Mutual Refraction of Electrons and Photons*

T. W. B. KIBBLE
Department of Physics, Imperial College of Science and Technology, London, England
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The equation of motion governing the time-averaged motion of a classical electron in an oscillating electromagnetic field is derived under very general conditions. It is shown that the motion is that of a relativistic particle of variable rest mass $m(1+\mu^2)^{1/2}$, where μ^2 is the parameter proportional to the field intensity, introduced earlier. Nonrelativistically, it is that of a particle with the effective potential-energy function $\frac{1}{2}mc^2\mu^2$. The complete analogy between the processes of refraction of light by electrons and of electrons by light is emphasized. It is shown that in the case of focused laser beams, effects substantially larger than those originally predicted are to be expected. The interaction of electrons with standing waves is discussed with particular reference to the Kapitza-Dirac effect, and it is shown that a modified effect may perhaps be expected at high intensities and low electron velocities. The use of classical electrodynamics is justified by showing, with the help of the WKB approximation, that specifically quantum effects should normally be negligible. A model which helps to explain the complementarity between the two refraction effects is presented in which the electrons and photons are treated as classical relativistic fluids. The relationship of this model to quantum electrodynamics is pointed out.

1. INTRODUCTION

THE effects of finite electron density on the interaction of light with an electron gas or plasma are well known. But it is only in the last few years, since the development of high-intensity laser beams, that the complementary effects of finite photon density have begun to receive much attention. In a recent paper[1] (here referred to as I) it was shown that one such effect that we may expect is the refraction of an electron beam passing through an intense electromagnetic wave. It is the aim of the present paper to study this effect in greater detail, and particularly to demonstrate its very close similarity to the more familiar phenomenon of refraction of light by electrons.

This work has developed out of earlier investigations of the interaction of electrons with intense plane-wave fields.[2–10] As was shown in I, the intensity-dependent frequency shift in high-intensity Compton scattering, predicted by Brown and Kibble[3] and by Goldman,[4] and the refraction effect discussed here may be regarded as two aspects of the same general phenomenon. They are both produced by forces depending on the spatial gradient or rate of change of the field intensity, of which the simplest example is the "field-gradient force" discussed by Phillips and Sanderson.[9]

We shall begin by presenting in Sec. 2 a much more thorough, and fully relativistic, derivation of the equation of motion obtained in I for the averaged motion of a classical electron in an electromagnetic wave. This derivation makes use of the vector potential rather than the electromagnetic field itself, and exhibits much more clearly the essential unity of the various effects under consideration. We shall in particular discuss the nature of the averaging procedure and the validity of the approximations involved.

In Sec. 3 we demonstrate that under suitable conditions the equation of motion we have obtained is precisely that of a relativistic particle with the variable rest mass $m^*=m(1+\mu^2)^{1/2}$, where μ^2 is the intensity parameter introduced in BK, K, and I. This effective mass has also been obtained for the case of plane-wave fields by semiclassical[3] and quantum-mechanical[10] arguments. Nonrelativistically, the motion is that of a particle with potential energy $\frac{1}{2}mc^2\mu^2$.

Section 4 is devoted to a comparison between the refraction of light by electrons, and of electrons by an electromagnetic wave. We show that both effects may be described by saying that the propagating particle acquires an effective mass, and that the expressions for the mass-squared shift are identical when written in terms of particle density and energy.

The way in which an electron acquires its increased mass as it enters a beam of light is discussed in Sec. 5. One may distinguish two basically different situations: Either the electron is overtaken by the beam or it enters from one side. It is the first of these which corresponds to the plane wave calculation of BK and G. The true physical situation will in general be intermediate between these two idealized cases, but we show that in

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the case of a focused beam it is normally a more reasonable approximation to say that the electron enters the beam from one side. Therefore the expected effects will be rather different from those originally predicted in BK and G, and in fact appreciably larger. They are moreover still of considerable interest.

In Sec. 6 we turn to the problem of the interaction of an electron with standing waves. This is a very interesting problem, particularly in connection with the Kapitza-Dirac effect.[11–13] It also turns out to be much more complicated than the analogous problem for the case of a traveling wave. The effective potential-energy function of the electron is periodic in space with period equal to one half-wavelength of the light. (The Kapitza-Dirac effect may be regarded as the coherent scattering of the electron by this periodic potential.) We show that especially for slow electrons a remarkable variety of possible effects may occur. In particular, it is shown that at sufficiently high intensities the electron can only just get over the crests of the periodic potential, the Kapitza-Dirac effect might very well be substantially modified. The intensity used in the experiment of Bartell, Thompson, and Roskos[12] is in fact in precisely this transition region and it is therefore peculiarly difficult to make reliable predictions. It is suggested that further experiments of this type with a range of intensities and electron velocities would yield some very interesting information.

Up to this point the discussion has been entirely classical. In order to justify this approach it is necessary to show that the same results would be obtained quantum-mechanically. (This is particularly true in view of the controversy which has surrounded this question.[5–8]) Since the equivalence of quantum-mechanical and semiclassical calculations has already been demonstrated,[6,7] our discussion in Sec. 7 will be mainly concerned with the question of whether a quantum-mechanical description of the electron is necessary. Using the WKB approximation, we shall establish conditions under which the classical description may be expected to suffice, and show that they are indeed met in reasonable physical situations.

Section 8 is devoted to a classical model which exhibits rather clearly the mutual influence of electrons and photons on each other. The electrons and photons are treated as classical relativistic fluids with an appropriate interaction. Energy and momentum are obviously conserved in the model, which therefore provides a suitable framework for the discussion of what happens to the energy and momentum in a particular process. The relationship of the model to quantum electrodynamics is briefly indicated in Sec. 9.

Finally, some of the conclusions are summarized and discussed in Sec. 10.

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2. CLASSICAL EQUATION OF AVERAGED MOTION

We present here an alternative derivation of the equation of motion obtained in I for the averaged motion of an electron in an oscillatory electromagnetic field.

The classical equation of motion for an otherwise free electron in an electromagnetic field described by the four-vector potential $A_\mu(x)$ is

$$\begin{aligned} m\dot{v}^\mu &= ev_\lambda F^{\lambda\mu}(x), \\ F_{\lambda\mu}(x) &= \partial_\mu A_\lambda(x) - \partial_\lambda A_\mu(x), \end{aligned} \tag{1}$$

where $v^\mu = \dot{x}^\mu$ is the velocity four-vector, and the dot denotes differentiation with respect to the proper time τ. (For the moment we set $c=1$ and adopt the "time-like" metric in which $v^2 \equiv v^\lambda v_\lambda = 1$.)

This equation may also be written in a form which gives directly the rate of change of the canonical momentum,

$$p_\mu \equiv \frac{d}{d\tau}[mv_\mu + eA_\mu(x)] = ev^\lambda \partial_\mu A_\lambda(x). \tag{2}$$

Now let us consider a series solution of this equation, starting in zeroth order with a constant four-velocity $v_{(0)}{}^\mu$, so that

$$x_{(0)}(\tau) = x^\mu(0) + v_{(0)}{}^\mu \tau. \tag{3}$$

The expansion parameter is the magnitude of eA_μ/m, a dimensionless quantity which is always much less than unity for physically realizable intensities. Our aim is to find an expression for the average acceleration of the electron in terms of electromagnetic field variables evaluated at the position defined by (3).

To first order, we may replace v_λ and x on the right-hand side of Eq. (1) [or in Eq. (2)] by $v_{(0)\lambda}$ and $x_{(0)}$. It is convenient to choose the gauge so that

$$v_{(0)} \cdot A \equiv v_{(0)}{}^\lambda A_\lambda = 0. \tag{4}$$

(Essentially, we are adopting the radiation gauge in the rest frame defined by $v_{(0)}$.) Then to first order we obtain the familiar result that the canonical momentum is a constant, which may be identified with $mv_{(0)}$. Thus

$$mv_{(1)}{}^\mu(\tau) = mv_{(0)}{}^\mu - eA^\mu[x_{(0)}(\tau)]. \tag{5}$$

Note that the relation $v^2 = 1$ is still satisfied to this order in virtue of (4).

If there were nonoscillatory terms in the field, then (5) would already reveal a secular acceleration, but we are interested in the case of purely oscillatory fields for which

$$\begin{aligned} m\langle \dot{v}_{(1)}{}^\mu \rangle &= -e\langle \dot{A}^\mu(x_{(0)}) \rangle \\ &= ev_{(0)\lambda} \langle F^{\lambda\mu}(x_{(0)}) \rangle = 0, \end{aligned}$$

where the angular brackets denote the time average over many oscillation periods. It is therefore both con-

sistent and desirable to impose the additional gauge condition

$$\langle A_\mu(x_{(0)})\rangle=0, \tag{6}$$

so that to first order the average velocity is $v_{(0)}$.

We now substitute from Eq. (5) into Eq. (2) and take the time average. What we are actually interested in is the time average with respect to a fixed reference frame. However, if t denotes the time in the rest frame defined by $v_{(0)}$, then to first order $dt/d\tau=v_{(0)}\cdot v_{(1)}=1$. Thus we can in fact take the average with respect to τ without introducing any error. In particular we can assume that $\langle \dot{A}_\mu(x_{(1)})\rangle=0$. Hence, correct to second order, we obtain

$$\begin{aligned}\langle \dot{v}_\mu\rangle&=-(e/m)^2\langle A^\lambda(x_{(0)})\partial_\mu A_\lambda(x_{(0)})\rangle\\&=\tfrac{1}{2}\partial_\mu\mu^2,\end{aligned} \tag{7}$$

where

$$\mu^2=(e/m)^2\langle -A^2(x_{(0)})\rangle. \tag{8}$$

This is the dimensionless intensity parameter defined in BK and in I. It is positive in our metric, since A_λ is a space-like vector. It is evidently a scalar, but in general depends not only on x but also on $v_{(0)}$, via the chosen gauge condition (4) and because $v_{(0)}$ specifies the direction in which the average is to be taken. However, we shall see shortly that in most cases of practical interest this dependence is unimportant, so that μ^2 may be taken to be a scalar function of position and time alone.

We conclude this section with a discussion of the validity of the approximations involved in the derivation of Eq. (7). For convenience we choose the time axis in the direction of $v_{(0)}$ and assume that the field possesses a well-defined angular frequency ω. (It is easy to include the case of a superposition of several distinct frequencies, since each contributes separately to the average.) Then, restoring the appropriate factors of c for later convenience, we may write

$$\mu^2=(e/mc^2)^2\langle \mathbf{A}^2\rangle=(e/mc\omega)^2\langle \mathbf{E}^2\rangle. \tag{9}$$

It is easy to verify firstly that if $\mu^2\ll 1$ then the expansion around the mean position of the electron is reasonable. For, the characteristic distance over which we may expect the field to vary in spatial directions is at least of the order of c/ω. But the amplitude x of the electron oscillation is of order $eE/m\omega^2\approx\mu c/\omega$, which is less than this by a factor μ.

A more important approximation is the neglect of radiative reaction. It has already been shown by Sanderson[8] that this could be important in the type of problem we are considering, and it should certainly be included. However, it is an essentially distinct effect which may be treated separately. We shall return to it later in Sec. 5.

It is also important to verify that the use of classical rather than quantum mechanics does not introduce any significant errors. It is convenient however to defer the discussion of this point to Sec. 7.

3. EFFECTIVE ELECTRON MASS

There are in practice two distinct types of electromagnetic waves in which we might be interested. First we have the case of a beam, represented by a traveling wave with a well-defined propagation direction at each point. This is the case we consider in the present section. Second, we may be interested in standing waves produced by multiple reflections. We defer this more complicated case to Sec. 6.

Let us then suppose that the vector potential has the form

$$A_\mu(x)=\mathrm{Re}[a_\mu(x)e^{i\psi(x)}], \tag{10}$$

where ψ is real and satisfies the eikonal equation

$$k^2=0,\quad k_\mu=\partial_\mu\psi, \tag{11}$$

and where k_μ and the complex amplitude function a_μ are slowly varying functions of position and time satisfying the gauge condition

$$k\cdot a=0. \tag{12}$$

Then to a good approximation we may write

$$\mu^2=\tfrac{1}{2}(e/m)^2[-a^*(x)\cdot a(x)]. \tag{13}$$

Moreover, the dominant part of the gauge transformation induced by a change in $v_{(0)}$ is the replacement

$$a_\mu\rightarrow a_\mu-k_\mu\frac{v_{(0)}\cdot a}{v_{(0)}\cdot k}, \tag{14}$$

and in virtue of (11) and (12) this does not change the value of μ^2 as given by (13). Thus, provided the relative variations in the propagation vector and the amplitude function over a few oscillations are small, we may legitimately treat μ^2 as a scalar function of x independent of $v_{(0)}$. (In the case of plane waves discussed in BK this is exactly true.)

With this assumption let us now investigate some of the immediate implications of Eq. (7). First we note that $\langle v\rangle^2\neq 1$ in general, although of course $v^2=1$ always. Indeed, using the boundary condition that $\langle v\rangle^2=1$ when $\mu^2=0$, we find

$$\langle v\rangle^2=\langle ds/d\tau\rangle^2=1+\mu^2, \tag{15}$$

where s denotes the proper time of an observer moving with the averaged velocity of the electron. The four-vector velocity of such an observer is

$$\frac{d\langle x^\mu\rangle}{ds}=\frac{\langle v^\mu\rangle}{(\langle v\rangle^2)^{1/2}}=\frac{\langle v^\mu\rangle}{(1+\mu^2)^{1/2}}. \tag{16}$$

These relations are more useful when re-expressed in terms of the average momentum of the electron. We have

$$\langle p^\mu\rangle=m\langle v^\mu\rangle=m_e{}^*d\langle x^\mu\rangle/ds, \tag{17}$$

where the effective mass m_e^* is defined by

$$\langle p\rangle^2=m_e^{*2}=m^2(1+\mu^2). \tag{18}$$

This expression for the effective mass was also obtained for the case of plane-wave fields in earlier semiclassical[3] and quantum-mechanical[10] calculations.

The equation of motion of a relativistic particle of variable rest mass m^*, a prescribed function of position and time may be obtained by variation of the action integral[14]

$$-\frac{1}{2}\int ds\, m^*(x)(\dot{x}^\mu\dot{x}_\mu)^{1/2}, \tag{19}$$

where s is an arbitrary parameter (not necessarily proper time) and $\dot{x}^\mu=dx^\mu/ds$. Imposing the constraint $\dot{x}^2=1$, which is consistent if $m^*\neq 0$, we may write it in the form

$$\frac{d}{ds}[m^*(x)\dot{x}_\mu]=\partial_\mu m^*(x). \tag{20}$$

On the other hand, using (15) and (17) we may write the equation of motion (7) in the form

$$\frac{d}{ds}\langle p_\mu\rangle=\frac{m\langle\dot{v}_\mu\rangle}{\langle ds/d\tau\rangle}=\frac{\frac{1}{2}m\partial_\mu\mu^2}{(1+\mu^2)^{1/2}}=\partial_\mu m_e^*. \tag{21}$$

The agreement between (20) and (21) shows that the averaged motion of the electron is precisely that of a classical relativistic particle with the variable rest mass m_e^*.

Nonrelativistically, the variable rest mass appears as an effective-potential-energy function

$$V=(m^*-m)c^2\approx\tfrac{1}{2}m\mu^2c^2. \tag{22}$$

Clearly the resulting force tends to push the electron away from regions of high field intensity, as noted in a special case by Phillips and Sanderson,[9] and also in I.

4. COMPARISON BETWEEN REFRACTION OF PHOTONS AND OF ELECTRONS

For convenience of application, we shall in this section (and until the end of Sec. 7) restore the appropriate factors of c, and also exhibit the factors of 4π which should appear in Gaussian units within square brackets.

Let us briefly recall some features of the familiar phenomenon of refraction of light by electrons. The dispersion equation for an electromagnetic wave of angular frequency ω and wave vector $\mathbf{k}$ is

$$\omega^2=c^2\mathbf{k}^2+\omega_p^2, \tag{23}$$

where the plasma frequency ω_p is related to the electron density n_e by

$$\omega_p^2=[4\pi]e^2n_e/m. \tag{24}$$

[14] This corresponds to the problem of finding the geodesics in a space with the metric tensor $g_{\mu\nu}=m^{*2}\eta_{\mu\nu}$, where $\eta_{\mu\nu}$ is the metric tensor of flat Minkowski space.

Thus a photon propagating in such a medium has an effective mass $m_{\rm ph}^*$ given by

$$m_{\rm ph}^{*2}=\left(\frac{\hbar\omega_p}{c^2}\right)^2=[4\pi]\left(\frac{e\hbar}{c}\right)^2\frac{n_e}{mc^2}. \tag{25}$$

On the other hand, if we express the effective mass of the electron found in the previous section in terms of the photon density $n_{\rm ph}$, we find

$$m_e^{*2}-m^2=\mu^2m^2=[4\pi]\left(\frac{e\hbar}{c}\right)^2\frac{n_{\rm ph}}{\hbar\omega}. \tag{26}$$

The close similarity between (25) and (26) is apparent. In each case, the effective-mass-squared shift is proportional to the particle density divided by the particle energy. This ratio n/E is essentially Lorentz-invariant provided that neither n nor E varies rapidly with position.

The analogy between the two cases extends beyond the expressions for the effective masses. Let us consider, for example, the refraction of a beam crossing a boundary whose normal we take to be the x direction. When a photon enters a medium with plasma frequency ω_p it is the normal component of the wave vector which changes. The amount of change may be found directly from (25), or

$$\Delta(k_x^2)=-\omega_p^2/c^2. \tag{27}$$

In particular, the photon will be totally reflected if initially $k_x<\omega_p/c$.

Precisely the same thing happens when an electron enters an electromagnetic wave. For convenience we shall drop the brackets on $\langle\mathbf{p}\rangle$ and denote the time-averaged momentum simply by $\mathbf{p}$. From (7) or (21) it is clear that only the normal component of the mean momentum is affected, and to be consistent with (18) this must change according to

$$\Delta(p_x^2)=-m^2c^2\mu^2. \tag{28}$$

Once again, if $p_x<mc\mu$ the electron will not be able to penetrate the beam.

Thus the laws governing the refraction of an electron passing through a photon beam are identical with those for a photon passing through a cloud of electrons. (Why this should be so will be discussed later in the context of the model introduced in Sec. 8.) Snell's law is valid, and we can define a frequency-dependent or energy-dependent refractive index in both cases, equal to the ratio of the velocity inside the medium to that outside. It may be written as

$$\left(\frac{E^2-(m_e^*c^2)^2}{E^2-(mc^2)^2}\right)^{1/2}=\left(1-\frac{(mc\mu)^2}{p^2}\right)^{1/2}, \tag{29}$$

where p is the momentum outside the beam. It may be noted that the quantity $mc^2\mu/\hbar$ is the true analog of the plasma frequency ω_p.

More generally we may consider a situation in which μ^2 is changing with time as well as position. If in some region of space μ^2 depends on the four coordinates x^μ only through a single linear combination $n\cdot x$, then the only component of p^μ which can change is that in the direction of n^μ. We can write (as in I)

$$p^\mu = p^\mu(0) + n^\mu f, \tag{30}$$

where f may be determined from (18) as before. This yields a quadratic equation for f,

$$n^2 f^2 + 2n\cdot p(0) f = m^2[\mu^2 - \mu^2(0)]. \tag{31}$$

5. ENTRY OF ELECTRON INTO A BEAM

Let us now consider the various ways in which an electron may enter an electromagnetic wave. We shall take the z axis in the beam direction, and suppose that the electron is initially moving with energy E and momentum p in the x direction. In BK the assumption of infinite plane-wave fronts was made, so that $n\cdot x = t - z/c$, say, and $n^2 = 0$. If this description is valid we find that the mean energy E^* and momentum p^* inside the beam are given by

$$E^* - E = cp_z^* = \mu^2(mc^2)^2/2E,$$
$$p_x^* = p.$$

In this case the electron is overtaken by the wave. The corresponding change in velocity to first order in μ^2 is given by

$$\Delta v_z = \tfrac{1}{2}\mu^2 c\left(\frac{mc^2}{E}\right)^2, \quad \Delta v_x = -\tfrac{1}{2}\mu^2 v_x\left(\frac{mc^2}{E}\right)^2. \tag{32}$$

On the other hand, if the electron enters the beam from one side, and if the intensity is almost independent of t and z, then we have the situation described by (28), so that

$$E^* = E, \quad p_z^* = 0,$$
$$p_x^* = [p^2 - \mu^2(mc)^2]^{1/2}.$$

For this case, we have to lowest order

$$\Delta v_x = -\tfrac{1}{2}\mu^2\frac{c^2}{v_x}\left(\frac{mc^2}{E}\right)^2, \quad \Delta v_z = 0. \tag{33}$$

This approximation is of course valid only if $v/c \gg \mu^2$. For slow electrons the change in velocity could be large.

In practice it is clear that neither of these idealized cases will provide a completely accurate description of the physical situation, and it is therefore important to ask which is likely to represent a closer approximation to reality. The answer to this question obviously depends on the values of the various parameters involved, but we can still give a fairly general discussion. Since large intensities can be attained at present only with pulsed beams, it is clear that the intensity must be varying with time. The rate of change is of order μ^2/T, where T is the duration of a pulse. Equally, it is necessary to focus the beam, so that there will be a transverse intensity gradient of order μ^2/L, where L is the width of the beam in the focal region. Consider an electron which arrives at the center of beam somewhere near the middle of the pulse duration. The question to ask is whether it will have entered from the side or have been overtaken by the beam. Clearly, it must have entered from the side if $v_x T \gg L$. Typically $T \approx 10^{-8}$ sec, and certainly $v_x \gtrsim 10^{-3}c$. (Indeed, it is more likely that v/c should be of order 10^{-1}.) Thus the condition is satisfied if L is much less than 3 mm, as indeed it usually is. In most cases of practical interest, therefore, it is likely that the transverse spatial gradient is much more important than the time rate of change. The typical change in velocity is then given by (35) rather than (32).

This means that the results given by BK will not normally apply to practical cases of focused beams. To attempt a direct verification of the original predictions it would be necessary to design the experiment so that the electrons were actually overtaken by the beam. One would need to use slow electrons and a beam with an extremely short pulse time and relatively poor focusing. Such an experiment does not appear to be feasible at the present time.

These remarks should not be taken to imply that it is uninteresting to measure the Doppler shifts of light scattered from an intense beam by free electrons. These shifts should reveal changes in velocity which both confirm the theory and provide information about the intensity distribution in the beam. Moreover, they should also be considerably larger (by a factor c/v at least) than those originally predicted, and therefore easier to observe.

Near a focus, there may be a substantial intensity gradient in the longitudinal direction as well as in transverse directions. As indicated in I, this will lead to the electrons being deviated away from the focus. If v/c is not too much larger than μ, this deviation can be quite large, and certainly readily measurable. The effect might provide a useful means of determining the intensity distribution near the focus of a laser beam, in the same way that the refraction of photons can be used to find the density distribution in an electron plasma.

In addition to the acceleration produced by the intensity gradient, we should also allow for the effect of radiative reaction.[8] This is of course always to accelerate the electron in the direction of propagation of the beam. It is interesting to compare the magnitudes of the two kinds of forces. In terms of the intensity I, the parameter μ^2 is

$$\mu^2 = [4\pi]e^2 I/m^2c^3\omega^2. \tag{34}$$

Thus the force due to the intensity gradient is

$$-\tfrac{1}{2}mc^2\nabla\mu^2 = [4\pi]\frac{e^2}{mc\omega^2}\nabla I$$
$$= 4\pi r_0 \lambda^2 \nabla I/c.$$

On the other hand, the force due to radiative reaction is the rate at which momentum is removed from the beam, namely, $\pi r_0^2 I/c$. Thus the ratio of the two is

$$\frac{4\lambda^2}{r_0}\frac{\nabla I}{I}=\frac{4\lambda^2}{r_0 L}, \tag{35}$$

if L is the typical length over which the intensity is varying. For optical photons this ratio would be close to unity only when $L=10^3$ cm. In most reasonable circumstances the intensity gradient force is much larger than the radiative reaction.

6. INTERACTION WITH STANDING WAVES

The interaction of electrons with standing electromagnetic waves exhibits some very interesting features, notably the Kapitza-Dirac effect.[11] This is the scattering of electrons at the Bragg angle corresponding to a lattice spacing equal to half the wavelength of the light. Its experimental observation has been reported by Bartell, Thompson, and Roskos,[12] and more recently Eberly[13] has suggested that this type of experiment might be a suitable place to look for effects similar to the predicted intensity-dependent frequency shift.

As we shall see, this problem is appreciably more complicated than the corresponding one in the case of traveling waves, and the variety of possible effects is considerably greater. The vector potential can no longer be written in the simple form (10); instead, we must take a sum of two terms representing the waves traveling in opposite directions. When we consider time averages it is then necessary to distinguish two distinct time scales.

Over a time interval short compared to the time taken for the electron to travel one wavelength (that is, if $\mathbf{k}\cdot\mathbf{v}t\ll 1$, where $\mathbf{k}$ is the wave vector), the two waves are oscillating with a fixed phase difference, and interfere in the time average. Thus the intensity parameter μ^2 has a form like

$$\mu^2=\mu_0^2[1+\cos 2\mathbf{k}\cdot\mathbf{x}], \tag{36}$$

where μ_0^2 is a slowly varying function of $\mathbf{x}$ and t. Over longer times, and provided that the electron velocity is not too small, the oscillatory term averages out to zero, giving an effective average potential

$$\mu^2=\mu_0^2. \tag{37}$$

We shall discuss the special case of slow electrons later.

Strictly speaking, μ^2 as given by (36) is no longer independent of the velocity $v_{(0)}$. In fact, a gauge transformation similar to (14) does not leave μ^2 unchanged, because although the two four-vectors $k_{\mu\pm}=(\omega, \pm\mathbf{k})$ satisfy $k_\pm^2=0$, they are not mutually orthogonal, $k_+\cdot k_-\neq 0$. However this is really irrelevant, since the extra terms are of the order v^2/c^2 and therefore negligible for nonrelativistic electrons. For relativistic electrons, there is no time interval to which (36) is appropriate, and we can always use (37), which is independent of $v_{(0)}$ because the two waves contribute separately.

Except in the case of very slow electrons, the averaged motion over the long times is that of a particle moving with the slowly varying potential-energy function $\frac{1}{2}mc^2\mu_0^2$. The electrons are therefore refracted in much the same way as in a traveling wave. In particular, the longitudinal component of velocity $\mathbf{k}\cdot\mathbf{v}=kv_z$, say, is essentially unchanged while the transverse component is reduced on entering the wave. The effect of the oscillatory term in (36) is to produce a short-period oscillation in v_z. The Kapitza-Dirac effect may be interpreted as the scattering of the electron by this periodic potential, with a momentum transfer $2\hbar\mathbf{k}$. (In terms of photons, it corresponds to the absorption of a photon from the beam and its re-emission into the beam in the opposite direction.) Since the refraction of the electron on entering the standing wave leaves its longitudinal momentum unaltered, the Bragg-scattering condition

$$p\sin\theta'=p\sin\theta=\hbar k, \tag{38}$$

where p is the electron momentum and θ, θ' are the incident and scattered angles, is actually unaffected by this refraction. We therefore conclude that the experiment suggested by Eberly[13] should not in fact reveal any effect.

Independently of the details of this mechanism it is clear from the symmetry of the problem that one should not expect a systematic difference between θ and θ'. Since there is some small spread of frequency $\Delta\omega$ in the light beam, it is actually possible to transfer a small amount of energy to or from the electron. However, one sign of energy transfer is in no way preferred over the other. (This is, of course, very different from the case where a photon is actually scattered out of the beam.) Such energy change corresponds for nonrelativistic electrons to a much larger change of momentum, so that in individual cases one need not expect θ' to be equal to θ, though the Laue condition

$$p(\sin\theta'+\sin\theta)=2\hbar k \tag{39}$$

should still be satisfied.[15] However, there is no reason to expect more electrons with $\theta'>\theta$ than with $\theta'<\theta$, or vice versa.

Very slow electrons constitute a special, and particularly interesting case. The relevant quantity is the magnitude of the longitudinal component of velocity, v_z say. The discussion above applies only if $v_z/c\gg\mu_0$. At the other extreme, if $v_z/c\ll\mu_0$, a classical electron would be confined within a single half-wavelength in the z direction, and would travel along this trough from one end to the other with a nearly constant velocity, v_x say,

[15] On this basis we would expect to find $|\theta'-\theta|/(\theta'+\theta)\approx(c/v)(\Delta\omega/\omega)$, whereas for fixed θ the relative spread in values of θ' is only of order $\Delta\omega/\omega$. The observations of Bartell, Thompson, and Roskos (Ref. 12) confirm the existence of nonzero values of $\theta'-\theta$, but appear to show values larger than one might expect from this formula.

and a superimposed oscillation in the z direction. Thus we see that electrons incident at angles much smaller than $\mu_0 c/v$ would never penetrate the regions of large intensity, and would be refracted in a very different way from those incident at larger angles.

When

$$\sin\vartheta \approx \sqrt{2}\mu_0 c/v, \tag{40}$$

we have an intermediate situation, in which classically the electrons are nearly able, or just able, to get over the crests of the periodic potential. Since such electrons spend a relatively long time on the crests rather than in the troughs, the effective average value of μ^2 may be enhanced. The classical motion in this case is rather complicated, but it is not difficult to see that the average z component of momentum might no longer be constant. However, this is one case in which a classical treatment is not wholly reliable, and we shall defer a complete discussion of it to a later publication, in which a fully quantum-mechanical treatment will be presented. (The argument justifying the use of classical electrodynamics which we give in the next section does not apply when the change in local velocity is too rapid.)

A rather interesting situation occurs when the Bragg scattering angle is about equal to the angle specified by (40). The condition for this is that μ_0 should be of the same order as $\hbar k/mc$, that is,

$$\mu_0 \approx \lambda\!\!\!^{-}_c/\lambda\!\!\!^{-}, \tag{41}$$

where λ_c is the electron Compton wavelength. For ruby photons this means $\mu_0^2 \approx 10^{-11}$, corresponding to an intensity around 6×10^7 W cm^{-2}. The intensity in the experiment of Bartell, Thompson, and Roskos[12] was in fact of about this order, so that one might perhaps expect some modification of the Bragg-scattering condition. A repetition of this experiment over a range of values of the intensity and electron velocity, and with improved resolution would be very valuable.

7. QUANTUM-MECHANICAL DISCUSSION

In order to justify the use of our purely classical methods it is necessary to show that a quantum-mechanical calculation would not give substantially different results. There are really two separate problems involved here, the quantum-mechanical treatment of the electrons and that of the photons.

An argument intended to show that the semiclassical calculations used by BK[3] must yield the same answer as a fully quantum-mechanical calculation was presented in K.[6] It made use of the so-called coherent states of the radiation field, and the correspondence established by Sudarshan[16] between the classical and quantum description of such a field. (In a sense this result may perhaps be regarded as a special case of the theorem of Senitzky.[17]) More recently, a much more thorough and general proof of the equivalence between the two methods of calculation was given by Frantz.[7] The theorem is valid provided that $\mu^2 \ll 1$, that the photon number is large enough, and that radiative corrections are negligible. All these conditions are likely to be satisfied in the cases of interest to us.

Next we have to consider whether a purely classical calculation should agree with this semiclassical calculation in which the electron is treated quantum mechanically. For simplicity, we shall discuss only the nonrelativistic case. First let us suppose that the electron is well localized in space initially. Ehrenfert's theorem[18] shows that the rates of change of the expectation values of position and momentum are given by

$$\frac{d}{dt}\langle \mathbf{x}\rangle = \frac{\langle \mathbf{p}\rangle}{m}, \tag{42}$$

$$\frac{d}{dt}\langle \mathbf{p}\rangle = e\langle \mathbf{E}(\mathbf{x})\rangle. \tag{43}$$

If the electron is localized in a small volume, near the origin say, then we can expand $\mathbf{E}$ about this point, and write approximately

$$\mathbf{E}(\mathbf{x}) = \mathbf{E}(0) + \mathbf{x}\cdot\nabla\mathbf{E}(0). \tag{44}$$

Then it follows that

$$\langle \mathbf{E}(\mathbf{x})\rangle = \mathbf{E}(\langle \mathbf{x}\rangle), \tag{45}$$

so that from (42) and (43) we see that the expectation value of position follows precisely a classical trajectory.

This proof cannot be applied directly to cases where the initial uncertainty in the electron position is large. Before discussing this case in detail it is worth asking what result we should expect to be able to prove. When the electron momentum distribution is fairly sharp but its position is very uncertain, it is not sufficient to consider merely the motion of the center of the wave packet. We should expect each part of the packet to follow approximately a classical trajectory, and to have its own characteristic velocity. The refraction effect we have been discussing will manifest itself not so much in a change in the mean velocity but rather in a correlation between velocity and position. If we find the electron in one region we may expect that a measurement of momentum will yield a result in some appropriate range; if it is in some other region the expected range would be different. (This is a very common situation. For example, for an electron scattered from a central potential there is a strong correlation between position and velocity in the final state, since the velocity is predominantly radially outwards.)

We can show that this does indeed happen by using the WKB approximation. We write the electron wave

[16] E. C. G. Sudarshan, Phys. Rev. Letters **10**, 277 (1963).
[17] I. R. Senitzky, Phys. Rev. Letters **15**, 233 (1965); **16**, 619 (1966).
[18] See, for example, L. I. Schiff, *Quantum Mechanics* (McGraw-Hill Book Company, Inc., New York, 1955), 2nd ed., p. 184.

function in the form

$$\psi=\phi\exp(iS/\hbar), \tag{46}$$

where ϕ and S are real functions of t and x. Then, using the radiation guage, the Schrödinger equation may be separated into two real equations

$$\frac{\partial}{\partial t}(\phi^2)+\nabla\cdot\left[\phi^2\frac{1}{m}\left(\nabla S-\frac{e}{c}\mathbf{A}\right)\right]=0, \tag{47}$$

and

$$\frac{\partial S}{\partial t}+\frac{1}{2m}\left(\nabla S-\frac{e}{c}\mathbf{A}\right)^2=\frac{\hbar^2}{2m\phi}\nabla^2\phi. \tag{48}$$

The first of these equations is clearly the continuity equation corresponding to the conservation of probability. The local velocity defined by the probability current density is

$$\mathbf{v}=\frac{1}{m}\left(\nabla S-\frac{e}{c}\mathbf{A}\right). \tag{49}$$

The WKB approximation consists in neglecting the right-hand side of Eq. (48). In this approximation the equations reduce to those for a classical fluid of density ϕ^2 and velocity $\mathbf{v}$. To see this, let us find the rate of change of $\mathbf{v}$. Using (48) and (49) we easily obtain

$$\frac{\partial}{\partial t}\mathbf{v}+\mathbf{v}\cdot\nabla\mathbf{v}=\frac{e}{m}\left(\mathbf{E}+\frac{1}{c}\mathbf{v}\times\mathbf{B}\right). \tag{50}$$

This is the equation of motion of a fluid composed of noninteracting classical electrons. Thus to find the effective local velocity in any region it is sufficient to follow the classical trajectories of electrons in the prescribed electromagnetic field.

Finally, we may examine the conditions under which the WKB approximation is valid. From (48) we see that it is sufficient to require that

$$\tfrac{1}{2}mv^2\gg\frac{\hbar^2}{2m}\frac{|\nabla^2\phi|}{\phi}. \tag{51}$$

If L is the characteristic distance over which the electron probability density ϕ^2 varies, then this condition may be written

$$p\gg\hbar/L. \tag{52}$$

The variation in ϕ^2 is attributable to two distinct causes. Firstly, the initial electron wave function will have some finite extent in space. In respect of this variation the condition is satisfied provided the electron momentum is much larger than the momentum uncertainty. This is a condition which is likely to be fulfilled in practice. Secondly, ϕ^2 varies because of inhomogeneities in the field which cause a bunching of electrons in some regions. For this case L is the typical length over which the field intensity varies. It is at least of the order of a wavelength of the light. Thus for optical wavelengths and all reasonable electron energies the condition (52) is well satisfied.

We have therefore shown that the WKB approximation is valid, and that in this approximation the results of a quantum-mechanical calculation agree with those of a classical one.

8. TWO-FLUID MODEL

In this section, we shall consider a model which exhibits rather clearly certain aspects of the interaction of photons and electrons. We first give a purely classical discussion, and later indicate how the model may be derived in a certain approximation from quantum electrodynamics. The model is intended to show what happens when a beam of photons and a beam of electrons, each with finite density, interact with each other. It is particularly useful in discussing the problem of energy and momentum conservation.

We shall assume that both the electrons and the photons behave like relativistic fluids. Specifically, we assume that at each point x of space-time there are well-defined energy-momentum vectors $p^\mu(x)$ and $k^\mu(x)$ such that all the electrons in the vicinity of x have momentum p^μ and all the photons have momentum k^μ.[19] It is convenient to introduce for each type of particle a Lorentz-invariant measure of density. We define

$$\rho=n_e/p^0, \quad \sigma=n_{\rm ph}/k^0. \tag{53}$$

Then the four-vectors $n_e{}^\mu$ and $n_{\rm ph}{}^\mu$ representing the electron-number and photon-number current densities are[20]

$$n_e{}^\mu=\rho p^\mu, \quad n_{\rm ph}{}^\mu=\sigma k^\mu. \tag{54}$$

Conservation of particle number requires that

$$\partial_\mu(\rho p^\mu)=0, \quad \partial_\mu(\sigma k^\mu)=0. \tag{55}$$

Next we assume that the mass of each particle varies in accordance with the density of particles of the other type, as described in Sec. 4. We then have

$$p^2=m_e{}^{*2}=m^2+e^2\sigma, \tag{56}$$

$$k^2=m_{\rm ph}{}^{*2}=e^2\rho, \tag{57}$$

where factors of $\hbar$ have been omitted. One may say either that we have set $\hbar=1$, or equivalently that e^2 here (and below) is an abbreviation for $[4\pi](e\hbar)^2$.

We have to rewrite the equation of motion (21) of a relativistic particle of variable rest mass $m_e{}^*$, namely,

$$\dot{p}_\mu=\partial_\mu m_e{}^*, \tag{58}$$

in a hydrodynamic form. Since the left-hand side is the

[19] This represents a minor change of notation k^μ here denotes what we have previously denoted by $\hbar k^\mu$.

[20] Note that this discussion does not depend on the existence of a velocity four-vector, and is therefore applicable to massless photons.

gradient of p_μ in the direction of the velocity four-vector, we have, on multiplying by $m_e{}^*$,

$$p^\lambda \partial_\lambda p_\mu = \tfrac{1}{2}\partial_\mu m_e{}^{*2} = \tfrac{1}{2}e^2\partial_\mu \sigma. \tag{59}$$

This is the equation of motion of the electrons. Similarly, the equation of motion of the photons may be written[21]

$$k^\lambda \partial_\lambda k_\mu = \tfrac{1}{2}\partial_\mu m_{\rm ph}{}^{*2} = \tfrac{1}{2}e^2\partial_\mu \rho. \tag{60}$$

The four equations (55), (59), and (60) together specify completely the time development of the system. It is easy to verify that they preserve the conditions (56) and (57).

The stress-energy tensor of the electrons may be written

$$T_e{}^{\mu\nu} = \rho p^\mu p^\nu. \tag{61}$$

Using (55) and (59) we find that its divergence is

$$\partial_\mu T_e{}^\mu{}_\nu = \rho p^\mu \partial_\mu p_\nu = \tfrac{1}{2}e^2\rho\partial_\nu\sigma. \tag{62}$$

Similarly, for the photons,

$$T_{\rm ph}{}^{\mu\nu} = \sigma k^\mu k^\nu \tag{63}$$

and from (55) and (60)

$$\partial_\mu T_{\rm ph}{}^\mu{}_\nu = \tfrac{1}{2}e^2\sigma\partial_\nu\rho. \tag{64}$$

Thus there is a conserved stress-energy tensor given by

$$T^\mu{}_\nu = \rho p^\mu p_\nu + \sigma k^\mu k_\nu - \tfrac{1}{2}e^2\rho\sigma\delta^\mu{}_\nu. \tag{65}$$

The last term may be regarded as due to the interaction, although of course the interaction does also affect the other terms to the extent of changing the masses. It is interesting to note that the mass shifts disappear if we take the trace,

$$T^\mu{}_\mu = \rho m^2. \tag{66}$$

Evidently, if we can find a solution of the basic equations (55), (59), and (60), then the formula (65) for $T^\mu{}_\nu$ provides complete information about what happens to the energy and momentum.

One interesting feature emerges when we try to solve these equations. It would be natural to look for a static solution representing two intersecting beams, and therefore to examine the boundary conditions across a surface of discontinuity. However, this is not quite as straightforward as it might appear, for it is easily seen that no time-independent solutions of this type exist.[22] In fact, more generally it is impossible to have a non-trivial solution in which all quantities in some region depend only on a single component of x^μ. For in such a case one can integrate the equations to obtain an algebraic equation for either ρ or σ, and it follows that both quantities are in fact constants in this region.[23] The physical reason for this is not hard to find. Consider, for example, the point where a beam of electrons enters a photon beam. Since their velocity is reduced on entry, the electron density is greater immediately inside than outside. The sharp gradient of the electron density thus provides a large outward force on the photons at the edge of the beam. A sharp boundary of this type is therefore highly unstable. Of course, this does not imply that time-independent solutions do not exist, but merely that they cannot have any surfaces of discontinuity.

9. RELATIONSHIP TO QUANTUM ELECTRODYNAMICS

It is interesting to examine the relationship between the model theory discussed in the preceding section and quantum electrodynamics. We shall for simplicity ignore the electron spin and therefore consider scalar electrodynamics, for which the Lagrangian density is

$$L = (\partial_\mu - ieA_\mu)\phi^*(\partial^\mu + ieA^\mu)\phi - m^2\phi^*\phi - \tfrac{1}{4}F^{\mu\nu}F_{\mu\nu}, \tag{67}$$

with $F_{\mu\nu} = \partial_\nu A_\mu - \partial_\mu A_\nu$, where we have now unequivocally set $\hbar = c = 1$.

It is easy to see that the effective-mass shifts arise from the interaction term $e^2A^2\phi^*\phi$. In a state with finite electron or photon density we can define the invariant (n/E) density functions by

$$\langle\phi^*\phi\rangle = \tfrac{1}{2}\rho, \quad \langle -A^\lambda A_\lambda\rangle = \sigma, \tag{68}$$

where of course the angular brackets now denote a quantum-mechanical expectation value.[24] Thus when there is a finite photon density we get a contribution

$$-e^2\langle -A^2\rangle\phi^*\phi = -e^2\sigma\phi^*\phi, \tag{69}$$

which changes the effective electron mass to $m_e{}^*$; while if there is a finite electron density we have

$$e^2\langle\phi^*\phi\rangle A^2 = \tfrac{1}{2}e^2\rho A^2, \tag{70}$$

which yields the effective photon mass $m_{\rm ph}{}^*$.

One can best see the relationship with our model theory by dropping the linear interaction terms from (67), keeping only this quadratic term. Since the linear terms are oscillatory in character, this is essentially a similar step to the time averaging we carried out earlier in Sec. 2.

This modified Lagrangian,

$$L = \partial_\mu\phi^*\partial^\mu\phi - m^2\phi^*\phi + e^2A^2\phi^*\phi - \tfrac{1}{4}F^{\mu\nu}F_{\mu\nu}, \tag{71}$$

[21] This equation cannot be written in the alternative form (58), because the proper time is undefined for a photon, though it could be written in the general form derived from variation of (19). Compare Ref. 20.

[22] It is easy to be misled by the equations specialized to the case of a discontinuity. One must remember that a physical solution must be the limit of a continuous solution. Thus for example the continuity condition $[\sigma\mathbf{n}\cdot\mathbf{k}]=0$ may be formally satisfied by setting $\sigma=0$ on one side of the boundary and $\mathbf{n}\cdot\mathbf{k}=0$ on the other; but it may nevertheless be impossible to find a solution through the boundary.

[23] The situation discussed in Ref. 22 corresponds to a sudden change from one root to another on crossing a boundary. This is not physically allowable.

[24] The relative factor of 2 between these equations arises from the fact that the field is real in one case and complex in the other. Both positive and negative frequency components of A_λ contribute, whereas only negative frequency components of ϕ do so.

yields the stress-energy tensor

$$T^{\mu}{}_{\nu}=\partial^{\mu}\phi^{*}\partial_{\nu}\phi+\partial^{\mu}\phi\partial_{\nu}\phi^{*}+F^{\mu\lambda}F_{\lambda\nu}+2e^{2}\phi^{*}\phi A^{\mu}A_{\nu}-\delta^{\mu}{}_{\nu}L. \quad (72)$$

Now let us consider the expectation value of this quantity. We shall make the assumption that within the region of interest only a very limited range of Fourier components of the fields contribute appreciably to this expectation value, so that we can for example replace $\partial_{\mu}\phi$ by $-ip_{\mu}\phi$. We shall also assume that the four-field expectation value can be factorized:

$$\langle\phi^{*}\phi A_{\mu}A_{\nu}\rangle=\langle\phi^{*}\phi\rangle\langle A_{\mu}A_{\nu}\rangle. \quad (73)$$

To be consistent with the equations of motion,

$$(\partial^{2}+m^{2})\phi=e^{2}A^{2}\phi, \quad (74)$$

and

$$\partial_{\nu}(\partial^{\nu}A^{\mu}-\partial^{\mu}A^{\nu})=-2e^{2}\phi^{*}\phi A^{\mu}, \quad (75)$$

it is necessary to choose p_{μ} and k_{μ} to satisfy the conditions (56) and (57). Thus we find

$$\langle T^{\mu}{}_{\nu}\rangle=\rho p^{\mu}p_{\nu}+\sigma k^{\mu}k_{\nu}-k^{2}\langle A^{\mu}A_{\nu}\rangle+e^{2}\langle A^{\mu}A_{\nu}\rangle-\delta^{\mu}{}_{\nu}[(p^{2}-m^{2})\rho-\tfrac{1}{2}e^{2}\rho\sigma+\tfrac{1}{2}k^{2}\sigma],$$

and using (56) and (57) we see that this is identical with (65).

It is obvious that to obtain our model theory we have to do considerable violence to quantum electrodynamics. Nevertheless, it does appear as a reasonable first approximation, at least for some purposes, in the case where the particle densities are large.

10. DISCUSSION

Earlier treatments by BK, G, and others have dealt almost exclusively with the case of plane waves. The methods developed here are far more general, and can be applied to all cases which are likely to arise in practice, including in particular sharply focused beams and standing waves. The approximations involved would break down only for intensities much higher than those currently available. For $\mu^{2}\approx 1$ (corresponding for optical frequencies to intensities around 10^{19} W cm^{-2}) the higher terms in the expansion in powers of μ^{2} would obviously be important. However, there is no immediate prospect of achieving such intensities. The approximations would also be somewhat dubious for really high-velocity electrons ($v\approx c$) and sharply focused beams, since we assumed that the amplitude was nearly constant over many oscillations of the field.

We have seen that the important effects in most cases of practical interest can all be understood in purely classical terms. The only significant exception is the Kapitza-Dirac effect, which is of course a quantum effect. The justification for the use of classical electrodynamics is provided by the work of Frantz[7] and the discussion of Sec. 8. However, in view of the controversy which has surrounded the question of the equivalence of classical and quantum-mechanical treatments, it may be desirable to supplement the considerations of this paper with a specifically quantum-mechanical calculation. Accordingly, we intend to present such a calculation in a future publication.

There are two kinds of experiments which are suggested by these investigations. One involves the use of sharply focused beams in which large intensity gradients appear. It should be possible to observe the refraction of electrons passing close to a focus either directly or by examining the spectrum of scattered light. To get a large effect one should use electrons of as low an energy as possible consistent with having a well-defined electron beam. The other category of experiments comprises those involving standing waves. As discussed earlier, it would be very interesting to see what happens in the type of experiment performed by Bartell, Thompson, and Roskos at other intensities.

Since the original prediction of BK and G refers, as we have seen, to the situation where the electron is overtaken by an electromagnetic wave rather than entering it from the side, an experiment to verify this prediction directly becomes rather difficult. It would certainly be interesting to perform such an experiment if one could be devised, but at the present time it does not seem to be possible.

Perhaps the most interesting conclusion of this paper from a theoretical point of view is the very precise similarity between the two refraction effects, of electrons by photons and of photons by electrons. The origin of both effects lies in the same interaction term, and both may be described by essentially the same equations. There is however one point of dissimilarity which was ignored in our earlier discussion, namely, the space-charge effect in the electron beam. For dense electron beams this is obviously an important factor which should be included in the discussion. It could be eliminated by using an ion-electron plasma in place of a beam of free electrons. However, the effect would then be considerably reduced in magnitude. The force on the electrons would remain the same but would yield accelerations reduced by a factor m/M. Nevertheless, with sufficiently high intensity the effect might still be observable.