Introduction to the N-Quantum Approximation for Bound States: the Deuteron in Pseudoscalar-Meson Theory

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The N-quantum approximation for bound states in relativistic quantum field theory is described by applying it to the deuteron in pseudoscalar-meson theory. Some properties of the covariant, single-time, N-quantum amplitude which plays the role of the deuteron wave function are given. A manifestly covariant single-time equation for this amplitude is derived. In the weak-binding, nonrelativistic limit, this equation is reduced to a Pauli-Schrödinger equation with the correct reduced mass and the usual second-order perturbation-theory central and tensor potentials. The interpretation of the N-quantum bound-state amplitude in the weak-binding, nonrelativistic limit is discussed. It is pointed out that a higher approximation will lead to a short-range force between nucleons.

1. INTRODUCTION

THE purpose of this article is to put forward a new approximate method of treating bound states. The framework in which the approximation operates is local relativistic quantum field theory with a specific Hamiltonian which characterizes the interaction. The method is an application of the N-quantum approximation[1] to bound states. For further discussion of this approximation we refer the reader to Ref. 1; however, we intend the present article to be self-contained.

The leading idea of the N-quantum approximation is to exploit the (assumed) existence of two irreducible sets of field operators, and the relation between them, to find approximate solutions of the Heisenberg field equations of the theory. The two irreducible sets of fields are the Heisenberg fields which appear in the Lagrangian and Hamiltonian but have no simple universal relation to particle states, and the in- (or out-) fields which do not appear in the Lagrangian or Hamiltonian but which are directly related to exact eigenstates of the Hamiltonian labeled by the quantum numbers of freely moving particles in the incoming (or outgoing) beam. The Heisenberg fields are given in advance; the in- (or out-) fields are not; on the contrary, which in- (or out-) fields occur is a dynamical question (which is identical with the question of which stable bound states are predicted by the theory) and must be answered by calculation of the dynamics of the theory. The link between the Heisenberg and, say, in-fields is given by the expansion of the Heisenberg fields in normal-ordered in-fields first given by Haag.[2] For the exact Heisenberg fields of any nontrivial local field theory, this expansion never terminates; nonetheless the expansion can terminate for approximations to the exact Heisenberg fields. Indeed, in the present article we will approximate the Heisenberg fields by the smallest number of normal-ordered terms which, when inserted in the Heisenberg equations of motion, will lead to a nontrivial equation for the amplitude which plays the role of the wave function of the bound state. The possibility which we are exploring, and the hope which motivates us, is that the dominant contributions to low-order amplitudes come from amplitudes of similar low order. In graphical language, this would mean that the connected graphs with few external lines dominate. We emphasize that manifest covariance can be maintained exactly within the approximation, and that positive and negative frequencies can be treated in a symmetric way. The N-quantum amplitudes correspond to connected graphs. Although in general the equations for the amplitudes are nonlinear, in the approximations discussed in the present article our equations will be linear.

The specific problem which we will consider will be the deuteron in pseudoscalar-meson theory.[3] Since the

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[1] O. W. Greenberg, Phys. Rev. **139**, B1038 (1965). This article refers to other applications of the N-quantum approximation. A preliminary version of this bound-state method was reported at the Washington meeting of the Americal Physical Society in 1965: O. W. Greenberg, Bull. Am. Phys. Soc. **10**, 484 (1965).

[2] R. Haag, Kgl. Danske Videnskab. Selskab, Mat. Fys. Medd. **29**, No. 12 (1955).
[3] We hope that the method illustrated here for the deuteron will have application to many bound-state problems, including those (quark or other triplet models of hadrons) with strong binding. For cases with strong binding, both pieces of our N-quantum amplitudes must be taken into account; in addition, higher amplitudes which describe the mesonic (and possibly baryonic) cloud effects may be important.

deuteron in-field first occurs in the expansion of, say, the proton Heisenberg field $N^{(p)}$ in a term containing the normal-ordered product $:\bar{N}_{\text{in}}^{(n)}D_{\text{in}}:$, we will keep this term in addition to the proton in-field. In schematic form, our ansatz for $N^{(p)}$ will be[4]

$$N^{(p)}\sim N_{\text{in}}^{(p)}+\int \mathfrak{N}^{(p)}:\bar{N}_{\text{in}}^{(n)}D_{\text{in}}:,$$

where $N^{(p)}$, $N_{\text{in}}^{(n)}$, and D_{in} are proton and neutron Heisenberg, and deuteron in-fields, respectively, and the bar and asterisk indicate Pauli and ordinary adjoint, respectively. The amplitude $\mathfrak{N}^{(p)}$ depends on two four-vectors, which can be chosen to be the momenta carried by $\bar{N}_{\text{in}}^{(n)}$ and D_{in}, each of which is on its mass shell. A typical higher order term which we have omitted is $:\bar{N}_{\text{in}}^{(n)}D_{\text{in}}\phi_{\text{in}}:$ which describes pionic cloud effects in the deuteron. Our amplitude $\mathfrak{N}^{(p)}$ is a vertex function which corresponds to the matrix elements ${}_{\text{in}}\langle N^{(n)}|N^{(p)}|D\rangle_{\text{in}}$, $\langle 0|N^{(p)}|\bar{N}^{(n)}D\rangle_{\text{in}}$, ${}_{\text{in}}\langle\bar{D}|N^{(p)}|\bar{N}^{(n)}\rangle_{\text{in}}$, and ${}_{\text{in}}\langle\bar{D}N^{(n)}|N^{(p)}|0\rangle$; i.e., to the virtual processes $D\rightleftharpoons n+p$, $D+\bar{n}\rightleftharpoons p$, $\bar{n}\rightleftharpoons p+\bar{D}$, and $0\rightleftharpoons p+n+\bar{D}$, respectively. The specific matrix element is determined by the choice of cones for the 4-vector momenta carried by $\bar{N}_{\text{in}}^{(n)}$ and D_{in}. For a weakly bound deuteron, which we will study in this article, kinematics indicates that ${}_{\text{in}}\langle N^{(n)}|N^{(p)}|D\rangle_{\text{in}}$ (${}_{\text{in}}\langle\bar{D}|N^{(p)}|\bar{N}^{(n)}\rangle_{\text{in}}$) can serve as a description of the deuteron (antideuteron), in contrast to a strongly bound state for which all four matrix elements may be necessary to describe the system. We will show in Sec. 2 that ${}_{\text{in}}\langle N^{(n)}|N^{(p)}|D\rangle_{\text{in}}$ reduces to the Schrödinger wave function of the deuteron in the nonrelativistic limit, and will call this object the "deuteron wave function."

We want to emphasize that in the rest frame of the deuteron, $\mathfrak{N}^{(p)}$ depends only on a single three-vector. *Thus $\mathfrak{N}^{(p)}$ provides a covariant description of the deuteron which is free of a relative time co-ordinate.* We believe that this makes $\mathfrak{N}^{(p)}$ a simpler amplitude to consider than the corresponding Bethe-Salpeter amplitude for which a relative time coordinate is necessary or, equivalently, the neutron is off the mass shell.

Apart from numerical factors, the N-quantum amplitude is the Bethe-Salpeter amplitude with the mass-shell singularity of the neutron propagator removed and with the momentum of the neutron restricted to the mass shell. The N-quantum amplitude has two pieces,

$$\langle 0|N_{\beta\,\text{in}}^{(n)}(p')\delta_{M_n}(p')N_\alpha^{(p)}(p)D_{\mu\,\text{in}}(b)^*\delta_D(b)|0\rangle \\ =-2M_n(2\pi)^{-6}\theta(p')\delta_{M_n}(p')\theta(b)\delta_D(b) \\ \times\delta(p+p'-b)\mathfrak{N}_\mu^{(p)}(-p',b)_{\alpha\beta},$$

and

$$\langle 0|N_\alpha^{(p)}(p)N_{\beta\,\text{in}}^{(n)}(p')\delta_{M_n}(p')D_{\mu\,\text{in}}(b)^*\delta_D(b)|0\rangle \\ =2M_n(2\pi)^{-6}\theta(-p')\delta_{M_n}(p')\theta(b)\delta_D(b) \\ \times\delta(p+p'-b)\mathfrak{N}_\mu^{(p)}(-p',b)_{\alpha\beta},$$

where $|0\rangle$ is the vacuum state and $\delta_M(q)=\delta(q^2-M^2)$. We define the Bethe-Salpeter amplitude by

$$\langle 0|T(N_\beta^{(n)}(x)N_\alpha^{(p)}(y))D_{\mu\,\text{in}}(b)^*\delta_D(b)|0\rangle \\ =\int dp'dp\, e^{-ip'\cdot x-ip\cdot y}\chi_\mu(p',p,b)_{\alpha\beta} \\ \times\delta(p+p'-b)\theta(b)\delta_D(b).$$

The reduction formula and the subsidiary conditions on $\mathfrak{N}^{(p)}$ lead to the precise relation

$$\theta(\pm p')\delta_{M_n}(p')\theta(b)\delta_D(b)\delta(p+p'-b)[(p'^2-M_n^2) \\ \times\chi_\mu(p',p,b)_{\alpha\beta}\pm 2M_n(2\pi)^{-7}\mathfrak{N}_\mu^{(p)}(-p',b)_{\alpha\beta}]=0, \quad (1)$$

which was stated in words above.

Blankenbecler and Cook[5] considered an amplitude similar to our $\mathfrak{N}^{(p)}$, but with the proton field replaced by the proton current, in their dispersion-theory treatment of the deuteron; however, except for this their method has little in common with ours.

In Sec. 2 we will describe the N-quantum approximation applied to a model of the deuteron in pseudoscalar-meson theory, give some properties of the N-quantum amplitude which plays the role of the wave function, find a covariant single-time equation for this amplitude, and show that the equation reduces, in the weak-binding, nonrelativistic limit, to a Pauli-Schrödinger equation with the correct expression for the reduced mass and with the central and tensor potentials which come from second-order perturbation theory applied to pseudoscalar-meson theory. We also will discuss the interpretation of the N-quantum bound state amplitude in the weak-binding, nonrelativistic limit. Section 3 contains some concluding remarks, including the observation that a higher approximation will lead to a short-range force between the nucleons.

2. THE MODEL; REDUCTION TO THE WEAK-BINDING, NONRELATIVISTIC LIMIT

A. Formal Preliminaries

We study a relativistic model of the deuteron. The Heisenberg fields of the underlying field theory are the charged nucleon field N of spin and isospin $\frac{1}{2}$, and the neutral pseudoscalar pion field ϕ, of isospin 1. From the assumption of asymptotic irreducibility, it must be possible to expand each Heisenberg field in normal-ordered products of in- (or out-) fields whose quantum numbers add up to the quantum numbers of the Heisenberg field. From this point of view, the Heisenberg fields are fundamental, and all particles, elementary

[4] We could have considered an amplitude $\mathfrak{N}^{(n)}$ occurring in the expansion of the neutron Heisenberg field, in which the roles of proton and neutron are interchanged. We will not discuss the relation of $\mathfrak{N}^{(p)}$ and $\mathfrak{N}^{(n)}$ when the proton and neutron have different masses in the present article. In Sec. 2 we will assume isospin invariance, in which case the two $\mathfrak{N}$'s are the same.

[5] R. Blankenbecler and L. F. Cook, Phys. Rev. **119**, 1745 (1960).

or composite, are treated on an equal footing in terms of asymptotic field operators. The asymptotic field of most interest here is the deuteron in-field, $D_{\rm in}{}^{\mu}$. One of these would be present for each possible stable state of the deuteron. Since the nucleon is stable, neglecting β decay, there is a nucleon in-field, $N_{\rm in}$. It is irrelevant in our approximation whether or not the pion is stable; if so there is a pion in-field, $\boldsymbol{\phi}_{\rm in}$. Since the pion is stable for strong interactions, we will carry the pion in-field.

The Lagrangian is $\mathcal{L}=\mathcal{L}_0+\mathcal{L}_1$, with

$$\mathcal{L}_0=\tfrac{1}{2}[\bar{N},\ (i\boldsymbol{\partial}-M)N]_-+\tfrac{1}{2}\textstyle\sum_1{}^3[(\partial_\mu\phi)(\partial^\mu\phi)-m^2\phi^2],$$

and

$$\mathcal{L}_I=\tfrac{1}{2}g[\bar{N},\boldsymbol{\tau}\gamma^5N]\cdot\boldsymbol{\phi},$$

where $p_\mu\gamma^\mu=\not{p}$, bold type for $\boldsymbol{\phi}$ indicates its isovector character, and the spinor and isospinor indices of N and $\bar{N}$ are suppressed. We use $[A,B]_\mp=AB\mp BA$. Our γ^μ satisfy $[\gamma^\mu,\gamma^\nu]_+=2g^{\mu\nu}$, where $g^{00}=-g^{st}=1$, $s=t=1,2,3$, $g^{\mu\nu}=0$, $\mu\neq\nu$. Our $\gamma^5=\gamma^0\gamma^1\gamma^2\gamma^3$, and later we will use a standard set of γ^μ with

$$\gamma^0=\begin{pmatrix}1&0\\0&-1\end{pmatrix},\quad \gamma^s=\begin{pmatrix}0&\sigma^s\\-\sigma^s&0\end{pmatrix},\quad \gamma^s=\begin{pmatrix}0&-i\\-i&0\end{pmatrix}.$$

The Heisenberg equations of motion in momentum space are

$$(\not{p}-M)N(p)=-\tfrac{1}{2}g\int dp_1dp_2\delta(p-p_1-p_2)\times[\boldsymbol{\tau}\cdot\boldsymbol{\phi}(p_1),\ \gamma^5N(p_2)]_+,\quad(2)$$

and

$$(m^2-k^2)\boldsymbol{\phi}(k)=\tfrac{1}{2}g\int dp_1dp_2\delta(k-p_1-p_2)\times[\bar{N}(p_1),\ \boldsymbol{\tau}\gamma^5N(p_2)]_-.\quad(3)$$

With our conventions, $N(p)$ annihilates energy-momentum p; to insure that $\bar{N}(p)$ also does this we define $\bar{N}(p)=N(-p)^*A$, where A (γ^0 in the standard set) adjoints the γ^μ, $A\gamma^\mu A^{-1}=\gamma^{\mu*}$. We have omitted renormalization counter-terms, because they do not enter in the weak-binding, nonrelativistic limit we will study later. The canonical and local commutation relations of the Heisenberg fields are not satisfied in our approximation.

To get the simplest nontrivial equation for the deuteron wave function we will use the ansätze

$$N(p)=N_{\rm in}(p)\delta_M(p)+\int dp'db\ \delta(p-p'-b)\times\mathfrak{N}_\mu(p',b):\bar{N}_{\rm in}(p')\delta_M(p')D_{\rm in}{}^\mu(b)\delta_D(b):,$$

$$\bar{N}(p)=\bar{N}_{\rm in}(p)\delta_M(p)+\int dp'db\ \delta(p-p'-b)\times:\bar{D}_{\rm in}{}^\mu(b)\delta_D(b)N_{\rm in}(p')\delta_M(p'):\bar{\mathfrak{N}}_\mu(p',b),$$

where

$$\bar{\mathfrak{N}}_\mu(p',b)=A^{-1}\mathfrak{N}_\mu(-p',\ -b)^*A,$$

and

$$\boldsymbol{\phi}(k)=\boldsymbol{\phi}_{\rm in}(k)\delta_m(k)+\int dp_1dp_2\delta(k-p_1-p_2)\times:\bar{N}_{\rm in}(p_1)\delta_M(p_1)\boldsymbol{\varphi}(p_1,p_2)N_{\rm in}(p_2)\delta_M(p_2):.$$

The commutation relations of the in-fields are

$$[N_{\rm in}(p)\delta_M(p),\bar{N}_{\rm in}(q)\delta_M(q)]_+=(2\pi)^{-3}\epsilon(p)\delta_M(p)\delta(p+q)(\not{p}+M),$$

$$[\phi_{\alpha\ \rm in}(k)\delta_m(k),\phi_{\beta\ \rm in}(l)\delta_m(l)]_-=(2\pi)^{-3}\delta_{\alpha\beta}\epsilon(k)\delta_m(k)\delta(k+l),$$

$$[D_{\rm in}{}^\mu(k)\delta_D(k),D_{\rm in}{}^\nu(l)\delta_D(l)]_-=-(2\pi)^{-3}(g^{\mu\nu}-D^{-2}k^\mu k^\nu)\epsilon(k)\delta_D(k)\delta(k+l),$$

where $\epsilon(p)=1$ (-1) in the forward (backward) cone and vanishes elsewhere.

The transformation properties of the Heisenberg and in-fields lead to covariance properties of the functions $\mathfrak{N}_\mu$ and $\boldsymbol{\varphi}$. For example, the spinor transformation laws[6] for N and $\bar{N}_{\rm in}$ and the vector transformation law for $D_{\rm in}{}^\mu$,

$$U(a,\Lambda)N(p)U(a,\Lambda)^{-1}=\pm e^{-i\Lambda p\cdot a}S(\Lambda)^{-1}N(\Lambda p),$$

$$U(a,\Lambda)\bar{N}_{\rm in}(p)\delta_M(p)U(a,\Lambda)^{-1}=\pm e^{-i\Lambda p\cdot a}\bar{N}_{\rm in}(\Lambda p)\delta_M(p)S(\Lambda),$$

$$U(a,\Lambda)D_{\rm in}{}^\mu(b)\delta_D(b)U(a,\Lambda)^{-1}=e^{-i\Lambda b\cdot a}(\Lambda^{-1})^\mu{}_\nu D_{\rm in}{}^\nu(\Lambda b)\delta_D(b),$$

where Λ preserves the sense of time, lead to

$$S(\Lambda)^{-1}\mathfrak{N}_\mu(p,b)S(\Lambda)^{T-1}=\Lambda_\mu{}^\nu\mathfrak{N}_\nu(\Lambda^{-1}p,\Lambda^{-1}b).\quad(4)$$

Using the matrix B ($\tfrac{1}{2}[\gamma^1,\gamma^3]_-$ in the standard set) which transposes the γ^μ, $B\gamma^\mu B^{-1}=\gamma^{\mu T}$, and $S(\Lambda)^{-1T}=BS(\Lambda)B^{-1}$, we find that $\mathfrak{M}_\mu(p,b)=\mathfrak{N}_\mu(p,b)B$ satisfies

$$S(\Lambda)^{-1}\mathfrak{M}_\mu(p,b)S(\Lambda)=\Lambda_\mu{}^\nu\mathfrak{M}_\nu(\Lambda^{-1}p,\Lambda^{-1}b).\quad(5)$$

Similarly, the parity (i_s) transformation laws,

$$U(i_s)N(p)U(i_s)^{-1}=\alpha_s\gamma^0N(i_sp),\quad|\alpha_s|=1,$$

$$U(i_s)\bar{N}_{\rm in}(p)\delta_M(p)U(i_s)^{-1}=\bar{\alpha}_s\bar{N}_{\rm in}(i_sp)\delta_M(p)\gamma^0,$$

$$U(i_s)D_{\rm in}{}^\mu(b)\delta_D(b)U(i_s)^{-1}=-(\alpha_s)^2g^{\mu\mu}D_{\rm in}{}^\mu(i_sb)\delta_D(b),$$

where $i_sp=(p_0,\ -\mathbf{p})$, and $g^{\mu\mu}D_{\rm in}{}^\mu$ is not summed, lead to

$$\gamma^0\mathfrak{M}_\mu(p,b)\gamma^0=-g_{\mu\mu}\mathfrak{M}_\mu(i_sp,i_sb),\quad(6)$$

and μ is not summed; and the anti-unitary time-reversal (i_t) transformation laws,

$$A(i_t)N(p)A(i_t)^{-1}=\alpha_tC\gamma^5\gamma^0N(i_sp),\quad|\alpha_t|=1,$$

$$A(i_t)\bar{N}_{\rm in}(p)\delta_M(p)A(i_t)^{-1}=-\bar{\alpha}_t\bar{N}_{\rm out}(i_sp)\delta_M(p)\gamma^0\gamma^5C^{-1},$$

$$A(i_t)D_{\rm in}{}^\mu(b)\delta_D(b)A(i_t)^{-1}=(\alpha_t)^2g^{\mu\mu}D_{\rm out}{}^\mu(i_sb)\delta_D(b),$$

[6] The operator $U(a,\Lambda)$ below is a unitary representation of the Poincaré group, a is the translation, and Λ is the (homogeneous) Lorentz transformation. Strictly speaking, we should use the covering group, $SL(2,C)$, of the Lorentz group, in which case we could remove the $\pm$ on the right-hand side of the transformation laws for N and $\bar{N}_{\rm in}$.

where μ is not summed, lead to

$$C\gamma^5\gamma^0\mathfrak{M}_\mu(p,b)(C\gamma^5\gamma^0)^{-1}=g_{\mu\mu}\mathfrak{M}_\mu{}^{\text{c.c.}}(i_sp,i_sb)\,, \quad (7)$$

for p and b in opposite cones, where μ is not summed, c.c. stands for complex conjugate, and C (γ^2 in the standard set) changes γ's to their negative complex conjugate, $C\gamma^\mu C^{-1}=-\bar{\gamma}^\mu$. The restriction of Eq. (7) to p and b in opposite cones is necessary because time reversal interchanges in and out, and only one-particle in and out states are the same. Time reversal relates $\mathfrak{M}_\mu$, $\mathfrak{M}_\mu{}^{\text{c.c.}}$, and the expression $\sum_j\langle 0|N|j\rangle_{\text{in in}}\langle j|S|\bar{N}D\rangle_{\text{in}}$, where S is the S operator, for b and p in the same cone. Charge-conjugation invariance relates our deuteron amplitude $\mathfrak{M}$ to an analogous amplitude for the antideuteron rather than placing a condition on $\mathfrak{M}$ itself. The isospin transformation laws,[7]

$$U(A)N(p)U(A)^{-1}=\mathfrak{D}^{(1/2)}(A)^{-1}N(p)\,,$$
$$U(A)\bar{N}_{\text{in}}(p)\delta_M(p)U(A)^{-1}=\bar{N}_{\text{in}}(p)\delta_M(p)\mathfrak{D}^{(1/2)}(A)\,,$$
$$U(A)D_{\text{in}}{}^\mu(b)\delta_D(b)U(A)^{-1}=\mathfrak{D}^{(I)}(A)_{\alpha\beta}{}^{-1}D_{\beta\ \text{in}}{}^\mu(b)\delta_D(b)\,,$$

for a deuteron of isospin I, where A is an element of $SU(2)$, lead to

$$\mathfrak{D}^{(1/2)}(A)^{-1}\mathfrak{M}_{\mu,\alpha}(p,b)\mathfrak{D}^{(1/2)}(A)^{T-1} =\mathfrak{D}^{(I)}(A)_{\alpha\beta}\mathfrak{M}_{\mu,\beta}(p,b)\,. \quad (8)$$

Using τ_2, which converts transpose to inverse via

$$\mathfrak{D}^{(1/2)}(A)^T\tau_2=\tau_2\mathfrak{D}^{(1/2)}(A)^{-1}\,,$$

we find that

$$\mathfrak{L}_{\mu,\alpha}(p,b)=\mathfrak{M}_{\mu,\alpha}(p,b)\tau_2$$

satisfies

$$\mathfrak{D}^{(1/2)}(A)^{-1}\mathfrak{L}_{\mu,\alpha}(p,b)\mathfrak{D}^{(1/2)}(A)=\mathfrak{D}^{(I)}(A)_{\alpha\beta}\mathfrak{L}_{\mu,\beta}(p,b)\,. \quad (9)$$

Equation (9) shows that $\mathfrak{L}_\mu$ is a multiple of the identity in isospin space for the isoscalar deuteron.

Because of the various amplitudes enter the ansätze for the Heisenberg fields as coefficients of in-fields, they are relevant only when their arguments are on appropriate mass shells. In addition, since $\bar{N}_{\text{in}}(p)\delta_M(p)$ satisfies the free Dirac equation, $\bar{N}_{\text{in}}(p)\delta_M(p)(p+M)=0$, the amplitude $\mathfrak{M}_\mu$ satisfies the subsidiary condition

$$\mathfrak{M}_\mu(p,b)(p+M)^T=0\,, \quad (10)$$

and $\mathfrak{L}_\mu$ (and $\mathfrak{M}_\mu$) satisfy

$$\mathfrak{L}_\mu(p,b)(p+M)=0\,. \quad (11)$$

The spin-one subsidiary condition for $D_{\text{in}}{}^\mu(b)\delta_D(b)$, $b_\mu D_{\text{in}}{}^\mu(b)\delta_D(b)=0$, leads to

$$b^\mu\mathfrak{L}_\mu(p,b)=0\,. \quad (12)$$

Standard covariance analysis based on Eqs. (5), (6), and (7) leads to a form of $\mathfrak{L}_\mu$ in terms of eight invariant

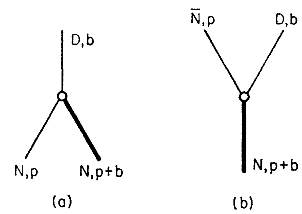


FIG. 1. Diagrams for $\mathfrak{L}_\mu$: (a) p and b in opposite cones; (b) p and b in the same cone.

functions for the spin-1 deuteron:

$$\mathfrak{L}_\mu(p,b)=\{(b+D)[M\mathfrak{L}^{(1)}(p\cdot b)\gamma_\mu+\mathfrak{L}^{(2)}(p\cdot b)p_\mu] +(b-D)[M\mathfrak{L}^{(3)}(p\cdot b)\gamma_\mu+\mathfrak{L}^{(4)}(p\cdot b)p_\mu]\}(p-M)\,, \quad (13)$$

where each of the $\mathfrak{L}^{(i)}$ stands for two functions; one (real valued) when p and b are in opposite cones, the other (complex valued) when they are in the same cone.

B. Diagrams

The amplitude $\mathfrak{L}_\mu$ can be represented by two diagrams (Fig. 1); one when p and b are in opposite cones, the other when they are in the same cone. In these diagrams the heavy line stands for the Heisenberg field (including the propagator which some authors factor off) and can be off the mass shell with a retarded boundary condition[1] (which in the low approximation to be used in this article need not be specified), and the light lines stand for the in-fields and are always on one of the two pieces of the appropriate mass shells. The amplitude $\mathfrak{L}_\mu$ for p and b in the same cone is the partner under crossing of the line associated with $\bar{N}_{\text{in}}$ in the other diagram. We will carry out the analysis in the lowest approximation where only this latter diagram enters, reserving the effect of including contributions from the other diagram to a later article.

C. Equation of Motion for the Deuteron Bound State

Finally we derive the equation of motion for $\mathfrak{L}_\mu$. To find this equation, we insert the ansätze for $N(p)$ and $\phi(k)$ in the equation for $N(p)$, renormal order, and look for the coefficient of the normal-ordered expression $:\bar{N}_{\text{in}}D_{\text{in}}{}^\mu:$. We find

$$(p+b-M)\mathfrak{L}_\mu(p,b)=-\tfrac{1}{2}(2\pi)^{-3}g\int dp'\delta_M(p')\gamma^5\boldsymbol{\tau} \times\mathfrak{L}_\mu(p',b)\psi(p,-p')(-p+M)\,, \quad (14)$$

where $\psi(p,p')=\tau_2B^{-1}\varphi(p,p')^TB\tau_2$. The factor $(-p+M)$ on the right-hand side of Eq. (14) insures that the subsidiary condition Eq. (11) is satisfied. See Fig. 2 for a diagram of Eq. (14). To find the simplest closed equation for $\mathfrak{L}_\mu$, we calculate φ from Eq. (3) for ϕ using only the first term in the ansätze for N and $\bar{N}$, which amounts to the Born approximation:

$$\varphi(p,p')=g\boldsymbol{\tau}\gamma^5[m^2-(p+p')^2]^{-1}\,, \quad (15)$$

[7] We added an isospin index, which is unnecessary for the isoscalar deuteron, here to show how the equations look for a composite particle with nonzero isospin.

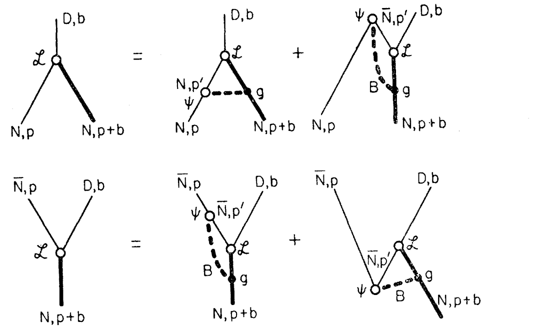


FIG. 2. Diagram of Eq. (14) with the two pieces of $\mathcal{L}$ separated. The heavy lines are part of $\mathcal{L}$ or ψ when connected to an open circle; otherwise the heavy line is a retarded denominator. The light line is a mass-shell δ function as an internal line, and indicates an in-field as an external line. Closed loops indicate $\int dp'$. The solid dot corresponds to the coupling constant g. The first term in each equation is the "direct" graph; the second term is the "crossed" graph.

and

$$\psi(p,p') = -g\boldsymbol{\tau}\gamma^5[m^2-(p+p')^2]^{-1}, \qquad (16)$$

since $\tau_2\boldsymbol{\tau}^T\tau_2 = -\boldsymbol{\tau}$. Replacing ψ according to Eq. (16), we find the following equation for $\mathcal{L}_\mu$ alone:

$$(p+b-M)\mathcal{L}_\mu(p,b) = -\tfrac{1}{2}(2\pi)^{-3}g^2\int dp'\delta_M(p') \times[m^2-(p-p')^2]^{-1}\gamma^5\boldsymbol{\tau}\cdot\mathcal{L}_\mu(p',b)\boldsymbol{\tau}\gamma^5(p-M). \qquad (17)$$

The piece of $\mathcal{L}_\mu$ which looks like a description of the deuteron as a two-nucleon bound state is the one [Fig. (1b)] with b and p in opposite cones, and, what is more important, this piece satisfies the Schrödinger equation in the nonrelativistic weak-binding limit. For the remainder of this article, we will drop the piece of $\mathcal{L}_\mu$ with both momenta in the same cone. This approximation restricts us to the weak-binding limit. Equation (17) changes to (choosing $b>0$, $p<0$)

$$(p+b-M)\mathcal{L}_\mu(p,b) = -(2\pi)^{-3}g^2\int dp'\theta(-p')\delta_M(p') \times[m^2-(p-p')^2]^{-1}\gamma^5\boldsymbol{\tau}\cdot\mathcal{L}_\mu(p',b)\boldsymbol{\tau}\gamma^5(p-M). \qquad (18)$$

See Fig. 3 for a diagram of Eq. (18). To see that the factor of $\frac{1}{2}$ in Eq. (17) should be omitted on the right hand in Eq. (18), notice that Eq. (18) can be found directly from Eq. (2) with $\boldsymbol{\tau}\cdot\boldsymbol{\phi}\gamma^5 N$ replacing $\frac{1}{2}[\boldsymbol{\tau}\cdot\boldsymbol{\phi}, \gamma^5 N]_+$ on the right-hand side. Since $\mathcal{L}_\mu$ is isoscalar, we can remove the $\boldsymbol{\tau}$ matrices from the right-hand side of both Eq. (17) and Eq. (18), using $\boldsymbol{\tau}\cdot\boldsymbol{\tau}=3$. However, we will carry the $\boldsymbol{\tau}$ matrices to show how the equations look for an isovector state.

D. Reduction to the Nonrelativistic Limit

The rest frame of the deuteron is convenient for the reduction to the weak-binding, nonrelativistic limit. For this reduction, we decompose the four-by-four matrix $\mathcal{L}_\mu$ into two-by-two blocks, using the explicit γ matrices given above. In the rest frame, $b=(D,\mathbf{0})$, the subsidiary condition, Eq. (12), leads to $\mathcal{L}_0=0$, and the other subsidiary condition, Eq. (11), can be satisfied by introducing the operator $S(\mathbf{p})=[E(\mathbf{p})+M]^{-1}\boldsymbol{\sigma}\cdot\mathbf{p}$, $E(\mathbf{p}) = (\mathbf{p}^2+M^2)^{1/2}$, and introducing two-by-two matrices $\mathcal{A}(\mathbf{p})$ and $\mathcal{B}(\mathbf{p})$ in terms of which

$$\mathcal{L}_j(p,b)|_{b=(D,\mathbf{0})} \equiv \mathcal{L}_j(\mathbf{p}) = \begin{pmatrix} \mathcal{A}_j(\mathbf{p}) & \mathcal{A}_j(\mathbf{p})S(\mathbf{p}) \\ \mathcal{B}_j(\mathbf{p}) & \mathcal{B}_j(\mathbf{p})S(\mathbf{p}) \end{pmatrix},$$

where $j=1, 2, 3$. Thus Eq. (11), which is the free Dirac equation for the on-shell nucleon, implies that in two-by-two block form $\mathcal{L}_j$ contains only two independent blocks rather than four. Inserting the block form for $\mathcal{L}_j$ into Eq. (18) specialized to the rest frame leads to coupled equations for $\mathcal{A}_j$ and $\mathcal{B}_j$:

$$(D-E-M)\mathcal{A}_j - \boldsymbol{\sigma}\cdot\mathbf{p}\mathcal{B}_j = (2\pi)^{-3}g^2\int d^3p'(2E')^{-1}[m^2-2M^2+2EE'-2\mathbf{p}\cdot\mathbf{p}']^{-1} \times\boldsymbol{\tau}\cdot\mathcal{B}_j'\boldsymbol{\tau}(S-S')(E+M), \qquad (19)$$

$$\boldsymbol{\sigma}\cdot\mathbf{p}\mathcal{A}_j + (E-D-M)\mathcal{B}_j = (2\pi)^{-3}g^2\int d^3p'(2E')^{-1}[m^2-2M^2+2EE'-2\mathbf{p}\cdot\mathbf{p}']^{-1} \times\boldsymbol{\tau}\cdot\mathcal{A}_j'\boldsymbol{\tau}(S-S')(E+M), \qquad (20)$$

where we have suppressed the arguments of the functions, for example, $\mathcal{A}_j\equiv\mathcal{A}_j(\mathbf{p})$, $\mathcal{A}_j'\equiv\mathcal{A}_j(\mathbf{p}')$, etc. We eliminate $\mathcal{B}_j$ from Eqs. (19) and (20) to find an equation for $\mathcal{A}_j$ alone:

$$[D-E-M-(D+M-E-i\epsilon)^{-1}\mathbf{p}^2]\mathcal{A}_j$$
$$= (2\pi)^{-3}g^2\int d^3p'(2E')^{-1}[m^2-2M^2+2EE'-2\mathbf{p}\cdot\mathbf{p}']^{-1}\{[D+M-E'-i\epsilon]^{-1}\boldsymbol{\sigma}\cdot\mathbf{p}'-[D+M-E-i\epsilon]^{-1}\boldsymbol{\sigma}\cdot\mathbf{p}\}\boldsymbol{\tau}$$
$$\times\mathcal{A}_j'\boldsymbol{\tau}(S-S')(E+M) - (2\pi)^{-6}g^4\int d^3p'd^3p''(4E'E'')^{-1}[m^2-2M^2+2EE'-2\mathbf{p}\cdot\mathbf{p}']^{-1}$$
$$\times[m^2-2M^2+2E'E''-2\mathbf{p}'\cdot\mathbf{p}'']^{-1}[D+M-E'-i\epsilon]^{-1}\{3\mathcal{A}_j''+2\boldsymbol{\tau}\cdot\mathcal{A}_j''\boldsymbol{\tau}\}$$
$$\times(S'-S'')(S-S')(E'+M)(E+M). \qquad (21)$$

In order to reduce Eq. (21) to the weak-binding, nonrelativistic limit, we use $E(p)\sim M+(2M)^{-1}\mathbf{p}^2$, keep only the lowest powers of $\mathbf{p}$ and $\mathbf{p}'$ on the right-hand side, and drop the last term on the right-hand side in which the effective potential acts twice. Using these approximations, standard manipulations, and going into position space via

$$\mathcal{A}_j(\mathbf{x})=\int d^3p\,\mathfrak{A}_j(\mathbf{p})\exp(i\mathbf{p}\cdot\mathbf{x})\,,$$

we find

$$-\frac{1}{M}\nabla^2\mathcal{A}_j(\mathbf{x})+\left(\frac{g^2}{4\pi}\right)\left(\frac{1}{2M}\right)^2\sigma^s\tau_\alpha\mathcal{A}_j(\mathbf{x})\tau_\alpha\sigma^t\nabla^s\nabla^t V(\mathbf{x})$$
$$=-(2M-D)\mathcal{A}_j(\mathbf{x})\,,\quad(22)$$

where $V(\mathbf{x})=|\mathbf{x}|^{-1}\exp(-m|\mathbf{x}|)$. This potential agrees with the order g^2 potential of pseudoscalar-meson theory.[8] In our formalism, the $\boldsymbol{\tau}$ matrices produce a factor of 3 for an isoscalar state ($\mathfrak{A}_j\sim 1$) and a factor of -1 for an isovector state ($\mathfrak{A}_j\sim\boldsymbol{\tau}$). The standard reduction $\nabla^s\nabla^t=\frac{1}{3}\delta^{st}\nabla^2+(\nabla^s\nabla^t-\frac{1}{3}\delta^{st}\nabla^2)$ separates the central and tensor forces:

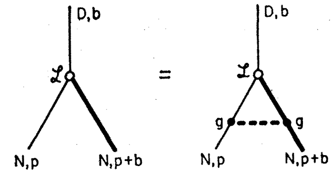
$$\sigma^s\mathcal{A}_j(\mathbf{x})\sigma^t\nabla^s\nabla^t V(\mathbf{x})=\tfrac{1}{3}m^2\sigma^s\mathcal{A}_j(\mathbf{x})\sigma^s\left(V(\mathbf{x})-\frac{4\pi}{m^2}\delta(\mathbf{x})\right)$$
$$+\sigma^s\mathcal{A}_j'(\mathbf{x})\sigma^t(\hat{x}^s\hat{x}^t-\tfrac{1}{3}\delta^{st})V(\mathbf{x})\left(1+\frac{3}{m|\mathbf{x}|}+\frac{3}{m^2\mathbf{x}^2}\right),$$

$\hat{x}^s=|\mathbf{x}|^{-1}x^s$. For the central part of the force, the $\boldsymbol{\sigma}$ matrices produce a factor of -1 for a triplet state and a factor of 3 for a singlet state.

The further reduction of Eq. (22) for the triplet state using the $^3S_1$ and $^3D_1$ radial wave functions is standard[9]; we will not repeat it.

To make an interpretation of the amplitudes $\mathfrak{N}$ and $\mathcal{L}$ in the nonrelativistic limit, we notice (see Sec. 1) that $\mathfrak{N}^{(p)}(-p,b)\sim\langle 0|N_{\rm in}{}^{(n)}(p)N^{(p)}D_{\rm in}(b)^*|0\rangle$, $p>0$ and $b>0$, where, for definiteness, we consider the amplitude which occurs in the expansion of the proton Heisenberg field, superscripts (p) or (n) indicate proton or neutron, and we have suppressed all irrelevant factors. Thus $\mathfrak{N}^{(p)}\sim\sum u^{(p)}u^{(n)T}$ where $u^{(p)}$ and $u^{(n)}$ are positive-energy Dirac spinors, and in the rest frame of the deuteron the three-vector $\mathbf{p}$ occurring in the first argument of $\mathfrak{N}^{(p)}$ is the momentum of the proton and minus the momentum of the neutron. Operators acting on $\mathfrak{N}^{(p)}$ from the left are proton operators; operators acting from the right are the transposes of neutron operators [see the transformation law, Eq. (5)]. The amplitude $\mathfrak{N}^{(p)}$ obeys a Dirac equation with interaction acting from the left [Eq. (18)], and the transpose of a free Dirac equation [Eq. (10)] acting from the right. The two-by-two block in the upper left-hand corner of $\mathfrak{N}^{(p)}$ contains the large components of the two-body wave function. Since we have used $\mathcal{L}=\mathfrak{N}B\tau_2$ in making our reduction to two-by-two form, we must take account of $B\tau_2$ in interpreting (neutron) operators acting from the right on either $\mathcal{L}$ or $\mathfrak{A}$. In our standard representation

Fig. 3. Diagram of Eq. (18).



$$B=\begin{pmatrix} i\sigma_2 & 0\\ 0 & i\sigma_2\end{pmatrix}.$$

From either $\sigma_2\boldsymbol{\sigma}^T\sigma_2=-\boldsymbol{\sigma}$, and the analogous equation for $\boldsymbol{\tau}$, or the transformation law of $\mathcal{L}$ [Eq. (5)] we see that operators acting from the left on $\mathcal{L}$ or $\mathfrak{A}$ are proton operators as before, while operators acting from the right are minus the corresponding neutron operator. This minus sign is the reason that the factors coming from the $\boldsymbol{\tau}$ and $\boldsymbol{\sigma}$ matrices which we gave above differ from the conventional ones.[10]

In keeping with our assumption of isospin invariance, we have taken the proton and neutron masses to be equal. We want to emphasize that we get the correct reduced mass, $\mu=M_pM_n(M_p+M_n)^{-1}$, in the weak-binding, nonrelativistic limit if the masses are taken to be different. In that case, using the equation of motion for the proton field, the bracket on the left-hand side of Eq. (21) changes, in the limit, to

$$D-E_n(\mathbf{p})-M_p-(D+M_p-E_n(\mathbf{p})-i\epsilon)^{-1}\mathbf{p}^2$$
$$\approx -B-(2\mu)^{-1}\mathbf{p}^2,$$

where

$$E_n(\mathbf{p})=(\mathbf{p}^2+M_n{}^2)^{1/2}.$$

3. CONCLUDING REMARKS

We have given a version of the N-quantum approximation for the deuteron described by pseudoscalar-meson theory. We found a manifestly covariant equation, Eq. (18), for the N-quantum amplitude which plays the role of the deuteron wave function, and showed that in the weak-binding, nonrelativistic limit this equation reduces to a Pauli-Schrödinger equation, Eq. (22), with the correct reduced mass and the usual central and tensor forces of pseudoscalar-meson theory. Although our covariant equation, Eq. (18), is logarithmically divergent and requires renormalization,[1] we neglected it because the renormalization effects vanish in the weak-binding, nonrelativistic limit. We expect that recoil and renormalization effects, which can be

[8] L. Hulthén and M. Sugawara, in *Handbuch der Physik*, edited by S. Flügge (Springer-Verlag, Berlin, 1957), Band XXXIX. See p. 21, Eq. (9.12), and p. 18, Eq. (8.9).

[9] Reference 8, pp. 64–66.

[10] Reference 8, p. 25, Eq. (10.4).

taken into effect in a higher approximation, will moderate the singularity of the potential at $r=0$. We also neglected some triangle graphs, including one which in the limit just mentioned leads to a nucleon–nucleon force with range $(4M^2-m^2)^{-1/2}$. This force arises in the following way from terms which couple the two different pieces of our amplitude (see Fig. 2). The Yukawa potential factor

$$[m^2-2M^2+2EE'-2\mathbf{p}\cdot\mathbf{p}']^{-1}\sim[m^2+|\mathbf{p}-\mathbf{p}'|^2]^{-1}$$

in the static limit of the direct graph is replaced by

$$[m^2-2M^2-2EE'-2\mathbf{p}\cdot\mathbf{p}']^{-1} \sim -[4M^2-m^2+|\mathbf{p}+\mathbf{p}'|^2]^{-1}$$

in the static limit of the crossed graph. Thus the crossed graph has an exchange potential with the short-range radial dependence $|\mathbf{x}|^{-1}\exp[-(4M^2-m^2)^{1/2}|\mathbf{x}|]$. This short-range force seems worthy of further study in connection with the deuteron; for problems involving strong binding[3] the short-range force may be very important. We hope to study this short-range force, as well as other effects which occur in the complete triangle equations for our amplitude, in a later article.

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N-Quantum Solution of the Derivative Coupling Model

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The derivative coupling model is solved using the N-quantum approximation. It follows by inspection of the form of the solution that there is no scattering. Although the Wightman and Green's functions of this model are nontemperate, the N-quantum amplitudes are temperate. The way in which the temperate N-quantum amplitudes lead to nontemperate Wightman and Green's functions is pointed out. It is suggested that the N-quantum approach may be useful in the study of nonrenormalizable field theories.

1. INTRODUCTION

WE have been studying a new method, the N-quantum approximation,[1] for finding approximate solutions of the Heisenberg field equations of specific quantum field theories. Here, we apply this method to an exactly soluble model, the derivative coupling model with $\mathcal{L}_1=g\bar{\psi}\gamma^\mu\psi\partial_\mu\phi$, where ψ is a charged spinor field and ϕ is a neutral scalar field.[2,3] We find that the N-quantum approximation yields the formal exact solution in a straightforward way, and gives a form of the solution which makes it clear by inspection that there is no scattering in this model. Although the Wightman functions and Green's functions of this model are nontemperate, the amplitudes which appear in the N-quantum form of the solution are temperate; indeed, they are constants. We point out the way in which the temperate N-quantum amplitudes, which (in momentum space) are multiple retarded commutator functions with all but one variable on the mass shell,[4] lead to nontemperate Wightman and Green's functions. If in nontrivial nonrenormalizable field theories the N-quantum amplitudes are also temperate, they might be useful in the study of such theories.

2. DERIVATIVE COUPLING MODEL IN THE HEISENBERG PICTURE

The Lagrangian of the gradient coupling model is

$$\mathcal{L}=\tfrac{1}{4}[\bar{\psi},(i\gamma^\mu\partial_\mu-M)\psi]_- -\tfrac{1}{4}[(i\partial_\mu\bar{\psi}\gamma^\mu+M\bar{\psi}),\psi]_- +\tfrac{1}{2}(\partial_\mu\phi\partial^\mu\phi-m^2\phi^2)+\tfrac{1}{4}g[[\bar{\psi},\gamma^\mu\psi]_-,\partial_\mu\phi]_+,$$

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[1] O. W. Greenberg, Phys. Rev. **139**, B1038 (1965); O. W. Greenberg and R. J. Genolio, preceding paper, Phys. Rev. **150**, 1070 (1966); and references cited in these articles.

[2] F. J. Dyson, Phys. Rev. **73**, 929 (1948), showed that this model does not lead to scattering.

[3] Green's functions of this model were determined and found to be nontemperate by R. Arnowitt and S. Deser, Phys. Rev. **100**, 349 (1955); and by S. Okubo, Nuovo Cimento **19**, 574 (1961). The model was also studied by T. Pradhan, Proc. Natl. Inst. Sci. India **A28**, 249 (1962); Nucl. Phys. **43**, 11 (1963). As far as we know, the most recent and most careful treatment of this model is by B. Klaiber, Nuovo Cimento **36**, 165 (1965).

[4] V. Glaser, H. Lehmann, and W. Zimmermann, Nuovo Cimento **6**, 1122 (1957).