can be done very simply with the help of the transformation formulas to the Schwarzschild form given by Raychaudhuri. Indeed, we shall be able to arrive at an expression in closed form and thus will not have to perform numerical integrations as FHN have. However, our final formulas are not new since Nariai and Tomita have also arrived at these results although they confined themselves mainly to the case of collapse.

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II. THE RAYCHAUDHURI METRIC AND TRANSFORMATION FORMULAS TO SCHWARZSCHILD COORDINATES

Here we shall recapitulate a little of Raychaudhuri's paper. Raychaudhuri considered a homogeneous distribution of dust and the isotropic form of the line element.

$$ds^2 = e^\nu dt^2 - e^\mu (dr^2 + r^2 d\Omega^2), \quad d\Omega^2 = d\theta^2 + \sin^2\theta d\varphi^2. \tag{1}$$

For the interior region $r \leqslant r_0$ one has the cosmological solution

$$e^\nu = 1, \quad e^\mu = \frac{e^q}{(1+zr^2/4R^2)^2}, \tag{2}$$

where $q = q(t)$; $z = 0, \pm 1$ and R^2 is a positive constant. The matter density ρ is given by

$$8\pi\rho = \frac{3ze^{-q}}{R^2} + \tfrac{3}{4}\dot{q}^2. \tag{2'}$$

The exterior solution for $r \geqslant r_0$ was given by

$$e^\mu = \xi^4/r^2, \quad e^\nu = (4\dot{\xi}/\dot{q}\xi)^2, \tag{3}$$

where ξ satisfied the differential equation

$$(\partial\xi/\partial x)^2 = \tfrac{1}{4}\xi^2 + \tfrac{1}{16}\dot{q}^2\xi^6 - \tfrac{1}{2}m, \tag{4}$$

in which $x = \ln r$ and m is the Schwarzschild mass given by

$$m = \frac{4\pi\rho}{3} \frac{(e^{3q/2})r_0^3}{(1+zr_0^2/4R^2)^3}. \tag{5}$$

ξ is subject to the boundary condition that ξ at r_0 must be such as to ensure the continuity of e^μ. The quantity ξ can obviously be expressed in terms of elliptic functions. However, for our purpose the transformation formulas to the Schwarzschild coordinates are of greater interest.

$$R = \xi^2,$$
$$dT = \frac{8r\dot{\xi}\xi'}{\dot{q}(\xi^2 - 2m)} dt + \frac{\xi^6 \dot{q}}{2r(\xi^2 - 2m)} dr, \tag{6}$$

where R and T are the Schwarzschild coordinates. $\dot{q}$ in the above formulas will be negative or positive as the dust distribution is collapsing or expanding.

III. CALCULATION OF THE FREQUENCY SHIFT

One has, for a radial light ray,

$$\int_{T_0}^{T} dT = \int_{R_0}^{R} \frac{dR}{1-2m/R} = \int_{\xi_0}^{\xi} \frac{2\xi d\xi}{1-2m/\xi^2},$$

so that

$$\frac{dT}{dT_0} - 1 = -\frac{2\xi_0}{1-2m/\xi_0^2}\left(\frac{d\xi_0}{dT_0}\right), \tag{7}$$

where the observer is considered to be stationary in the Schwarzschild frame ($R=\xi^2=\text{const}$) and ξ_0 is the value of ξ at the boundary of our collapsing (or exploding) system at the instant of emission.

Using (6) and remembering that at the boundary $dr=0$, we have

$$\frac{d\xi_0}{dT_0} = \frac{(\dot{q})_0(\xi_0^2-2m)}{8(\xi_x)_0}. \tag{8}$$

The subscript 0 indicates that the values at the boundary are to be taken at the instant of emission.

Combining (7) and (8)

$$\frac{dT}{dT_0} = 1 - \frac{\xi_0^3(\dot{q})_0}{4(\xi_x)_0}. \tag{9}$$

Also the proper times at reception and emission corresponding to dT and dT_0 are, respectively,

$$ds = \left(1-\frac{2m}{R}\right)^{1/2} dT = \left(1-\frac{2m}{\xi^2}\right)^{1/2} dT, \tag{10}$$

and

$$ds_0 = \left(1-\frac{2m}{\xi_0^2}\right)^{1/2}\left(1-\frac{(\dot{q})_0^2\xi_0^6}{16(\xi_x)_0^2}\right)^{1/2} dT_0$$

$$= \frac{\xi_0}{2(\xi_x)_0}\left(1-\frac{2m}{\xi_0^2}\right) dT_0 \tag{11}$$

the two forms being equal by virtue of (4).

Combining (9), (10), and (11) one gets

$$\frac{ds}{ds_0} = \frac{1}{(1-2m/\xi_0^2)^{1/2}} \frac{(1-\beta)^{1/2}}{(1+\beta)^{1/2}}, \tag{12}$$

where we have taken the observer at spatial infinity $\xi^2 = R \to \infty$ and written

$$\beta = \xi_0^3(\dot{q})_0/4(\xi_x)_0. \tag{13}$$

Obviously the first factor in the denominator corresponds to the usual gravitational shift and the factor involving β indicates the effect of the motion of the source (a typical Doppler effect) and would give rise to a violet or red shift according as β is positive or negative (i.e., $\dot{q}_0$ is positive or negative corresponding to explosion or implosion, respectively).

The expression for β in terms of the mass m of the system, ξ_0, and the constants r_0 and R is, however, not very simple. We shall consider the cases $z=0$ and ± 1 separately.

Case I. $z=0$

In this case using (2), (2′), (3), (4), and (5) $(\xi_x)_0 = \frac{1}{2}\xi_0$ and $\dot{q}_0\xi_0^3 = \pm(8m)^{1/2}$, the upper sign before the radical corresponding to positive $\dot{q}_0$ (explosion) and the lower to negative $\dot{q}_0$ (implosion).

Hence from (12) and (13)

$$\frac{ds}{ds_0} = \frac{1}{1 \mp (2m/\xi_0^2)^{1/2}}; \tag{14}$$

the expression showing the cutoff for the collapsing case when the Schwarzschild singularity is reached ($\xi_0^2 = 2m$). However, for the explosion case, there is a violet shift, the frequency being doubled when $\xi_0^2 = 2m$ and increasing indefinitely as ξ_0 further decreases.

Case II. $z=\pm 1$

Using (2), (2′), (3), (4), (5), and applying the continuity condition for e^{μ}, one can reduce (12) to the following form:

$$\frac{ds}{ds_0} = \frac{1}{(1-2m/\xi_0^2)^{1/2}} \frac{[1 \mp \lambda(2m/\xi_0^2)^{1/2}]^{1/2}}{[1 \pm \lambda(2m/\xi_0^2)^{1/2}]^{1/2}}, \tag{15}$$

where

$$\lambda^2 = 1 + \left(1-\frac{\xi_0^2}{2m}\right)\frac{zr_0^2}{R^2}\left(1-\frac{zr_0^2}{4R^2}\right)^2, \tag{16}$$

the upper signs occurring for explosion ($\dot{q}$ positive) and the lower ones for implosion ($\dot{q}$ negative).

The general forms of (15) and (16) are complicated. However, as $\xi_0^2 \to 2m$ (Schwarzschild singularity) $\lambda \to 1$ and (15) goes over to (14), so that one has again a violet shift for the exploding case.

IV. THE FREQUENCY SHIFT IN THE FHN CASE

We can, by the procedure outlined above, obtain an analytical expression for the frequency shift for the case considered in FHN. We give below the final formula, omitting details of the calculations:

$$\frac{\nu_{\text{source}}}{\nu_{\text{reception}}}=\left(1-\frac{R_c}{R_i}\frac{R_i}{R}\right)^{1/2}\Bigg/\left\{\left(1-\frac{R_c}{R_i}\right)^{1/2}+\left[\frac{R_c}{R_i}\left(\frac{1}{s}-1\right)\right]^{1/2}\right\}, \quad (17)$$

where the symbols have the same meanings as in FHN. Some trial calculations show that the calculated values from (17) agree with the values given in FHN, even when the surface is well inside the Schwarzschild radius. This is interesting for in our deduction we have utilized the Schwarzschild line element, the applicability of which seems questionable for $\xi^2<2m$.

V. CONCLUDING REMARKS

Our formulas have been deduced on the hypothesis of a homogeneous dust system. Should one introduce nonhomogeneity and pressure, the results would surely be quantitatively different although the general features would apparently be preserved.

ACKNOWLEDGMENTS

The author is deeply indebted to Professor A. K. Raychaudhuri of Presidency College, Calcutta for his continuous help and useful guidance.

PHYSICAL REVIEW VOLUME 150, NUMBER 4 28 OCTOBER 1966

Production of Low-Energy Cosmic-Ray Electrons

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(Received 16 May 1966)

The production of cosmic-ray electrons of characteristically low energies is investigated. Secondary sources, other than that of meson decay, are considered, and constraints are placed on both secondary and primary sources. (1) Calculations are made of the intensity of low-energy knock-on and beta-decay electrons which are secondary to cosmic-ray interactions. In particular, knock-on production is calculated in the 100-KeV to 50-BeV kinetic-energy interval. Interstellar losses due to ionization, leakage from the galaxy, and synchrotron, bremsstrahlung, and inverse Compton effects are considered, as well as those due to plasma excitation, the red shift and synchrotron, bremsstrahlung, and inverse Compton effects in the intergalactic medium. The intensity of low-energy relativistic electrons from these sources is not negligible compared with the low energy $\pi\rightarrow\mu\rightarrow e$ intensity, but it is shown not to account for the observed interplanetary electron intensity. (2) Energy inputs to the injected secondary electrons by a possible solar electric field of low magnitude and by a possible galactic Fermi acceleration are investigated. It is shown that at least one such input is necessary if the observed low-energy interplanetary electron intensity is to be attributed to secondary production alone. A heliocentric field which does allow for a fit to the low-energy data cannot, however, account for the high-energy BeV electrons found to be in excess of those from $\pi\rightarrow\mu\rightarrow e$ production. The Fermi acceleration shown to be necessary to provide a fit is greater than that usually postulated for cosmic-ray protons, and also requires that the ratio of escape losses to acceleration λ/α be much smaller than is usually assumed for protons. This distinction is acceptable only if one postulates a significant difference between interstellar proton and electron propagation. (3) The observation that the velocity spectrum of electrons in the energy-per-unit-mass region of 7–25 closely approximates that of the cosmic-ray protons, and the necessity of constraints on the secondary-electron hypothesis outlined above, suggest that most of the low-energy electrons are of primary origin. The similarity between this conclusion and the conclusion (based on the measurement of the charge ratio of electrons) that the higher energy electrons are mostly primary is discussed.

I. INTRODUCTION AND DISCUSSION OF OBSERVATIONS

THE study of cosmic radiation has been, for the most part, the measurement of the intensities and energy spectra of the protons and other nuclei which possess the bulk of the cosmic-ray energy content. Recently, the electromagnetic component began to be investigated: Earl[1] and Meyer and Vogt[2] found electrons which have typical cosmic-ray energies, but which, in the BeV region, have only a small fraction of the proton intensity. Also, DeShong, Hildebrand, and Meyer[3] later found that the electron flux is partially composed of positrons. In addition, Kraushaar *et al.*[4] set a new upper limit to the high-energy gamma-ray intensity. Since one can assume that some of these electrons and gamma rays may be primary cosmic rays and

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