Derivation of the Schrödinger Equation from Newtonian Mechanics*

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We examine the hypothesis that every particle of mass m is subject to a Brownian motion with diffusion coefficient $\hbar/2m$ and no friction. The influence of an external field is expressed by means of Newton's law $\mathbf{F}=m\mathbf{a}$, as in the Ornstein-Uhlenbeck theory of macroscopic Brownian motion with friction. The hypothesis leads in a natural way to the Schrödinger equation, but the physical interpretation is entirely classical. Particles have continuous trajectories and the wave function is not a complete description of the state. Despite this opposition to quantum mechanics, an examination of the measurement process suggests that, within a limited framework, the two theories are equivalent.

I. INTRODUCTION

WE shall attempt to show in this paper that the radical departure from classical physics produced by the introduction of quantum mechanics forty years ago was unnecessary. An entirely classical derivation and interpretation of the Schrödinger equation will be given, following a line of thought which is a natural development of reasoning used in statistical mechanics and in the theory of Brownian motion.

Consider an electron in an external field. The electron is regarded as a point particle of mass m in the sense of Newtonian mechanics. Our basic assumption is that any particle of mass m constantly undergoes a Brownian motion with diffusion coefficient inversely proportional to m. We write the diffusion coefficient as $\hbar/2m$ and later identify $\hbar$ with Planck's constant divided by 2π. As in the theory of macroscopic Brownian motion, the influence of the external force is expressed by means of Newton's law $\mathbf{F}=m\mathbf{a}$, where $\mathbf{a}$ is the mean acceleration of the particle. The chief difference is that in the study of macroscopic Brownian motion in a fluid, friction plays an important role. For the electron we must assume that there is no friction in order to preserve Galilean covariance. The kinematical description of Brownian motion with zero friction is the same as the description used in the Einstein-Smoluchowski theory[1] (the approximate theory of macroscopic Brownian motion in the limiting case of infinite friction).

The picture which emerges is the following. If we have, for example, a hydrogen atom in the ground state, the electron is in dynamical equilibrium between the random force causing the Brownian motion and the attractive Coulomb force of the nucleus. Its trajectory is very irregular. Most of the time the electron is near the nucleus, sometimes it goes farther away, but it always shows a general tendency to move toward the nucleus, and this is true no matter which direction we take for time. This behavior is quite analogous to that of a particle in a colloidal suspension, in dynamical equilibrium between osmotic forces and gravity. However, the electron in the hydrogen atom has other states of dynamical equilibrium, at the usual discrete energy levels of the atom.

The equations of motion which we derive are nonlinear, but if the wave function ψ is introduced, in a way simply related to the kinematical description of the motion, we find that ψ satisfies the Schrödinger equation. Every solution of the Schrödinger equation arises in this way.

Our theory is by no means a causal theory, but probabilistic concepts enter in a classical way. The description of atomic processes is by means of classical ideas of motion in space-time, and so is contrary to quantum mechanics. However, we show that for observations which may be reduced to position measurements, the two theories give the same predictions. This, and a discussion of von Neumann's theorem[2] on the impossibility of hidden variables, is contained in Sec. IV. The same argument shows that some features of our description are incapable of observation. It is well known[3] that macroscopic Brownian motion imposes limits on the precision of measurements if the measuring instruments are subject to it. If, as we are assuming, every system is subject to a Brownian motion, this implies an absolute limit restricting some measurements.

The discussion in this paper is restricted to the nonrelativistic mechanics of particles without spin, in the presence of external fields.

Our work has close connections with some previous work on classical interpretations of the Schrödinger equation. A comparison with other hidden-variable theories is contained in Sec. V.

II. STOCHASTIC MECHANICS

Stochastic processes occur in a number of classical physical theories.[4] Statistical mechanics is based on a

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[1] A. Einstein, *Investigations on the Theory of the Brownian Movement*, translated by A. D. Cowper (Methuen and Company, Ltd., London, 1926); M. v. Smoluchowski, *Abhandlungen über die Brownsche Bewegung und verwandte Erscheinungen* (Akademische Verlagsgesellschaft, Leipzig, 1923).

[2] J. von Neumann, *Mathematical Foundations of Quantum Mechanics*, translated by R. T. Beyer (Princeton University Press, Princeton, New Jersey, 1955).

[3] R. B. Barnes and S. Silverman, Rev. Mod. Phys. **6**, 162 (1934).

[4] A more detailed account of this subject is contained in lecture notes by the author, *Dynamical Theories of Brownian Motion* (Princeton University Press, Princeton, New Jersey, to be published).

stationary stochastic process, the measure-preserving flow in phase space. The Einstein-Smoluchowski[1] theory of Brownian motion involves a Markoff process in coordinate space, and the more refined Ornstein-Uhlenbeck[5] theory of Brownian motion is expressed in terms of a Markoff process in phase space.

Let $x(t)$ be a stochastic process (for example, the x coordinate of a particle at time t). It is well known that for many important processes $x(t)$ is not differentiable. This is the case for the Wiener process,[6] which is the process occurring in Einstein's theory of Brownian motion. The same difficulty occurs with the velocity in the Ornstein-Uhlenbeck theory. To discuss the kinematics of stochastic processes, therefore, we need a substitute for the derivative.

We define the mean forward derivative $Dx(t)$ by

$$Dx(t)=\lim_{\Delta t\to 0+} E_t\frac{x(t+\Delta t)-x(t)}{\Delta t}, \tag{1}$$

where E_t denotes the conditional expectation (average) given the state of the system at time t. Thus $Dx(t)$ is again a stochastic process. The symbol $0+$ means that Δt tends to 0 through positive values. Similarity, we define the mean backward derivative $D_*x(t)$ by

$$D_*x(t)=\lim_{\Delta t\to 0+} E_t\frac{x(t)-x(t-\Delta t)}{\Delta t}. \tag{2}$$

If $x(t)$ is differentiable, then of course $Dx(t)=D_*x(t)=dx/dt$, but in general $D_*x(t)$ is not the same as $Dx(t)$.

The Ornstein-Uhlenbeck Theory

As an example, consider the Ornstein-Uhlenbeck theory of Brownian motion with friction in the presence of a potential V. We denote by $\mathbf{x}(t)$ the position of the Brownian particle at time t, by $\mathbf{v}(t)$ its velocity, by m its mass, and by

$$\mathbf{K}=-(1/m)\,\mathbf{grad}V, \tag{3}$$

the acceleration of the particle produced by V. We assume that the system is in equilibrium, having the Maxwell-Boltzmann distribution. We let $m\beta$ be the friction coefficient. Then the Langevin equations are

$$d\mathbf{x}(t)=\mathbf{v}(t)dt, \tag{4a}$$

$$d\mathbf{v}(t)=-\beta\mathbf{v}(t)dt+\mathbf{K}(\mathbf{x}(t))dt+d\mathbf{B}(t). \tag{4b}$$

Here $\mathbf{B}$ is a Wiener process representing the residual random impacts. The $d\mathbf{B}(t)$ are Gaussian with mean 0, mutually independent, and

$$Ed\mathbf{B}(t)^2=6(\beta kT/m)dt, \tag{5}$$

where E denotes expectation (average), k is Boltzmann's constant, and T is the absolute temperature. The $d\mathbf{B}(t)$ are independent of all of the $\mathbf{x}(s)$, $\mathbf{v}(s)$ with $s\leq t$ (see Doob's discussion[5]). There is clearly an asymmetry in time here: we may also write the Langevin equations with (4b) replaced by

$$d\mathbf{v}(t)=\beta\mathbf{v}(t)dt+\mathbf{K}(\mathbf{x}(t))dt+d\mathbf{B}_*(t), \tag{6}$$

where the $d\mathbf{B}_*(t)$ are independent of all of the $\mathbf{x}(s)$, $\mathbf{v}(s)$ with $s\geq t$.

If we apply our definitions (1) and (2), we find

$$D\mathbf{x}(t)=D_*\mathbf{x}(t)=\mathbf{v}(t), \tag{7}$$

since $\mathbf{x}(t)$ is differentiable with $d\mathbf{x}/dt=\mathbf{v}(t)$.

By (5), $d\mathbf{B}(t)$ is of the order $dt^{1/2}$, so that $\mathbf{B}(t)$, and consequently $\mathbf{v}(t)$, is not differentiable. However, $D\mathbf{B}(t)=0$, since $\mathbf{B}(t+\Delta t)-\mathbf{B}(t)$ for $\Delta t>0$, is independent of the pair $\mathbf{x}(t)$, $\mathbf{v}(t)$ (the state of the system at time t), and has expectation 0. Since D is a linear operation, (4b) implies that

$$D\mathbf{v}(t)=-\beta\mathbf{v}(t)+\mathbf{K}(\mathbf{x}(t)). \tag{8}$$

Similarly, we find from (6) that

$$D_*\mathbf{v}(t)=\beta\mathbf{v}(t)+\mathbf{K}(\mathbf{x}(t)). \tag{9}$$

In the case of a free particle ($\mathbf{K}=0$), we see that $D\mathbf{v}(t)=-D_*\mathbf{v}(t)=-\beta\mathbf{v}(t)$. Because of the damping effect of friction, the velocity has a general trend toward 0, no matter which direction of time we take.

It follows from (8) and (9) that

$$\tfrac{1}{2}DD_*\mathbf{x}(t)+\tfrac{1}{2}D_*D\mathbf{x}(t)=\mathbf{K}(\mathbf{x}(t)). \tag{10}$$

We define the mean second derivative of a stochastic process to be

$$a(t)=\tfrac{1}{2}DD_*x(t)+\tfrac{1}{2}D_*Dx(t). \tag{11}$$

If $\mathbf{x}(t)$ is a position vector, we call $D\mathbf{x}(t)$ the mean forward velocity, $D_*\mathbf{x}(t)$ the mean backward velocity, and $\mathbf{a}(t)$ the mean acceleration.

Thus we see that in the Ornstein-Uhlenbeck theory, Newton's law $\mathbf{F}=m\mathbf{a}$ holds if $\mathbf{F}$ is the external force and $\mathbf{a}$ is the mean acceleration.

Kinematics of Markoff Processes

For a time scale large compared to the relaxation time β^{-1}, the macroscopic Brownian motion of a free particle in a fluid is adequately described by the Wiener process $\mathbf{w}(t)$. The $d\mathbf{w}(t)$ are Gaussian with mean 0, mutually independent, and

$$Ed\mathbf{w}_i(t)d\mathbf{w}_j(t)=2\nu\delta_{ij}dt, \tag{12}$$

where ν is the diffusion coefficient $kT/m\beta$. (We write ν instead of D to avoid confusion with mean forward derivatives.) In general, if there are external forces or currents in the containing fluid, the position $\mathbf{x}(t)$ of the

[5] See S. Chandrasekhar, G. E. Uhlenbeck, and L. S. Ornstein, Ming Chen Wang and G. E. Uhlenbeck, and J. L. Doob, in *Selected Papers on Noise and Stochastic Processes*, edited by N. Wax (Dover Publications, Inc., New York, 1954).

[6] The Wiener process is discussed under the name "Brownian motion" in J. L. Doob, *Stochastic Processes* (John Wiley & Sons, Inc., New York, 1953).

Brownian particle satisfies

$$d\mathbf{x}(t)=\mathbf{b}(\mathbf{x}(t),t)dt+d\mathbf{w}(t), \tag{13}$$

where $\mathbf{w}$ is as before and $\mathbf{b}$ is a vector-valued function on space-time. This is the description used in the approximate theory of Brownian motion due to Einstein and Smoluchowski, and is the limiting case of the Ornstein-Uhlenbeck theory for large β. The $d\mathbf{w}(t)$ are independent of the $\mathbf{x}(s)$ with $s\leq t$, so by (1) $\mathbf{b}$ is the mean forward velocity:

$$D\mathbf{x}(t)=\mathbf{b}(\mathbf{x}(t),t). \tag{14}$$

This description is asymmetrical in time. We may also write

$$d\mathbf{x}(t)=\mathbf{b}_*(\mathbf{x}(t),t)dt+d\mathbf{w}_*(t), \tag{15}$$

where $\mathbf{w}_*$ has the same properties as $\mathbf{w}$ except that the $d\mathbf{w}_*(t)$ are independent of the $\mathbf{x}(s)$ with $s\geq t$. Thus

$$D_*\mathbf{x}(t)=\mathbf{b}_*(\mathbf{x}(t),t) \tag{16}$$

is the mean backward velocity.

We wish to study this type of process in some detail, as we shall use this kinematical description for the motion of an electron.

Let $\rho(\mathbf{x},t)$ be the probability density of $\mathbf{x}(t)$. Then ρ satisfies the forward Fokker-Planck equation

$$\partial\rho/\partial t=-\operatorname{div}(\mathbf{b}\rho)+\nu\Delta\rho, \tag{17}$$

where $\Delta\equiv\nabla^2$, and the backward Fokker-Planck equation

$$\partial\rho/\partial t=-\operatorname{div}(\mathbf{b}_*\rho)-\nu\Delta\rho. \tag{18}$$

The average of (17) and (18) yields the equation of continuity

$$\partial\rho/\partial t=-\operatorname{div}(\mathbf{v}\rho), \tag{19}$$

where we define $\mathbf{v}$ by

$$\mathbf{v}=\tfrac{1}{2}(\mathbf{b}+\mathbf{b}_*). \tag{20}$$

We call $\mathbf{v}$ the current velocity.

Let f be a function of $\mathbf{x}$ and t. To compute $Df(\mathbf{x}(t),t)$, expand f in a Taylor series up to terms of order two in $d\mathbf{x}(t)$:

$$df(\mathbf{x}(t),t)=\frac{\partial f}{\partial t}(\mathbf{x}(t),t)dt+d\mathbf{x}(t)\cdot\nabla f(\mathbf{x}(t),t)+\tfrac{1}{2}\sum_{i,j}dx_i(t)dx_j(t)\frac{\partial^2 f}{\partial x_i\partial x_j}(\mathbf{x}(t),t), \tag{21}$$

plus terms of higher order. By (13), we may replace the $dx_i(t)$ by $dw_i(t)$ in the last term. When we take the average [given $\mathbf{x}(t)$], we may replace $d\mathbf{x}(t)\cdot\nabla f(\mathbf{x}(t),t)$ by $\mathbf{b}(\mathbf{x}(t),t)\cdot\nabla f(\mathbf{x}(t),t)$, since $d\mathbf{w}(t)$ is independent of $\mathbf{x}(t)$ and has mean 0. Using (12), we obtain

$$Df(\mathbf{x}(t),t)=(\partial/\partial t+\mathbf{b}\cdot\nabla+\nu\Delta)f(\mathbf{x}(t),t). \tag{22}$$

In the same way we obtain

$$D_*f(\mathbf{x}(t),t)=(\partial/\partial t+\mathbf{b}_*\cdot\nabla-\nu\Delta)f(\mathbf{x}(t),t). \tag{23}$$

The distribution $\rho d^3x\,dt$ is invariant on space-time. For this reason,[7] $(\partial/\partial t+\mathbf{b}\cdot\nabla+\nu\Delta)$ and $(-\partial/\partial t-\mathbf{b}_*\cdot\nabla+\nu\Delta)$ are adjoints of each other with respect to $\rho d^3x\,dt$. That is,

$$\rho^{-1}(\partial/\partial t+\mathbf{b}\cdot\nabla+\nu\Delta)^+\rho=-\partial/\partial t-\mathbf{b}_*\cdot\nabla+\nu\Delta, \tag{24}$$

where the superscript $+$ denotes the Lagrange adjoint (with respect to $d^3x\,dt$). If we compute the left-hand side of (24) and use the forward Fokker-Planck equation (17), we find

$$\mathbf{b}_*=\mathbf{b}-2\nu(\mathbf{grad}\rho/\rho) \tag{25}$$

or

$$\mathbf{u}=\nu(\mathbf{grad}\rho/\rho), \tag{26}$$

where we define

$$\mathbf{u}=\tfrac{1}{2}(\mathbf{b}-\mathbf{b}_*). \tag{27}$$

According to Einstein's theory[1] of Brownian motion, (26) is the velocity acquired by a Brownian particle, in equilibrium with respect to an external force, to balance the osmotic force. For this reason, we call $\mathbf{u}$ the osmotic velocity. Notice that by subtracting (17) from (18) we obtain

$$0=\operatorname{div}(\mathbf{u}\rho)-\nu\Delta\rho=\operatorname{div}[\mathbf{u}\rho-\nu\,\mathbf{grad}\rho], \tag{28}$$

which also follows from (26).

By (26),

$$\mathbf{u}=\nu\,\mathbf{grad}\ln\rho. \tag{29}$$

Using the equation of continuity (19), we may compute $\partial\mathbf{u}/\partial t$:

$$\partial\mathbf{u}/\partial t=-\nu\,\mathbf{grad}(\operatorname{div}\mathbf{v})-\mathbf{grad}(\mathbf{v}\cdot\mathbf{u}). \tag{30}$$

If we apply (22) to $\mathbf{b}_*$ and (23) to $\mathbf{b}$, and recall the definition (11) of the mean acceleration $\mathbf{a}$, we find

$$\mathbf{a}=\frac{1}{2}\frac{\partial}{\partial t}(\mathbf{b}+\mathbf{b}_*)+\tfrac{1}{2}(\mathbf{b}\cdot\nabla)\mathbf{b}_*+\tfrac{1}{2}(\mathbf{b}_*\cdot\nabla)\mathbf{b}-\tfrac{1}{2}\nu\Delta(\mathbf{b}-\mathbf{b}_*). \tag{31}$$

By (20) and (27), $\mathbf{b}=\mathbf{v}+\mathbf{u}$ and $\mathbf{b}_*=\mathbf{v}-\mathbf{u}$, so that (31) is equivalent to

$$\partial\mathbf{v}/\partial t=\mathbf{a}-(\mathbf{v}\cdot\nabla)\mathbf{v}+(\mathbf{u}\cdot\nabla)\mathbf{u}+\nu\Delta\mathbf{u}. \tag{32}$$

III. THE HYPOTHESIS OF UNIVERSAL BROWNIAN MOTION

We wish to examine the hypothesis that particles in empty space, or let us say the ether, are subject to Brownian motion. Macroscopic bodies do not appear to exhibit such behavior, so we shall assume that the diffusion coefficient ν is inversely proportional to the mass. We set

$$\nu=\hbar/2m. \tag{33}$$

The constant $\hbar$ has the dimensions of action; we shall see later that it can be identified with Planck's constant divided by 2π.

[7] E. Nelson, Duke Math. J. 25, 671 (1958).

We cannot attribute any friction to the ether, for then we could distinguish absolute rest from uniform motion. This means that the Brownian motion will not be smooth, and velocities will not exist. Hence we cannot describe the state of a particle by a point in phase space. As in the Einstein-Smoluchowski theory, the motion will be described by a Markoff process in coordinate space, as in the previous section but with $\nu=\hbar/2m$.

The mean acceleration $\mathbf{a}$ has no dynamical significance in the Einstein-Smoluchowski theory. That theory applies in the limit of large friction, so that an external force $\mathbf{F}$ does not accelerate a particle but merely imparts a velocity $\mathbf{F}/m\beta$ to it. In our theory the dynamics will be given by Newton's law $\mathbf{F}=m\mathbf{a}$. In other words, to study Brownian motion in a medium with zero friction we adopt the kinematics of the Einstein-Smoluchowski theory but use Newtonian dynamics as in the Ornstein-Uhlenbeck theory.

Consider, then, a particle of mass m in an external force $\mathbf{F}$. The particle performs a Markoff process, so its stage at time t_0 is given by a point $\mathbf{x}(t_0)$ in coordinate space. However, to know the particle's motion we also need to know what the Markoff process is. That is, we need to know $\mathbf{b}(\mathbf{x},t)$ and $\mathbf{b}_*(\mathbf{x},t)$, or, equivalently, $\mathbf{u}(\mathbf{x},t)$ and $\mathbf{v}(\mathbf{x},t)$ for all t. But $\mathbf{u}$ and $\mathbf{v}$ satisfy (30) and (32) and the term $\mathbf{a}$ in (32) is known; it is $\mathbf{F}/m$. Thus $\mathbf{u}$ and $\mathbf{v}$ satisfy

$$\partial\mathbf{u}/\partial t=-(\hbar/2m)\,\mathbf{grad}(\mathrm{div}\mathbf{v})-\mathbf{grad}(\mathbf{v}\cdot\mathbf{u}), \tag{34a}$$

$$\partial\mathbf{v}/\partial t=(1/m)\mathbf{F}-(\mathbf{v}\cdot\nabla)\mathbf{v}+(\mathbf{u}\cdot\nabla)\mathbf{u}+(\hbar/2m)\Delta\mathbf{u}. \tag{34b}$$

Consequently, if $\mathbf{u}(\mathbf{x},t_0)$ and $\mathbf{v}(\mathbf{x},t_0)$ are known and we can solve the Cauchy problem for the coupled nonlinear partial differential equations (34), then the Markoff process will be completely known. Thus the state of a particle at time t_0 is described by its position $\mathbf{x}(t_0)$ at time t_0, the osmotic velocity $\mathbf{u}$ at time t_0, and the current velocity $\mathbf{v}$ at time t_0. Notice that $\mathbf{u}(\mathbf{x},t_0)$ and $\mathbf{v}(\mathbf{x},t_0)$ must be given for all values of $\mathbf{x}$ and not just for $\mathbf{x}(t_0)$.

The Real Time-Independent Schrödinger Equation

Let us seek some special solutions of (34) in the case that the force comes from a potential, $\mathbf{F}=-\mathbf{grad}V$. Suppose first that $\mathbf{v}=0$. This implies, by (19) and (26), that ρ and $\mathbf{u}$ are independent of t and that the solution is stationary. In this case (34a) says that $\partial\mathbf{u}/\partial t=0$ and (34b) becomes

$$\mathbf{u}\cdot\nabla\mathbf{u}+(\hbar/2m)\Delta\mathbf{u}=(1/m)\,\mathbf{grad}V. \tag{35}$$

By (29), $\mathbf{u}$ is a gradient, so that $(\mathbf{u}\cdot\nabla)\mathbf{u}=\frac{1}{2}\,\mathbf{grad}\,\mathbf{u}^2$ and $\Delta\mathbf{u}=\mathbf{grad}(\mathrm{div}\mathbf{u})$. Thus (35) becomes

$$\mathbf{grad}\left(\tfrac{1}{2}\mathbf{u}^2+\frac{\hbar}{2m}\,\mathrm{div}\mathbf{u}\right)=\frac{1}{m}\,\mathbf{grad}V, \tag{36}$$

or

$$\tfrac{1}{2}\mathbf{u}^2+\frac{\hbar}{2m}\,\mathrm{div}\mathbf{u}=\frac{1}{m}V-\frac{1}{m}E, \tag{37}$$

where E is a constant with the dimensions of energy. If we multiply by $m\rho$ and integrate, we obtain

$$\int\frac{1}{2}m\mathbf{u}^2\rho d^3x-\frac{\hbar}{2}\int(\mathbf{u}\cdot\mathbf{grad}\rho)d^3x=\int V\rho d^3x-E. \tag{38}$$

By (26), the left-hand side is $-\int\frac{1}{2}m\mathbf{u}^2\rho d^3x$, so that

$$E=\int\frac{1}{2}m\mathbf{u}^2\rho d^3x+\int V\rho d^3x. \tag{39}$$

Thus E is the average value of $\frac{1}{2}m\mathbf{u}^2+V$, and so may be interpreted as the mean energy of the particle. Notice that in this case $\mathbf{b}=-\mathbf{b}_*=\mathbf{u}$, so we could replace $\mathbf{u}$ by $\mathbf{b}$ or $\mathbf{b}_*$ in computing the mean kinetic energy. [If we compute the mean of $\frac{1}{2}m(d\mathbf{x}/dt)^2$ there is the additional infinite term $\frac{1}{2}m(d\mathbf{w}/dt)^2$. Since this does not depend on the potential V, differences of energy levels will not be affected, and we can neglect it. In any case, the potential V contains an arbitrary additive constant.]

The equation (37) is nonlinear, but it is equivalent to a linear equation by a change of dependent variable. By (29),

$$R=\tfrac{1}{2}\ln\rho \tag{40}$$

is the potential of $m\mathbf{u}/\hbar$. Let

$$\psi=e^R. \tag{41}$$

Then ψ is real and $\rho=\psi^2$. It is immediately seen that (37) is equivalent to the time-independent Schrödinger equation

$$[-(\hbar^2/2m)\Delta+V-E]\psi=0 \tag{42}$$

for real ψ. Thus for the hydrogen atom the hypothesis of Brownian motion leads to the correct energy levels for bound states of the atom, and interprets them as states of dynamical equilibrium. This is discussed further in Sec. V.

The Time-Dependent Schrödinger Equation

There are in general other stationary (that is, $\partial\mathbf{u}/\partial t=0$, $\partial\mathbf{v}/\partial t=0$) solutions of (34) for which $\mathbf{v}\neq0$ (magnetic atoms). However, it is just as easy to discuss the general time-dependent case.

At this point we need a further kinematical assumption. Suppose we were to assume that $\mathrm{div}\mathbf{v}=0$. Since $\mathbf{u}$ is always a gradient, we could then write (34b) as

$$\partial\mathbf{v}/\partial t=-(\mathbf{v}\cdot\nabla)\mathbf{v}-\mathbf{grad}p, \tag{43}$$

which is the Euler equation of motion for a nonviscous incompressible fluid of unit density, if p is the pressure. Thus this situation appears to be related to diffusion processes in fluids with currents flowing in them. We do not want this, so we make the opposite assumption that $\mathbf{v}$ is a gradient, and set

$$\mathbf{grad}S=(m/\hbar)\mathbf{v}. \tag{44}$$

Keeping R as before [Eq. (40)], we let

$$\psi = e^{R+iS}. \tag{45}$$

We wish to show that ψ satisfies the Schrödinger equation

$$\frac{\partial\psi}{\partial t} = i\frac{\hbar}{2m}\Delta\psi - \frac{1}{i\hbar}V\psi + i\alpha(t)\psi. \tag{46}$$

[Since $|\psi|^2=\rho$, if we multiply by $\bar{\psi}$, integrate over space, and take real parts, we see that if (46) is satisfied then $\alpha(t)$ is real. We can always choose the potential S so that the phase factor $\alpha(t)$ is 0.] That is, we wish to show that

$$\left(\frac{\partial R}{\partial t}+i\frac{\partial S}{\partial t}\right)\psi = i\frac{\hbar}{2m}(\Delta R+i\Delta S+[\mathbf{grad}(R+iS)]^2)\psi - \frac{1}{i\hbar}V\psi+i\alpha(t)\psi. \tag{47}$$

If we divide by ψ, take gradients, and separate real and imaginary parts: this is equivalent to the pair of equations

$$\frac{\partial\mathbf{u}}{\partial t} = -\frac{\hbar}{2m}\Delta\mathbf{v} - \mathbf{grad}(\mathbf{v}\cdot\mathbf{u}), \tag{48a}$$

$$\frac{\partial\mathbf{v}}{\partial t} = \frac{\hbar}{2m}\Delta\mathbf{u} + \tfrac{1}{2}\,\mathbf{grad}(\mathbf{u}^2) - \tfrac{1}{2}\,\mathbf{grad}(\mathbf{v}^2) - \frac{1}{m}\,\mathbf{grad}V. \tag{48b}$$

Since $\mathbf{u}$ and $\mathbf{v}$ are gradients, (48) is equivalent to (34).

Conversely, if we have any solution to the Schrödinger equation (46), normalized so that $\int|\psi|^2d^3x=1$, we may write $\psi=\exp(R+iS)$, $\mathbf{u}=\hbar\,\mathbf{grad}R/m$, $\mathbf{v}=\hbar\,\mathbf{grad}S/m$, $\mathbf{b}=\mathbf{v}+\mathbf{u}$, $\mathbf{b}_*=\mathbf{v}-\mathbf{u}$, and $\rho=|\psi|^2$. The Markoff process with diffusion coefficient $\hbar/2m$, forward velocity $\mathbf{b}$, and backward velocity $\mathbf{b}_*$ has probability density ρ and mean acceleration $\mathbf{a}=-\mathbf{grad}V/m$. That is, every solution of the Schrödinger equation arises in this way.

The same considerations apply to systems of several particles with n-body potentials.

The Schrödinger Equation in an External Electromagnetic Field

Finally, we consider the Brownian motion (without friction) of a particle of charge e and mass m in the presence of an external electromagnetic field. As usual, we denote by $\mathbf{A}$ the vector potential, by φ the scalar potential, by $\mathbf{E}$ the electric field strength, and by $\mathbf{H}$ the magnetic field strength, so that

$$\mathbf{H}=\mathbf{curl}\,\mathbf{A}, \tag{49}$$

$$\mathbf{E}+\frac{1}{c}\frac{\partial\mathbf{A}}{\partial t} = -\mathbf{grad}\varphi, \tag{50}$$

where c is the speed of light. The force on the particle is

$$\mathbf{F}=e[\mathbf{E}+(1/c)\mathbf{v}\times\mathbf{H}], \tag{51}$$

where $\mathbf{v}$ is the current velocity. (The force should be symmetrical under $t\to -t$. Under time inversion, $\mathbf{v}\to-\mathbf{v}$ and $\mathbf{H}\to-\mathbf{H}$, so that this is the case. Note that $\mathbf{u}\to\mathbf{u}$ under time inversion.) To solve (34) with this force, we proceed as before except that instead of assuming the momentum $m\mathbf{v}$ to be a gradient, we assume the generalized momentum $m\mathbf{v}+e\mathbf{A}/c$ to be a gradient. This assumption is gauge-independent. We set

$$\mathbf{grad}S=(m/\hbar)[\mathbf{v}+(e/mc)\mathbf{A}], \tag{52}$$

and define $\psi=\exp(R+iS)$ as before, where R is given by (40).

We claim that ψ satisfies the Schrödinger equation

$$\frac{\partial\psi}{\partial t} = -\frac{i}{2m\hbar}\left(-i\hbar\nabla-\frac{e}{c}\mathbf{A}\right)^2\psi - \frac{ie}{\hbar}\varphi\psi + i\alpha(t)\psi. \tag{53}$$

[As before, $\alpha(t)$ must be real, and we can choose S so that $\alpha(t)=0$.] That is, we claim that

$$\left(\frac{\partial R}{\partial t}+i\frac{\partial S}{\partial t}\right)\psi = i\frac{\hbar}{2m}(\Delta R+i\Delta S+[\mathbf{grad}(R+iS)]^2)\psi + \frac{e}{mc}[\mathbf{A}\cdot\mathbf{grad}(R+iS)]\psi + \frac{1}{2}\frac{e}{mc}\,\mathrm{div}\mathbf{A}\psi - \frac{ie^2}{2m\hbar c^2}\mathbf{A}^2\psi - \frac{ie}{\hbar}\varphi\psi + i\alpha(t)\psi. \tag{54}$$

Divide (54) by ψ and take gradients. The real part then yields (34a), as the two terms involving $\mathbf{grad}\,\mathrm{div}\mathbf{A}$ cancel and the two terms involving $\mathbf{grad}(\mathbf{A}\cdot\mathbf{u})$ cancel. If we use (50), the imaginary part yields, after simplification,

$$\frac{\partial\mathbf{v}}{\partial t} = \frac{e}{m}\mathbf{E} + \frac{\hbar}{2m}\Delta\mathbf{u} + \tfrac{1}{2}\,\mathbf{grad}(\mathbf{u}^2) - \tfrac{1}{2}\,\mathbf{grad}(\mathbf{v}^2). \tag{55}$$

By the vector identity

$$\tfrac{1}{2}\,\mathbf{grad}(\mathbf{v}^2) = \mathbf{v}\times(\mathbf{curl}\,\mathbf{v}) + (\mathbf{v}\cdot\nabla)\mathbf{v}, \tag{56}$$

and the fact that $\mathbf{u}$ is a gradient, this becomes

$$\frac{\partial\mathbf{v}}{\partial t} = \frac{e}{m}\mathbf{E} - \mathbf{v}\times(\mathbf{curl}\,\mathbf{v}) + (\mathbf{u}\cdot\nabla)\mathbf{u} - (\mathbf{v}\cdot\nabla)\mathbf{v} + \frac{\hbar}{2m}\Delta\mathbf{u}. \tag{57}$$

Now $\mathbf{curl}(\mathbf{v}+e\mathbf{A}/mc)=0$, so that

$$-\mathbf{v}\times(\mathbf{curl}\,\mathbf{v}) = \frac{e}{mc}\mathbf{v}\times(\mathbf{curl}\,\mathbf{A}) = \frac{e}{m}\left(\frac{1}{c}\mathbf{v}\times\mathbf{H}\right). \tag{58}$$

Therefore (57) is (34b) with the force (51). As before, every solution of (53) arises in this way.

IV. COMPARISON WITH QUANTUM MECHANICS

The theory we are proposing is so radically different from quantum mechanics, and the latter is so well

verified, that it should be a simple matter to show that it is wrong. Consider, however, an experiment which might distinguish between the two theories. It can be maintained that all measurements are reducible to position measurements (pointer readings). We consider an experiment which can be described within the framework of the nonrelativistic mechanics of systems of finitely many degrees of freedom.

At time t_0 we have N particles with wave function ψ. We assume that they interact through given two-body and n-body potentials. We apply an external field and at a later time t_1 we measure the positions of the N particles (pointer readings, which may include a record of readings made at previous times). We call U the unitary operator (depending on the external field chosen) which takes ψ into $\psi(t_1)$, and if x is any coordinate of one of the particles we call the value of x at time t_1, which we denote by ξ, a primary observable. If it is true that all measurements can be reduced to position measurements, then only primary observables have an operational meaning as observables. In quantum mechanics, ξ is represented by the self-adjoint operator $U^{-1}xU$. In stochastic mechanics it is represented by the random variable $x(t_1)$. If f is a bounded real function, we can make the measurement ξ and then compute $f(\xi)$. Quantum mechanics represents $f(\xi)$ by the self-adjoint operator $f(U^{-1}xU)=U^{-1}f(x)U$, so that the expected value of $f(\xi)$ is $[\psi,U^{-1}f(x)U)]=[\psi(t_1),f(x)\psi(t_1)]$, which is equal to

$$\int f(x)\rho(x,z,t_1)dx\,d^{3N-1}z\,, \tag{59}$$

where z represents the remaining coordinates. Stochastic mechanics represents $f(\xi)$ by the random variable $f[x(t_1)]$, so that the expected value of $f(\xi)$ is again (59). So far, therefore, the statistical interpretations of the two theories agree. If one accepts the view that all measurements are reducible to position measurements, this means that the two theories give the same predictions.

Suppose now that we had performed a different experiment, applying a different external field giving the unitary operator V, and observing a possibly different coordinate y at time t_1. Call this observation η. Quantum mechanics represents this observable by the self-adjoint operator $V^{-1}yV$, stochastic mechanics by a random variable $y(t_1)$, and again the statistical interpretations agree.

Now consider $\xi+\eta$. This has no clear operational meaning. If we perform an experiment to measure ξ, it is not clear that the question of what would have happened had we performed a different experiment to measure η has meaning. However, in quantum mechanics it is customary to consider the operator $U^{-1}xU+V^{-1}yV$ as corresponding to the "observable" $\xi+\eta$. (Considerations of domains of operators are irrelevant here. We may replace x and y by bounded functions of x and y.) The statistical interpretation is the same as for primary observables: If f is a bounded real function, the expected value of $f(\xi+\eta)$ is $[\psi, f(U^{-1}xU+V^{-1}yV)\psi]$. No operational meaning is assigned to this definition of $\xi+\eta$. It is the natural definition in the framework of operators on Hilbert space.

Stochastic mechanics also leads to a natural interpretation of $\xi+\eta$. The observable ξ is represented by the random variable $x(t_1)$ and η is represented by the random variable $y(t_1)$, so we may represent $\xi+\eta$ by the random variable $x(t_1)+y(t_1)$. The random variables $x(t_1)$, $y(t_1)$ depend on the different external fields, but they are defined on the same probability space, that of the Weiner processes for the N particles, so that it is meaningful to add them. If f is a bounded real function, the expectation of $f[x(t_1)+y(t_1)]$ is well-defined. No operational meaning is assigned to this definition of $\xi+\eta$. It is the natural definition in the framework of stochastic processes.

However, the statistical interpretations of the two definitions of $\xi+\eta$ do not agree. In fact, we may easily find U and V (unitary operators, arising from external fields, which connect the wave functions at time t_0 and t_1) such that $U^{-1}xU$ and $V^{-1}yV$ do not commute. This means (and this is the content of von Neuman's proof[2] of the impossibility of hidden variables) that for a suitable ψ there is no pair of random variables $x(t_1)$, $y(t_1)$ such that $\alpha U^{-1}xU+\beta V^{-1}yV$ in the state with wave function ψ has the same probability distribution as $\alpha x(t_1)+\beta y(t_1)$, for all real α and β.

The situation is clarified by considering the motion of a free particle. Let $\xi(t)$ be the position of the particle at time t. In the quantum-mechanical description, let $\mathbf{X}(t)$ be the position operator at time t and in the stochastic description, let $\mathbf{x}(t)$ be the random variable giving the position at time t. The $\mathbf{x}(t)$ are the random variables of a Markoff process, the precise form of which depends on the wave function ψ at a given time.

Now consider two different times t_1 and t_2. We know, by von Neumann's theorem,[2] that $\alpha\mathbf{X}(t_1)+(1-\alpha)\mathbf{X}(t_2)$ cannot give the same statistics as $\alpha\mathbf{x}(t_1)+(1-\alpha)\mathbf{x}(t_2)$ for all α. On the other hand, since the particle is free,

$$\alpha\mathbf{X}(t_1)+(1-\alpha)\mathbf{x}(t_2)=\mathbf{X}(t_3)\,, \tag{60}$$

where $t_3=\alpha t_1+(1-\alpha)t_2$, and we have seen that $\mathbf{X}(t_3)$ gives the same statistics as $\mathbf{x}(t_3)$. There is no contradiction because $\alpha\mathbf{x}(t_1)+(1-\alpha)\mathbf{x}(t_2)$ and $\mathbf{x}(t_3)$ are different random variables, although their expectations are the same, for any ψ.

The quantum and stochastic descriptions give different statistics for the triple $\xi(t_1)$, $\xi(t_2)$, $\xi(t_3)$, but the question as to which description is correct is moot unless an operational meaning can be given to $\alpha\xi(t_1)+(1-\alpha)\times\xi(t_2)$. Von Neumann, in his proof of the impossibility of hidden variables, explicitly assumes that there is a one-to-one correspondence between observables and self-

adjoint operators, but this is not a necessary requirement for a physical theory. It should be remarked, however, that the class of Hamiltonians that we have been able to treat by the stochastic method is quite limited, all of them being of second order in the momentum.

One conclusion which may be drawn from this is that the additional information which stochastic mechanics seems to provide, such as continuous trajectories, is useless, because it is not accessible to experimental varification.

Another conclusion which this analysis suggests is that the practice of regarding the sum of two noncommuting operators corresponding to two observables as being the operator corresponding to the sum of the observables is a matter of convention which is not accessible to experimental verification. If one wishes, for convenience, to add observables, stochastic mechanics provides a simpler (although by now less familiar) framework than does Hilbert space, a framework in which observables are represented by random variables which may be freely added and multiplied together, and which all have joint probability distributions.

V. DISCUSSION

In this paper we have made no attempt at mathematical rigor. In deriving the Schrödinger equation we assumed $\mathbf{u}$, $\mathbf{v}$, and ρ to be smooth functions. This is the case if and only if ψ is smooth and $\rho=|\psi|^2$ does not vanish [see Eq. (40)]. For real solutions of the time-independent Schrödinger equation, other than the ground-state solution, ρ has nodal surfaces on which $\mathbf{u}$ becomes infinite. The definition (41) produces an equivalence between Eqs. (37) and (42) only within a region bounded by nodal surfaces. However, it can be shown that the associated Markoff process is well defined in each such region, and that a particle performing the Markoff process never reaches a nodal surface. The question remains as to the proper definition of ρ throughout space, which is essential for the identification of the constant E as the mean energy. Let us take the smooth solution ψ of the Schrödinger equation (42), let ψ_0 be the ground-state solution, and let

$$\psi_\epsilon=(\psi+i\epsilon\psi_0)/(1+\epsilon^2)^{1/2}. \tag{61}$$

The solution of the time-dependent Schrödinger equation with ψ_ϵ as initial value corresponds to a Markoff process in which $\mathbf{u}$ becomes infinite on a surface only at isolated times, and there is no problem with uniqueness. The probability density is $\rho_\epsilon=|\psi_\epsilon|^2$. As ϵ tends to 0, this Markoff process becomes approximately stationary, and has as limit the Markoff process associated with ψ and probability density $\rho=|\psi|^2$. We could also think of taking a different wave function ψ_1 which in each region bounded by nodal surfaces is a constant multiple of $\exp R$, but then if we approximate ψ_1 by a smooth $\psi_{1\epsilon}$ with nonvanishing $|\psi_{1\epsilon}|^2$, the corresponding Markoff process will no longer be approximately stationary. For this reason we take the usual probability density $\rho=|\psi|^2$, and so have the usual energy levels.

An interpretation of the Schrödinger equation in terms of particle trajectories was first proposed by de Broglie[8] and later developed by Bohm.[9] In this work the particle velocity was identified with what we call the current velocity $\mathbf{v}$, and Bohm interpreted the deviation from the Newtonian equations of motion as being due to a quantum-mechanical potential associated with the wave function. Bohm and Vigier[10] introduced the notion of random fluctuations arising from interaction with a subquantum medium. Since completing this work, the author became aware of the work of Fényes,[11] Weizel,[12] and Kershaw.[13] Fényes showed that instead of assuming a quantum-mechanical potential the motion could be understood in terms of a Markoff process. This work was developed by Weizel, who also proposed a model for the random aspects of the motion in terms of interaction with hypothetical particles, which he calls zerons. The case $\mathbf{v}=0$ was also discussed by Kershaw. The theory which we have developed is just the Fényes-Weizel theory from a different point of view. Our aim has been to show how close it is to classical theories of Brownian motion and Newtonian mechanics, and how the Schrödinger equation might have been discovered from this point of view.

A formal analogy between Brownian motion and the Schrödinger equation was noticed by Fürth[14] and developed recently by Comisar.[15] In this work the diffusion coefficient is imaginary. A parallelism between quantum mechanics and stochastic processes has been noticed in a number of recent papers.[16]

Only a very small part of the subject matter of quantum mechanics has been discussed here. Relativity, spin, statistics of identical particles, and systems of infinitely many degrees of freedom have all been ignored. Consequently, no firm conclusions can be drawn. However, it appears that the phenomena which first led to the abandonment of classical physics admit a simple classical interpretation which is only in a limited sense equivalent to quantum mechanics.

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