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Gravitational Fields in Finite and Conformal Bondi Frames

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We generalize the Bondi-Sachs treatment of the initial-value problem using null coordinate systems. This treatment is applicable in both finite and asymptotic regions of space whose sources are bounded by a finite world tube. Using the conformal techniques developed by Penrose, we rederive the results of Bondi and co-workers and of Sachs in conformal-space language. Definitions of asymptotic symmetry "linkages" are developed which offer an invariant way of labeling the properties of finite regions of space, e.g., energy and momentum. These linkages form a representation of the Bondi-Metzner-Sachs asymptotic symmetry group.

I. INTRODUCTION

NULL coordinate systems have been very helpful in studying the properties of asymptotically flat gravitational fields by means of a characteristic initial-value formulation.[1–4] The investigations of Bondi *et al.* and of Sachs have led to the extremely important result that a gravitationally radiating system must lose mass and to the identification of the Bondi-Metzner-Sachs (BMS) asymptotic symmetry group. These investigators applied their techniques mainly to questions concerning the behavior of gravitational fields at null infinity (in the limit of infinite luminosity distances along null hypersurfaces). As a consequence, it is not clear just how important the role played by boundary conditions at null infinity is to their formulation of the characteristic initial-value problem. In Secs. II–V, we carry over their formulation to finite regions of space in terms of a null coordinate system based upon a finite world tube rather than upon null infinity.

Penrose[5–7] has provided a rigorous geometrical foundation to the asymptotic methods which have been used in studying the behavior of gravitational fields at null infinity. His treatment maps null infinity into a finite world tube which forms the boundary of a space conformal to the physical space. In Secs. VI–VIII, we use Penrose's techniques to carry back our formulation of the characteristic initial-value problem based upon finite world tubes to a formulation based upon the world tube at null infinity. We thus rederive the results of Bondi and co-workers and of Sachs in conformal space language. In this way, a clear picture of the role of null infinity in their approach emerges. Of particular importance, we have been able to give a natural extension and geometrization of their mass expression.[8]

There are three items where caution with regard to the notation should be exercised. Firstly, we use signature +2 for the metric. This results in various minus-sign differences with Ref. 8. Secondly, in Secs. II–V, $g_{\mu\nu}$ symbolizes the usual physical space metric. In Secs. VI–VIII, we use $g_{\mu\nu}$ for the conformal space metric and $\tilde{g}_{\mu\nu}$ for the physical space metric. When reading the latter sections, any equations brought over from the first sections should be regarded as equations with $\tilde{g}_{\mu\nu}$ substituted for $g_{\mu\nu}$ and, say, $\tilde{R}_{\mu\nu}$ substituted for $R_{\mu\nu}$. We shall re-emphasize this point later. Finally, we apply the generalized Gauss integral theorems mainly in the dual form, such as

$$\int_\Gamma A^{[\mu\nu]}{}_{;\nu} dS_\mu = \oint_\Sigma A^{[\mu\nu]} dS_{\mu\nu}.$$

We use the standard dual hypersurface elements, except for the incorporation of a factor or $(4\pi)^{-1}$. For

[1] H. Bondi, M. G. J. van der Burg, and A. W. K. Metzner, Proc. Roy. Soc. (London) **269**, 21 (1962).
[2] E. Newman and R. Penrose, J. Math. Phys. 3, 566 (1962).
[3] R. K. Sachs, Proc. Roy. Soc. (London) 270, 103 (1962).
[4] R. K. Sachs, Phys. Rev. **128**, 2851 (1962).
[5] R. Penrose, Phys. Rev. Letters **10**, 66 (1963).
[6] R. Penrose, in *Relativity, Groups and Topology* (Gordon and Breach, Science Publishers, Inc., New York, 1964), p. 565.
[7] R. Penrose, Proc. Roy. Soc. (London) **284**, 159 (1965).
[8] J. Winicour and L. Tamburino, Phys. Rev. Letters **15**, 601 (1965).

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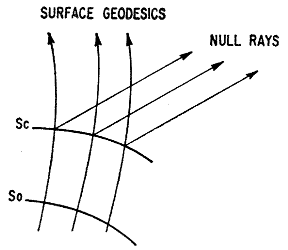

FIG. 1. This diagram illustrates the construction of our null coordinate system. The hypersurface Γ is delineated by the three surface geodesics, which are normal to the geodesically parallel slices S_0 and S_c. The null hypersurface N_c, the locus of null rays emanating from S_c, is delineated by the three null rays.

instance,

$$dS_\mu = \left(\frac{1}{4\pi}\right)\frac{1}{6}(-g)^{1/2}\,\epsilon_{\mu\nu\rho\sigma}d\tau^{\nu\rho\sigma},$$

where $d\tau^{\nu\rho\sigma}$ is the tensor volume element.[9] Brackets and parentheses denote antisymmetrization and symmetrization, respectively.

II. NULL COORDINATE SYSTEMS

The characteristic surfaces for the gravitational field equations are null hypersurfaces in a curved space. Solutions of the equations

$$g^{\mu\nu}u_{,\mu}u_{,\nu}=0 \qquad (2.1)$$

define a family of $u=$const null hypersurfaces. The normal directions to the surface, $k^\mu = g^{\mu\nu}u_{,\nu}$, are also tangent to the surface since null vectors are self-orthogonal. A two-parameter system of null geodesics tangent to k^μ, called rays, generate a single null hypersurface.

Null coordinate systems incorporate a family of null hypersurfaces as coordinate surfaces $x^0=u$. Two additional coordinates x^A are chosen as parameters constant along each ray:

$$x^A{}_{,\nu}k^\nu=0\,,\quad (A=2,3)\,. \qquad (2.2)$$

This choice of coordinates leads to three algebraic conditions on the metric:

$$g^{00}=g^{0A}=0 \Longleftrightarrow g_{11}=g_{1A}=0\,. \qquad (2.3)$$

Only the x^1 coordinate varies along a given ray.

As an example, consider a null coordinate system in Minkowski space. The retarded time $u=t-r$ labels a family of null cones which are generated by light rays emanating from the vertices. The Minkowski line element in null polar coordinates is

$$ds^2=-du^2-2dudr+r^2(d\theta^2+\sin^2\theta d\varphi^2)\,, \qquad (2.4)$$

where $x^1=r$ is the radial distance and $x^2=\theta$ and $x^3=\varphi$ are the usual polar angles.

We shall construct a particular type of null coordinate system based upon a three-dimensional time-like hypersurface Γ. We begin by establishing geodesic coordinates on Γ. These in turn determine a unique family of null hypersurfaces in the 4-space. Assume we are given a fixed two-dimensional space-like slice S_0 of Γ on which there is a coordinate net x^A $(A=2,3)$. The *inner* geometry of Γ determines a family of time-like geodesics normal to S_0 which we label by their intersections x^A. We take the arc length along these surface geodesics to be the time coordinate $x^0=u$ and the labels x^A to be spatial coordinates. Thus we now have a family of parallel slices S_c, given by $x^0=c$ and coordinatized by x^A. Using these coordinates, the inner metric of Γ takes the geodesic form,

$$ds^2=-(dx^0)^2+q_{AB}dx^Adx^B\,, \qquad (2.5)$$

which is completely specified by three functions q_{AB}. Below, we shall relate these functions to the equation of the surface Γ and to two functions that characterize the dynamics of the gravitational field. The latter are analogous to Bondi's "news function" and comprise part of the data for the initial-value problem.

We extend our coordinate system by constructing a family of null hypersurfaces N_c emanating from the family of parallel slices S_c. We label the null rays on N_c by their intersections x^A with S_c (Fig. 1).

From the point of view of differential equations, the coordinates on Γ are the initial data for the integration of Eqs. (2.1) and (2.3). The existence of solutions to these differential equations is justified by physical considerations. We can imagine the null rays to be light beams from flashlights moving along the surface geodesics and pointing normal to Γ.

We prescribe the coordinate $x^1=r$ to be a luminosity distance along the null rays by the algebraic definition

$$[r^4f(x^A)^2]^{-1}\equiv |g^{\mu\nu}x^A{}_{,\mu}x^B{}_{,\nu}| = |g^{AB}|\,, \qquad (2.6)$$

where we have introduced an abbreviated notation for the 2-dimensional determinant and where $f(x^A)$ is some definite (known) function characterizing the type of angular variable x^A used to label the null rays. For instance, $f(x^A)=1$ for ray labels corresponding to $x^2=\cos\theta$, $x^3=\varphi$. The significant feature of this condition is that in the new coordinate system $|g_{AB}|$ is both determined and independent of time, so that the 2-metric g_{AB} manifestly contains only *two* dynamical degrees of freedom. This feature distinguishes the Bondi type null coordinate systems[1] considered here from other versions.[2] We may describe Γ in our coordinate system by an equation $r=\eta$. Equation (2.6), evaluated on Γ, relates η to the inner metric:

$$\eta(u,x^A)=[|g_{AB}|/f(x^A)^2]^{1/4}. \qquad (2.7)$$

The full metric obeys Eqs. (2.3) and (2.6) and has the form

$$ds^2=g_{00}du^2+2[g_{01}dr+g_{0A}dx^A]du+g_{AB}dx^Adx^B\,, \qquad (2.8)$$

so that its determinant satisfies

$$g=-(g_{01})^2|g_{AB}|=-(g_{01})^2r^4f^2. \qquad (2.9)$$

[9] J. L. Synge and A. Schild, *Tensor Calculus* (University of Toronto Press, Toronto, 1949).

Furthermore, the following conditions hold *on* Γ:

$$r=\eta(u,x^A), \quad (2.10)$$

$$q_{AB}=g_{AB}, \quad (2.11)$$

$$q_{00}=g_{00}+2\eta_{,0}g_{01}=-1, \quad (2.12)$$

$$q_{0A}=g_{0A}+\eta_{,A}g_{01}=0. \quad (2.13)$$

The general coordinate transformation that preserves the null coordinate conditions which we have tailored to Γ contains three arbitrary functions of two variables. The subgroup that keeps S_0 fixed is

$$\bar{x}^A=f^A(x^B), \quad \bar{u}=u, \quad \bar{r}^2=r^2/|\bar{x}^A{}_{,B}|, \quad (2.14)$$

where the two functions f^A represent the freedom of relabeling the surface geodesics on Γ and, in turn, the null rays. The transformations that alter S_0 are more complicated because the new initial slice $\bar{S}_0$ defines a different family of surface geodesics and therefore a different family of null rays. The third arbitrary function describes the location of $\bar{S}_0$.

III. THE ELECTROMAGNETIC FIELD

Maxwell's equations in a Minkowski space with line element (2.4) provide a simple model for our treatment of the gravitational field equations in the next section. Because both sets of field equations contain two dynamical degrees of freedom and propagate along null characteristic hypersurfaces, there is a striking analogy in the formulations of the characteristic initial-value problem for the two fields. We shall consider electromagnetic fields whose sources are bounded by a closed time-like world tube Γ. In order to facilitate our discussion, we consider a cylindrical world tube given by $r=r_0$. Furthermore, we relate the null polar coordinate system to a family of null cones N_c having spherical intersections S_c with Γ as depicted in Fig. 2. The nonvanishing components of the metric (2.4) are

$$g^{11}=g_{00}=-g^{01}=-g_{01}=1,$$
$$g_{22}=(g^{22})^{-1}=r^2, \quad (3.1)$$
$$g_{33}=(g^{33})^{-1}=r^2\sin^2\theta,$$

where $(x^0,x^1,x^A)\equiv(u,r,\theta,\varphi)$ and $(-g)^{1/2}=r^2\sin\theta$.

The Maxwell field equations *in vacuo* are

$$L^\mu\equiv[(-g)^{1/2}F^{\mu\nu}]_{,\nu}=0, \quad (3.2)$$

where $F_{\mu\nu}$ is expressed in terms of the 4-potential Φ_μ by

$$F_{\mu\nu}=\Phi_{\nu,\mu}-\Phi_{\mu,\nu}. \quad (3.3)$$

This formalism is invariant under the gauge transformation

$$\Phi_\mu\to\Phi_\mu+G_{,\mu}, \quad (3.4)$$

which involves one arbitrary function $G(x^\mu)$. The flat-space metric components are ancillary entities. In the next section, the metric components are the potentials of the gravitational field and the analog of Φ_μ.

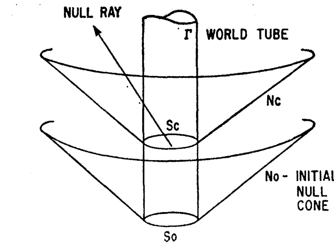

FIG. 2. This diagram illustrates a cylindrical world tube with null hypersurfaces emanating from the spherical sections S_c.

We shall adopt the following gauge conditions:

$$\Phi_1=0 \quad \text{for} \quad r\geq r_0, \quad (3.5)$$

$$\Phi_0=0 \quad \text{for} \quad r=r_0, \quad (3.6)$$

which are analogous to the coordinate conditions (2.3) and (2.5), respectively. The remaining gauge freedom is

$$\Phi_A\to\Phi_A+G(x^B)_{,A}. \quad (3.7)$$

The four field equations L^μ are not independent. They satisfy the identity

$$((-g)^{1/2}L^\mu)_{,\mu}\equiv[(-g)^{1/2}F^{\mu\nu}]_{,\mu\nu}\equiv0. \quad (3.8)$$

It is sufficient to require L^1 only *on* Γ, for it will remain satisfied off Γ provided the other three field equations are satisfied everywhere. It is convenient to group these equations into the following sets:

L^0: hypersurface equation,
L^A: dynamical equations, (3.9)
L^1: conservation condition *on* Γ.

We now write out the field equations and their integrals in our particular gauge. Constants of integration on Γ and S_0 are denoted by $[\ ]_\Gamma$ and $[\ ]_{S_0}$.

$$L^0: \quad [r^2\Phi_{0,1}]_{,1}=J_{0,1},$$

$$\Phi_0=\int_{r_0}^{r}(r')^{-2}J_0dr'+C_\Gamma(1/r_0-1/r), \quad (3.10)$$

where

$$J_0\equiv\csc\theta[(\sin\theta)\Phi_2]_{,2}+(\csc^2\theta)\Phi_{3,3},$$

$$C_\Gamma\equiv r_0^2[\Phi_{0,1}]_\Gamma-[J_0]_\Gamma,$$

$$[\Phi_0]_\Gamma=0.$$

$$L^A: \quad \Phi_{A,01}=J_A,$$

$$\Phi_A=\int_{r_0}^{r}J_Adr'+[\Phi_{A,0}]_\Gamma, \quad (3.11)$$

where

$$J_A\equiv\tfrac{1}{2}(\Phi_{A,1}+\Phi_{0,A})_{,1}+\frac{\csc\theta}{2r^4}[(\Phi_{A,B}-\Phi_{B,A})\csc\theta]_{,B}g_{AA},$$

and where triplex A is not summed.

$$L^1: \ \Phi_{0,10}=J_1,$$

$$[\Phi_{0,1}]_\Gamma=\int_{u_0}^{u} J_1 du'+[\Phi_{0,1}]_{S_0}, \tag{3.12}$$

where

$$J_1\equiv(\csc\theta)\sum_B[g^{BB}(\Phi_{B,1}-\Phi_{B,0})\sin\theta]_{,B},$$

and where the integration is *on* Γ.

In the above grouping, the hypersurface equation defines Φ_0 in terms of the dynamical variables Φ_A; the dynamical equations determine the evolution of Φ_A off the null cones; and the conservation condition determines the evolution of the integration constant $[\Phi_{0,1}]_\Gamma$ along the world tube.

The mixed data required to integrate the field equations are:

$$[\Phi_{A,0}]_\Gamma \text{ on world tube}, \tag{3.13a}$$

$$[\Phi_A]_{N_0} \text{ on initial null cone}, \tag{3.13b}$$

$$[\Phi_{0,1}]_{S_0} \text{ on initial slice}. \tag{3.13c}$$

We can evolve a solution in time by using an iterative integration process. This scheme commences on the initial null cone N_0. Given the data (3.13b) and (3.13c), we determine Φ_0 from the hypersurface equation L^0. With this result and the data (3.13a), we use the dynamical equations L^A to find the retarded time derivatives of Φ_A. The retarded time derivative of $[\Phi_{0,1}]_\Gamma$ at S_0 follows directly from the conservation condition L^1. This completes the initial cycle. Next, using the retarded time derivatives, we determine the same data given in (3.13b) and (3.13c) on a neighboring hypersurface separated by an infinitesimal u displacement. Through repeated application of this process, we can formally generate a solution in the region between Γ and N_0.

This formal iterative approach guarantees the existence of a unique solution only in the analytic case. From the standpoint of mathematical rigor, it offers much weaker knowledge than other techniques which are applicable to linear systems, such as Maxwell's equations. An additional shortcoming peculiar to this mixed initial-value formulation is that it is not correctly set. The "news" (3.13a) cannot be assigned on Γ with complete arbitrariness. On the contrary, because of the time-like character of Γ, prescribing the news on a small region of Γ automatically determines the news on a larger region of Γ. This is a consequence of what John[10] has called the "coherency" of data on time-like surfaces. Although there are no constraints of a purely differential form on the news, there are constraints of a functional nature which must be satisfied on Γ. In the analytic case, these functional constraints are automatically satisfied. But even in the nonanalytic case, the functional class of the news must still possess *unique continuation* properties similar to those of analytic functions. Friedlander[11,12] has investigated the consequences of the incorrectly set nature of the mixed characteristic initial-value problem with the particular geometry considered here. Despite these analytic weaknesses, the strong point of this approach remains: It provides a formulation of the initial-value problem with which Einstein's nonlinear equations can be analyzed without any more inherent difficulties than in the foregoing analysis of Maxwell's linear equations.

[10] F. John, Commun. Pure Appl. Math. 2, 209 (1949).

IV. EINSTEIN'S EQUATIONS

The empty-space equations for the gravitational field are

$$G_{\mu\nu}=0, \tag{4.1}$$

where the Einstein tensor is defined in terms of the Ricci tensor and the curvature scalar by

$$G_{\mu\nu}=R_{\mu\nu}-\tfrac{1}{2}g_{\mu\nu}R. \tag{4.2}$$

These ten equations are not independent. Because of the general covariance of the theory, they satisfy four Bianchi identities

$$G^{\mu\nu}{}_{;\nu}=0. \tag{4.3}$$

We will work in the null-coordinate system constructed in Sec. II. Recall that the metric (2.8) can be specified by six functions, $g_{0\mu}$ and g_{AB}. The transverse metric g_{AB} is determined by two functions which are the analogs of the Φ_A in the previous section.

Combining Eqs. (4.1) and (4.3) with our choice of coordinate conditions leads to the following grouping of the field equations:

$$G_{1\mu}=0 \text{ (4 hypersurface equations)}, \tag{4.4}$$

$$G_{AB}-\tfrac{1}{2}g_{AB}g^{CD}G_{CD}=0 \text{ (2 dynamical equations)}, \tag{4.5}$$

$$[G_{00}]_\Gamma=[G_{0A}]_\Gamma=0 \text{ (3 conservation conditions)}, \tag{4.6}$$

$$g^{CD}G_{CD}=0 \text{ (1 superfluous equation)}. \tag{4.7}$$

In Eqs. (4.4) and (4.5), we have six equations for the six unknown metric components. The hypersurface equations determine $g_{0\mu}$ in terms of the transverse metric to within constants of integration of S_0. The conservation conditions determine the time dependence of these constants of integration. The dynamical equations determine the evolution of the transverse metric off the initial null hypersurface. Two constants of integration on Γ, analogous to what Bondi *et al.*[1] have called the "news function," comprise part of the initial data for the dynamical equations. The superfluous equation, listed for completeness, is automatically satisfied when Eqs. (4.4) and (4.5) are satisfied (see Appendix A).

[11] F. G. Friedlander, Proc. Roy. Soc. (London) **269**, 53 (1962).
[12] F. G. Friedlander, Proc. Roy. Soc. (London) **279**, 386 (1964).

We can construct an integration scheme for the gravitational-field equations in analogy to that in the previous section for the electromagnetic field. We shall exhibit its important features by denoting by J lengthy expressions which are determined by the initial data. The definitions of the J's follow merely from the form of the field equations. The order of considering the latter is designed so that at each step the J's are known either from the initial data or from the results of the preceding equations. Note that all the equations below have been decoupled as shown in the corresponding integrations.

$G_{11}=0$: $(\ln g_{01})_{,1}=J_{11}\geq 0$

$$g_{01}=[g_{01}]_\Gamma \exp\left[\int_\eta^r J_{11}dr'\right]. \quad (4.8)$$

$G_{1A}=0$: $(r^2g^{1\mu}g_{\mu A,1})_{,1}=J_{1A}$,

$$g_{0C}=g_{CD}\left\{[g^{DA}g_{0A}]_\Gamma+[r^2g^{1\mu}g_{\mu A,1}]_\Gamma\int_\eta^r g^{DA}g_{01}(r')^{-2}dr' + \int_\eta^r\left[g^{DA}g_{01}(r')^{-2}\left(\int_\eta^{r'}J_{1A}dr''\right)\right]dr'\right\}. \quad (4.9)$$

$G_{01}=0$: $(g_{00}g^{01}r)_{,1}=J_{01}$,

$$g_{00}=\frac{g_{01}}{r}\left\{\int_\eta^r J_{01}dr'+[g_{00}g^{01}r]_\Gamma\right\}. \quad (4.10)$$

To discuss the dynamical equations, it is convenient to introduce a complex polarization dyad t^A, which is defined up to a phase transformation by the following properties:

$$g_{AB}=t_A\bar{t}_B+\bar{t}_At_B, \quad (A,B=2,3), \quad (4.11)$$

$$t^A=g^{AB}t_B, \quad t^At_A=\bar{t}^At_A-1=0. \quad (4.12)$$

Equation (4.5) is then equivalent to the one complex equation

$$G_{AB}t^At^B=0 \quad (4.13)$$

of the form

$$\dot{c}_{,1}-2\dot{c}[r^{-1}-t^A\bar{t}_{A,1}]=J, \quad (4.14)$$

where

$$\dot{c}\equiv\tfrac{1}{2}rg_{AB,0}t^At^B. \quad (4.15)$$

The integral of Eq. (4.14) is

$$\dot{c}=[\dot{c}]_\Gamma\exp[-V(r,\eta)] + \int_\eta^r J(r')\exp[V(r',r)]dr', \quad (4.16)$$

where

$$V(s,t)=2\int_t^s (t^A\bar{t}_{A,1}-r^{-1})dr.$$

The integration constant $[\dot{c}]_\Gamma$ is the news function for Γ. A particular choice of polarization dyad is discussed in Appendix B.

There are six constants of integration that appear in the hypersurface equations, $[g_{0\mu}]_\Gamma$ and $[g_{0A,1}]_\Gamma$. These are not independent. They satisfy Eqs. (2.12) and (2.13), which relate the inner metric of Γ to the 4-space metric. It is convenient to choose $[g_{00}]_\Gamma$ and $[g_{0A,1}]_\Gamma$ as the three independent integration constants. The three conservation conditions take the same differential form.

$$[G_{00}]_\Gamma=0: \quad D[g_{00}]_\Gamma=[J_{00}]_\Gamma, \quad (4.17)$$

$$[G_{0A}]_\Gamma=0: \quad D[g_{0A,1}]_\Gamma=[J_{0A}]_\Gamma, \quad (4.18)$$

where

$$D\equiv[\partial/\partial u+\eta_{,0}\partial/\partial r]_\Gamma$$

is the u derivative along the time-like geodesics in Γ. Solutions to this system require specification of $[g_{00}]_{S_0}$ and $[g_{0A,1}]_{S_0}$ on the initial slice S_0.

With our choice of coordinates, those equations which govern the r dependence of the metric take on the form of ordinary differential equations whose integrals we have exhibited above. Those equations which govern the u dependence have no such simple features. However, an iterative integration process in complete analogy to that of the preceding section may be applied to formally generate a solution. The mixed initial data required consist of four functions of three variables and three functions of two variables:

$$(\dot{c})_\Gamma, \quad (4.19a)$$

$$[g_{AB}]_{N_0}, \quad (4.19b)$$

$$[g_{00}]_{S_0}, \text{ and } [g_{0A,1}]_{S_0}. \quad (4.19c)$$

The news function for Γ (4.19a) characterizes the time behavior of the sources within Γ. The initial null hypersurface data (4.19b) describe incoming fields entering the region under consideration through N_0. The initial-slice integration constants (4.19c) describe characteristics of the initial configuration of the sources within Γ in analogy with Bondi's mass aspect.[1] The discussion in Sec. III of the analytic weakness of this approach applies here, although it is not clear what coherency requirements may be derived for these *nonlinear* equations. In addition, there are geometrical limitations associated with possible self-intersections in the families of geodesics used to construct the coordinate system.

V. THE CONSERVATION CONDITIONS

The initial-value formulation developed in the previous sections can be applied most effectively when Γ is a world tube of topology $S^2\times E^1$ which completely surrounds the sources of the gravitational field. In this case, the conservation conditions (4.6) are equivalent to a set of integral flux conservation laws. We first note that these conditions may be written as

$$[G_{0\mu}\xi^\mu]_\Gamma=0, \quad (5.1)$$

where ξ^μ ranges over a complete vectorial set of directions lying in Γ. If in addition we let ξ^μ range over a complete functional set we may replace Eq. (5.1) by the integral equations

$$\int_\Gamma G_\mu{}^\nu \xi^\mu dS_\nu = 0. \qquad (5.2)$$

Since the hypersurface and dynamical equations ensure the vanishing of the Ricci scalar, we may equivalently write Eq. (5.2) as

$$\int_\Gamma R_\mu{}^\nu \xi^\mu dS_\nu = 0. \qquad (5.3)$$

Using the Ricci identities, Eq. (5.3) becomes

$$\int_\Gamma \{\xi^{[\mu;\nu]}{}_{;\mu} + \xi^{(\mu;\nu)}{}_{;\mu} - \xi^\mu{}_{;\mu}{}^{;\nu}\} dS_\nu = 0. \qquad (5.4)$$

Applying the conservation conditions in the form of Eq. (5.4) to a portion of Γ bounded by two slices Σ_1 and Σ_2 of topology S^2 and utilizing the generalized Gauss theorem then gives

$$K_\xi(\Sigma_2) - K_\xi(\Sigma_1) = \int_\Gamma [\xi^{(\mu;\nu)}{}_{;\mu} - \xi^\mu{}_{;\mu}{}^{;\nu}] dS_\nu, \qquad (5.5)$$

where

$$K_\xi(\Sigma) \equiv \oint_\Sigma \xi^{[\mu;\nu]} dS_{\mu\nu}. \qquad (5.6)$$

Equation (5.5) is the integral form of a covariant flux conservation law previously considered by Komar.[13] It is a "weak" conservation law in the sense that its validity depends upon the field equations being satisfied. In this integral form, the conservation law relates the change in the functional $K_\xi(\Sigma)$ evaluated at two slices of the world tube to a flux integral across the world tube.

The scalar functional $K_\xi(\Sigma)$ is determined by the specification of a vector field and a closed 2-space. In order to attach physical meaning to this functional we are at first led to associate the vector field ξ^μ with the descriptor of an infinitesimal transformation. For this to be effective, we need a means of geometrically selecting descriptor fields. If the space admits a global symmetry transformation whose descriptor satisfies Killing's equation

$$\xi^{(\mu;\nu)} = 0, \qquad (5.7)$$

we are then provided with a preferred descriptor field. This is not a sufficiently effective approach because global symmetries do not exist in a general space.

There is, however, an approach which is applicable to the class of asymptotically flat spaces. Such spaces admit asymptotic symmetries.[4] In the following sections, we will precisely define the notions of asymptotically flat spaces and asymptotic symmetries. For the present, it is sufficient to state that these spaces define preferred descriptor fields at null infinity. There is no geometrically intrinsic way to propagate descriptors at null infinity throughout space-time. The 2-space Σ, however, does determine a geometrical prescription for propagating descriptors from null infinity to Σ. Each point on Σ geometrically determines two null directions which are orthogonal to the local 2-space and to themselves. The entirety of these null directions on Σ define two null hypersurfaces which, for simple topologies, emanate out from Σ to future and past null infinity. In this paper we concentrate on the future null hypersurface N. Let k^μ denote its normal vector field. Then the propagation law

$$\xi^{(\mu;\rho)} k_\rho = \tfrac{1}{2} \xi^\rho{}_{;\rho} k^\mu \qquad (5.8)$$

uniquely determines ξ^μ *on* the null hypersurface N in terms of its value at future null infinity. This can be readily seen by writing Eq. (5.8) in a Bondi coordinate system for which N is a u=constant surface. In fact, in such a coordinate system Eq. (5.8) describes an infinitesimal transformation which preserves the Bondi coordinate conditions [Eqs. (2.3) and (2.6)].[4] Furthermore, Eq. (5.8) is the only covariant propagation law which (1) is linear in the descriptor, (2) is of first differential order, (3) involves derivatives only in directions lying in N, (4) is automatically satisfied by global symmetry descriptors, and (5) uniquely determines the descriptor on N in terms of its value at null infinity.

Using this propagation law, we still cannot evaluate Komar's functional on Σ because $K_\xi(\Sigma)$ involves derivatives of the descriptor in directions lying *out* of N, while the descriptor field has only been defined *on* N. However, we can find a geometrical modification of Komar's integral which is defined on Σ. Represent by k^μ and m^μ, respectively, the outgoing and incoming null directions which Σ picks out at each point on its surface. Normalize these vectors by

$$k^\mu m_\mu = -1. \qquad (5.9)$$

Although Eq. (5.9) does not uniquely determine the extensions of k^μ and m^μ it does completely fix bilinear products such as the bivector

$$B^{\mu\nu} = k^{[\mu} m^{\nu]}. \qquad (5.10)$$

If we now look for the most general geometrically defined integral which is linear in the descriptor, is of first differential order, and involves derivatives only in directions lying *in* N, we are uniquely led to the following functional:

$$L_\xi(\Sigma) = \oint_\Sigma (\xi^{[\mu;\nu]} - \xi^\rho{}_{;\rho} k^{[\mu} m^{\nu]}) dS_{\mu\nu}. \qquad (5.11)$$

[13] A. Komar, Phys. Rev. 113, 934 (1959).

For global symmetry descriptors, it reduces to Komar's functional.

When the asymptotic symmetry descriptor represents, say, a time translation, we can correlate the corresponding functional with energy. The closed 2-space Σ, however, does not pick out any spanning space-like 3-space in which we could say the energy resides. In fact, we have not even made an attempt to describe the topology of the region close to the sources inside Γ. We can avoid this problem by interpreting the functional as an energy linkage through Σ. Linkage[14] is meant here in its topological sense as the 4-dimensional analog of the 3-dimensional concept of a trajectory passing through a closed loop. For a generic descriptor ξ^μ we refer to the corresponding functional $L_\xi(\Sigma)$ as the ξ linkage through Σ.[8] It represents that aspect corresponding to the ξ asymptotic symmetry belonging to those sources which link the closed 2-space Σ.

The modification of Komar's integral which we have been led to on geometrical grounds has strong physical justification. The total mass of a gravitationally radiating system defined this way turns out to be the monotonically decreasing mass defined by Bondi (see Sec. VIII). On the other hand, Komar's integral leads to a total mass which does not *decrease* for a *radiating* system.

In order to obtain a conservation law for $L_\xi(\Sigma)$ analagous to Eq. (5.5) we must define the local flux

$$F_\xi{}^\mu \equiv (\xi^{[\mu;\nu]} - \xi^\rho{}_{;\rho} k^{[\mu} m^{\nu]})_{;\nu}. \quad (5.12)$$

This entails extending the domain of definition of ξ^μ, k^μ, and m^μ. There does not appear to be any way which is not somewhat geometrically artificial to define these quantities on Γ. What does assume intrinsic geometrical importance is the local flux across the null hypersurface N. ξ^μ and k^μ are already defined along N. We define m^μ along N by

$$[m_\mu m^\mu]_N = [m_\mu k^\mu]_N + 1 = 0. \quad (5.13)$$

This defines m^μ on N up to a null rotation.[15] We are interested in the component of the local flux across N,

$$F_\xi \equiv F_\xi{}^\mu k_\mu. \quad (5.14)$$

From Eqs. (5.12) and (5.13) we have

$$F_\xi = \xi^{[\mu;\nu]}{}_{;\nu} k_\mu - \tfrac{1}{2}(\xi^\nu{}_{;\nu} k^\mu)_{;\mu}. \quad (5.15)$$

It is manifestly independent of the null rotation freedom in m^μ. Using the Ricci identities and the propagation law, we also have

$$F_\xi = -\xi^{(\mu;\nu)} k_{\mu;\nu} - \xi^\mu{}_{;\mu\nu} k^\nu. \quad (5.16)$$

The flux conservation law between two closed 2-spaces on N takes the form

$$L_\xi(\Sigma_2) - L_\xi(\Sigma_1) = \int_N F_\xi dV, \quad (5.17)$$

where

$$k_\mu dV \equiv dS_\mu. \quad (5.18)$$

It is also convenient to rewrite the linkage expression as

$$L_\xi(\Sigma) = -\oint_\Sigma \xi^{\nu;\mu} k_\mu m_\nu dS, \quad (5.19)$$

where

$$k_{[\mu} m_{\nu]} dS = dS_{\mu\nu}. \quad (5.20)$$

VI. CONFORMAL BONDI COORDINATES

Using conformal techniques, Penrose[5–7] has provided a precise geometrical formulation of the boundary conditions at null infinity which have been so successful[1–4] in analyzing the radiative structure of asymptotically flat gravitational fields. Penrose envisages null infinity as a boundary to space-time consisting of the limit points of null rays. The metric structure of space-time offers no satisfactory basis for discussing "points at infinity." However, by means of a coordinate transformation we could at least assign finite coordinate values to these points. Obviously, in such a coordinate system the metric must be singular at the points representing null infinity in order that points in their neighborhood be infinitely distant. By performing a conformal transformation on this metric it might be possible to eliminate these singularities and arrive at a conformal metric which is regular at null infinity. It is the realizability of just this procedure which Penrose postulates as the key requirement for a space to be asymptotically flat. For most purposes, the requirement that the conformal metric be of class C^3 at null infinity ensures sufficient regularity, although we will not emphasize this point here. In addition, a topological requirement must also be made. The set of points at future null infinity called J^+ which represent the future end points of null rays and the set J^- at past null infinity must each have the topology $S^2 \times E^1$. Geometrically then, we may summarize the boundary conditions appropriate to asymptotically flat spaces by the requirement that there exist a manifold conformal to the physical manifold with this differentiable and topological structure at null infinity.

We have shown in the preceding sections how the adaptation of a Bondi coordinate system to a finite world tube leads to a reasonably straightforward formulation of the characteristic initial-value problem. Using Penrose's conformal techniques to adapt a conformal Bondi coordinate system to the world tube at null infinity, we now connect this finite version of the initial-value problem with the asymptotic treatments of Bondi *et al.*[1] and Sachs.[3] We confine our attention to future null infinity J^+.

[14] P. Alexandroff, *Elementary Concepts in Topology* (Dover Publications, New York, 1961), p. 16.

[15] R. K. Sachs, Proc. Roy. Soc. (London) **264**, 309 (1961).

Let x^μ represent a successful candidate for a coordinate system satisfying Penrose's requirements. Then points at J^+ take on finite coordinate values and there exists a conformal space metric, now represented by $g_{\mu\nu}(x)$, that is regular at J^+ and related to the physical space metric, now represented by $\tilde{g}_{\mu\nu}(x)$, by

$$g_{\mu\nu}=\Omega^2\tilde{g}_{\mu\nu}. \tag{6.1}$$

Points on J^+ satisfy $\Omega(x^\mu)=0$. We further require[7] that Ω be regular at J^+ and that

$$[\Omega_{;\mu}]_{J^+}\neq 0. \tag{6.2}$$

We now use symbols such as $\tilde{\nabla}_\tau\tilde{R}_{\mu\nu\rho\sigma}$ to denote geometrical quantities and covariant derivatives constructed from the physical metric; and $\nabla_\tau R_{\mu\nu\rho\sigma}$ or $R_{\mu\nu\rho\sigma;\tau}$ to denote the corresponding constructions from the conformal metric. The Einstein equations then take the form[16]

$$\tilde{G}^\alpha{}_\beta=\Omega^2 G^\alpha{}_\beta-2\Omega\Omega^{;\alpha}{}_\beta+\delta^\alpha{}_\beta(2\Omega\Omega^{;\rho}{}_\rho-3\Omega_{;\rho}\Omega^{;\rho})=0. \tag{6.3}$$

Evaluating Eq. (6.3) at $\Omega=0$ immediately tells us that J^+ is a null hypersurface

$$[\Omega_{;\rho}\Omega^{;\rho}]_{J^+}=0. \tag{6.4}$$

We must accordingly modify the procedure of Sec. II which used the arc length along time-like geodesics to "coordinatize" a time-like hypersurface.

We begin by assigning labels x^A $(A=2,3)$ to each point of some initial space-like slice of J^+. Through each of these points x^A there passes exactly one null geodesic lying in J^+. Let u be an affine parameter defined along this family of null geodesics. Assign to each point on the same null geodesic the same labels x^A. Then $x^0=u$ and x^A coordinatize J^+.

Next consider the family of space-like slices of J^+ given by $u=$constant. Let N denote the unique outgoing null hypersurface intersecting J^+ in a *particular* slice. Emanating from each point x^A of this slice there is exactly one null geodesic which is orthogonal to the local 2-space of the slice and lies in N. Assign the same labels x^A to all points on this null geodesic and the same label u to all points on N. Applying this procedure to the entire family of $u=$constant slices defines the coordinates $x^0=u$ and x^A off J^+. Define an *inverse* luminosity distance $x^1=r$ by

$$r^4=\Omega^4|g^{\alpha\beta}x^A{}_{,\alpha}x^B{}_{,\beta}|. \tag{6.5}$$

Then in the conformal Bondi frame so constructed the conformal metric takes the form

$$g_{\mu\nu}=\begin{bmatrix} g_{00} & g_{01} & g_{0A} \\ g_{01} & 0 & 0 \\ g_{0A} & 0 & g_{AB}\end{bmatrix}. \tag{6.6}$$

Furthermore, by performing the conformal transformation $g_{\mu\nu}\rightarrow(r^2/\Omega^2)g_{\mu\nu}$ we require

$$|g_{AB}|=1. \tag{6.7}$$

The physical metric is now related to the conformal metric by

$$g_{\mu\nu}=r^2\tilde{g}_{\mu\nu}, \tag{6.8}$$

and J^+ is specified by $r=0$. In this coordinate system, the Einstein equations (6.3) become

$$r^2G^\alpha{}_\beta-2rr^{;\alpha}{}_\beta+\delta^\alpha{}_\beta(2rr^{;\rho}{}_\rho-3r_{;\rho}r^{;\rho})=0. \tag{6.9}$$

Because u is an affine parameter for null geodesics on J^+ labeled by $x^A=$constant, we have *at* J^+

$$g_{00}=g_{0A}=2g_{01,0}-g_{00,1}=0. \tag{6.10}$$

Also, by differentiating and contracting Eq. (6.9), we have *at* J^+

$$r_{;\mu\nu}-\tfrac{1}{4}g_{\mu\nu}r^{;\rho}{}_\rho=0. \tag{6.11}$$

Moreover, by applying the coordinate conditions (6.6), (6.7), and (6.10) to Eq. (6.11), we find *at* J^+

$$r_{;\mu\nu}=0, \tag{6.12}$$

or more specifically the following conditions *at* J^+:

$$g_{00,1}=g_{01,1}=g_{AB,0}=g_{0A,1}+g_{01,A}=0. \tag{6.13}$$

There still remains the affine freedom in our original choice of affine parameter u on J^+ and the freedom of assigning different labels x^A to the initial slice of J^+. Taking advantage of these freedoms and Eqs. (6.10) and (6.13), we may impose the following coordinate conditions *at* J^+:

$$\begin{gathered} g_{00}=g_{00,1}=g_{0A}=g_{0A,1}=g_{01,1}=0, \\ g_{01}=1,\ g_{AB}=q_{AB}, \end{gathered} \tag{6.14}$$

where q_{AB} is time-independent. In particular, because J^+ has topology $S^2\times E^1$ we may put

$$q_{AB}=\begin{pmatrix}(1-W^2)^{-1} & 0 \\ 0 & 1-W^2\end{pmatrix} \tag{6.15}$$

for the choice of ray labels $x^2=W=\cos\theta$, $x^3=\varphi$. The global requirements, Eqs. (6.6), (6.7), and (6.8), and the asymptotic requirements, Eqs. (6.14) and (6.15), summarize our coordinate and conformal conditions. Our construction of a conformal metric satisfying these conditions can be justified for some finite neighborhood of J^+. From Eq. (6.12) we see that the null geodesics on J^+ are shear free and divergenceless, so that null geodesics with different labels x^A cannot intersect.[15] Thus, the affine parameter u and the labels x^A constitute *bona fide* coordinates throughout J^+. Furthermore, there exists some neighborhood of J^+ in which the null hypersurfaces and geodesics used to extend u and x^A off J^+ do not intersect. Thus, u and x^A are well defined in some neighborhood of J^+. Accordingly, there exists a neighborhood of J^+ in which the inverse lumi-

[16] L. R. Eisenhart, *Riemannian Geometry* (Princeton University Press, Princeton, New Jersey, 1926).

nosity distance r can be defined by Eq. (6.5) and used as a coordinate by virtue of Eq. (6.2). This finite region of conformal space which we have coordinatized corresponds to an infinite region of physical space bounded by some finite world tube and future null infinity.

The field equations (6.3) were essential in the preceding construction to recognize the null, shear-free, and divergenceless properties of J^+. The nullness followed immediately from applying the field equations at J^+. Applying the derivative of the field equations at J^+, we obtain

$$[(\nabla_\mu\nabla_\nu-\tfrac{1}{4}g_{\mu\nu}\nabla_\rho\nabla^\rho)\Omega]_{J^+}=0. \tag{6.16}$$

Geometrically, Eq. (6.16) expresses the integrability conditions for the existence of a conformal transformation

$$\Omega^{-2}g_{\mu\nu}=\bar{\Omega}^{-2}\bar{g}_{\mu\nu} \tag{6.17}$$

such that

$$[\bar{\nabla}_\mu\bar{\nabla}_\nu\bar{\Omega}]_{J^+}=0. \tag{6.18}$$

Thus, there exist preferred conformal factors for which J^+ is manifestly shear free and divergenceless. The inverse luminosity distance is such a preferred conformal factor, as evidenced by Eq. (6.12).

Written in terms of the contravariant conformal metric components, the global coordinate and conformal conditions become

$$r^{-2}\tilde{g}^{\mu\nu}=g^{\mu\nu}=\begin{bmatrix}0 & g^{01} & 0\\ g^{01} & g^{11} & g^{1A}\\ 0 & g^{1A} & g^{AB}\end{bmatrix}, \tag{6.19a}$$

$$|g^{AB}|=1, \tag{6.19b}$$

and the asymptotic conditions *at* J^+ become

$$g^{11}=g^{11}{}_{,1}=g^{1A}=g^{1A}{}_{,1}=g^{01}{}_{,1}=0, \tag{6.20a}$$

$$g^{01}=1, \tag{6.20b}$$

$$g^{AB}=q^{AB}=\begin{pmatrix}1-W^2 & 0\\ 0 & (1-W^2)^{-1}\end{pmatrix}. \tag{6.20c}$$

This conformal space metric is obviously not finite at the spherical poles $W=\pm1$. This shortcoming arises from the impossibility of covering a sphere with a *single* regular coordinate patch. Putting $Z^2=1-W^2$, the matrix

$$h^{\mu\nu}=\begin{bmatrix}0 & g^{01} & 0 & 0\\ g^{01} & g^{11} & g^{12}/Z & Zg^{13}\\ 0 & g^{12}/Z & g^{22}/Z^2 & g^{23}\\ 0 & Zg^{13} & g^{23} & Z^2g^{33}\end{bmatrix} \tag{6.21}$$

is free of these polar singularities. Similarly, the singularities at the poles arising from differentiation with respect to polar coordinates do not arise from the differential operators

$$\frac{D}{DW}\equiv Z\frac{\partial}{\partial W},\quad \frac{D}{D\varphi}\equiv\frac{1}{Z}\frac{\partial}{\partial\varphi}. \tag{6.22}$$

Regularity at the spherical poles must be interpreted with respect to Eqs. (6.21) and (6.22).

Some coordinate freedom still remains in the conformal Bondi frame which we have tailored to suit the structure of null infinity. We defer a treatment of the corresponding transformation group until we have more thoroughly analyzed the asymptotic content of the field equations.

VII. THE CONFORMAL-SPACE SOLUTIONS

The null coordinate system of the preceding section leads to the same integration scheme for the conformal-space field equations as presented in Sec. IV. The analysis in terms of data on an initial null hypersurface $u=u_0$ and on J^+ parallels that of Sachs.[3] The major difference is that the requirements of differentiability *at* J^+ of the conformal space metric replace requirements for the existence of an asymptotic series expansion.

First, the $\tilde{G}_{11}$ hypersurface equation is

$$g_{01}g^{01}{}_{,1}=-\tfrac{1}{8}rg^{AB}{}_{,1}g_{AB,1}. \tag{7.1}$$

Equation (7.1) gives g^{01} in terms of the initial hypersurface data g_{AB} [the asymptotic condition (6.20b) determines the integration constant]:

$$g^{01}=\exp\left\{-\frac{1}{8}\int_0^r g^{AB}{}_{,1}g_{AB,1}rdr\right\}. \tag{7.2}$$

The $\tilde{G}_{1A}$ hypersurface equation is

$$r[g_{DC}g^{01}(g^{1C}g_{01})_{,1}]_{,1}-2g_{DC}g^{01}(g^{1C}g_{01})_{,1}=rK_D, \tag{7.3}$$

where K_D is given by

$$K_D=r^{-2}(r^2g_{01}g^{01}{}_{,D})_{,1}-(g^{AB}{}_{,1}g_{BD})_{,A}+\tfrac{1}{2}g^{AB}{}_{,1}g_{AB,D}. \tag{7.4}$$

Applying the differentiability requirements *at* J^+ to Eq. (7.3) gives

$$[K_{D,1}]_0=0, \tag{7.5}$$

where we have introduced the notation $F(u,s,x^A)=[F]_s$. Equation (7.5) is an asymptotic constraint on the hypersurface data. By algebraic manipulations it becomes

$$\epsilon^{AB}b_{DA:B}=0, \tag{7.6}$$

where

$$b_{AB}=[g_{AB,11}-\tfrac{1}{2}g_{AB}g^{CD}g_{CD,11}]_0, \tag{7.7}$$

where ϵ^{AB} is the unit 2-dimensional antisymmetric matrix, and where the colon represents 2-dimensional covariant differentiation with respect to the polar metric $q_{AB}=[g_{AB}]_0$. Equation (7.6) leads to the pair of equations

$$b_{3A}=b_{,A}\quad\text{and}\quad b^{:A}{}_A=0,$$

whose only regular solution defined on the sphere is $b=$constant. Consequently, the asymptotic constraint requires

$$[b_{AB}]_0=0. \tag{7.8}$$

Equation (7.3) further gives

$$[g^{1A}{}_{,11}]_0 = -q^{AB}[K_B]_0 = c^{AB}{}_{:B}\,, \quad (7.9)$$

where

$$c^{AB} = [g^{AB}{}_{,1}]_0.$$

Equation (7.9) and the regularity conditions associated with Eqs. (6.21) and (6.22) lead to the regularity condition at the spherical poles

$$[h^{AB}{}_{,1}]_0 \to 0 \quad \text{as} \quad Z \to 0. \quad (7.10)$$

Using l'Hospital's rule and integration by parts, the solution to Eq. (7.3) assumes the form

$$g^{1A} = \tfrac{1}{2}[g_{BC}g^{1C}{}_{,111} + 2g_{BC,1}g^{1C}{}_{,11}]_0 g^{01}\int_0^r dt\, t^2[g_{01}g^{BA}]_t - g^{01}\int_0^r dt\, t[g^{BA}g_{01}K_B]_t + g^{01}\int_0^r dt\, t^2[g^{BA}g_{01}]_t \int_0^t ds\, s^{-1}[K_{B,1}]_s \quad (7.11)$$

and determines g^{1C} in terms of the hypersurface data and the integration constant

$$N^A(u_0,x^B) \equiv -\tfrac{1}{4}g^{1A}{}_{,111}(u_0,0,x^B). \quad (7.12)$$

The $\tilde{G}_{10}$ hypersurface equation introduces no new constraints due to regularity conditions at J^+. Its integration gives

$$g^{11} = -r^2g^{01}K + \tfrac{1}{6}r^3g^{01}[g^{11}{}_{,111}]_0 + r^3g^{01}\int_0^r dr\, r^{-1}K_{,1}, \quad (7.13)$$

where K is given by

$$K = \tfrac{1}{2}g_{01}g^{AB}R_{AB} - \frac{2}{r}(g_{01}g^{1A})_{,A} \quad (7.14)$$

($g^{AB}R_{AB}$ involves neither g^{11} nor u derivatives). K satisfies

$$[K]_0 = -1\,, \quad [K_{,1}]_0 = 0. \quad (7.15)$$

Equation (7.13) determines g^{11} in terms of the hypersurface data and the integration constant

$$M(u_0,x^A) = -\tfrac{1}{12}g^{11}{}_{,111}(u_0,0,x^A). \quad (7.16)$$

The dynamical equations introduce no new constraints. Introducing a polarization dyad

$$g^{AB} = t^A\bar{t}^B + \bar{t}^At^B\,, \quad t^A = g^{AB}t_B \quad (7.17)$$

and the complex news function

$$C \equiv \frac{1}{2r}g_{AB,0}t^At^B, \quad (7.18)$$

we find

$$\dot{c} = [\dot{c}]_0 \exp\left\{2\int_0^r dt[\bar{t}_Bt^B{}_{,1}]_t\right\} + \frac{1}{2}\int_0^r dS\,\frac{J(S)}{S}\exp\left\{2\int_s^r dt[\bar{t}_Bt^B{}_{,1}]_t\right\}, \quad (7.19)$$

where J satisfies

$$[J]_0 = [J_{,1}]_0 = 0. \quad (7.20)$$

The actual form is quite complicated:

$$J = -g_{01}K_{AB}t^At^B + \frac{1}{r}g_{AB,C}t^At^Bg_{01}g^{1C} - \frac{1}{r}g_{01}g^{11}g_{AB,1}t^At^B - \frac{2}{r}t_A(g_{01}g^{1A})_{,B}t^B\,,$$

where K_{AB} is equal to the hypersurface part of R_{AB} (those terms in R_{AB} not containing u derivatives). Equation (7.19) determines the time development of the hypersurface data in terms of the news at J^+.

The conservation conditions applied at J^+ give

$$M_{,0} = -[|\dot{c}|^2 + \tfrac{1}{4}g^{1A}{}_{,110A}]_0\,, \quad (7.21)$$

$$g_{AB}N^B{}_{,0} = -\tfrac{1}{4}[g_{01,11A0} - 2(g^{1C}{}_{,11}g_{AC,1})_{,0} - 4M_{,A} + g_{AD}(g^{1B}{}_{,11}{}^{:D} - g^{1D}{}_{,11}{}^{:B})_{:B} - (g^{BC}g_{AC,0})_{,11:B} - g_{BC,01}g^{BC}{}_{,1:A}]_0. \quad (7.22)$$

These equations determine the time development of the integration constants M and N^A.

The result of Bondi *et al.* and Sachs now follows:

A specification of g_{AB} on an initial null hypersurface intersecting J^+, of M and N^A on the initial slice of J^+, and of $\dot{c}$ on J^+, determines a solution of the field equations in the neighborhood of J^+.

In physical space, this corresponds to the determination of a solution in the region between some finite world tube and future null infinity. Alternatively, the constants of integration in Eqs. (7.2), (7.11), (7.13), and (7.19) could be assigned on this finite world tube. This connects the asymptotic version of the initial value problem with the finite version discussed in Sec. IV.

VIII. THE ASYMPTOTIC SYMMETRIES

The results of the last two sections are sufficient to show that the differential and topological requirements of J^+ lead to those asymptotic properties which have previously been associated with outgoing radiation fields. For instance, the structure of the conformal Bondi metric directly leads to the vanishing of the conformal curvature tensor[16] at J^+:

$$[C_{\mu\nu\rho\sigma}]_{J^+} = 0. \quad (8.1)$$

When transformed back into physical space using a Bondi coordinate system based upon a luminosity distance, Eq. (8.1) becomes equivalent to one of the various forms (given by Sachs[3]) of stating the peeling-off property of the Weyl tensor.

Another important feature of asymptotically flat spaces is the asymptotic symmetries investigated by Sachs.[4] An isometric transformation

$$\bar{x}^\mu{}_{,\alpha}\bar{x}^\nu{}_{,\beta}\tilde{g}^{\alpha\beta}(x) = \tilde{g}^{\mu\nu}(\bar{x}) \quad (8.2)$$

expresses a symmetry of the physical space. While a general physical space does not possess such symmetries, in asymptotically flat spaces there does exist a transformation group, the BMS group, which asymptotically satisfies Eq. (8.2). Penrose[6] has interpreted the BMS group geometrically as a group of conformal isometries of J^+. In terms of the conformal space metric, the conditions for an isometry of physical space Eq. (8.2), become conditions for a conformal isometry satisfying

$$\bar{x}^\mu{}_{,\alpha}\bar{x}^\nu{}_{,\beta}g^{\alpha\beta}(x)=[\Omega^2(\bar{x})/\Omega^2(x)]g^{\mu\nu}(\bar{x}). \tag{8.3}$$

Unlike Eq. (8.2), however, Eq. (8.3) is geometrically well defined at J^+ for those allowed transformations that preserve the differentiability of J^+. In this way, we can geometrically define the asymptotic symmetry transformations as the group of transformations of conformal space satisfying Eq. (8.3) *at* J^+. The conditions for an infinitesimal asymptotic symmetry transformation $\bar{x}^\mu=x^\mu+\xi^\mu$ then become

$$[\xi^{(\mu;\nu)}-(\Omega_{,\rho}\xi^\rho/\Omega)g^{\mu\nu}]_{J^+}=0. \tag{8.4}$$

We immediately have the regularity condition

$$[\Omega_{,\rho}\xi^\rho]_{J^+}=0. \tag{8.5}$$

The analysis of these conditions becomes particularly simple in our conformal Bondi frame. For this frame, $\Omega=r$ and

$$[g^{\mu\nu}]_{J^+}=\begin{bmatrix}0 & 1 & 0\\ 1 & 0 & 0\\ 0 & 0 & q^{AB}(x^C)\end{bmatrix}. \tag{8.6}$$

Equation (8.5) gives

$$[\xi^1]_{J^+}=0 \tag{8.7}$$

and, using l'Hospital's rule, Eq. (8.4) may be written out as

$$[g^{\mu\rho}\xi^\nu{}_{,\rho}+g^{\nu\rho}\xi^\mu{}_{,\rho}-g^{\mu\nu}{}_{,\rho}\xi^\rho-2\xi^1{}_{,1}g^{\mu\nu}]_0=0. \tag{8.8}$$

From Eq. (8.8), we find the following conditions *at* J^+:

$$\xi^A=f^A(x^B),\quad f^{(A:B)}=\tfrac{1}{2}q^{AB}f^C{}_{:C},\quad \xi^0=\tfrac{1}{2}uf^A{}_{:A}+\alpha(x^B), \tag{8.9a}$$

and

$$\xi^0{}_{,1}=0,\quad \xi^B{}_{,1}=\tfrac{1}{2}f^A{}_{:A},\quad \xi^B{}_{,1}=-(\tfrac{1}{2}uf^A{}_{:A}+\alpha)^{:B}. \tag{8.9b}$$

Equations (8.7) and (8.9a) restrict the arbitrariness of the values of the new coordinates *at* J^+. Equations (8.9a) are precisely the defining equations of BMS transformations as given by Sachs.[4] The functions f^A and α determine the transformation freedom at J^+. The transformations with $\alpha=0$ describe the 6-parameter subgroup of conformal transformations of the unit sphere. This subgroup is isomorphic to the orthochronous homogeneous Lorentz group. The transformations with $f^A=0$ form the normal subgroup of supertranslations. In particular, the supertranslations given by

$$\alpha=Y_a\epsilon^a, \tag{8.10a}$$

where

$$Y_0=1,\ Y_1+iY_2=-(1-W^2)^{1/2}e^{i\varphi},\ Y_3=-W, \tag{8.10b}$$

form the normal 4-parameter translation subgroup. The parameter choices $\epsilon^a=\delta^{a\beta}$ represent unit translations along four orthogonal axes, with $b=0$ corresponding to a time-like translation.

Equations (8.9b) uniquely determine the first r derivative of the new coordinates *at* J^+. There are no conditions on the higher order r derivatives. Hence all transformations of the interior with $f^A=\alpha=0$ are automatically asymptotic symmetry transformations. They form an invariant subgroup of the asymptotic symmetry group whose factor group is the BMS group.

The defining equations of the asymptotic symmetry group are covariant under the allowed differentiable transformations of J^+. The structure of the asymptotic symmetry group is consequently an invariant property of J^+. The conformal Bondi frame merely played the role of a calculational aid. Had we chosen a different conformal coordinate frame for our analysis, the resulting asymptotic symmetry group would turn out to be isomorphic to the group which we have found. This presentation of the BMS group has been motivated by ideas once suggested by Bergmann.[17] Penrose[6] has come to the same conclusions from a slightly different geometric viewpoint.

We may now define a basis

$$\eta_Q{}^\alpha(x)=[\xi_Q{}^\alpha(x)]_{J^+}\quad (Q=0,1,2,\cdots)$$

for BMS descriptors on J^+. Given a closed 2-space Σ as in Sec. V, we can use the propagation law to define a descriptor field $\xi_Q{}^\alpha$ on Σ corresponding to each BMS descriptor $\eta_Q{}^\alpha$ at its image Σ^+ on J^+. In this way, we may form a linkage $L_Q(\Sigma)$ corresponding to each asymptotic symmetry η_Q. We now formulate these ideas analytically in terms of the conformal space geometry. First, we re-express the null vectors associated with Σ in Sec. V (now written as $\tilde{k}^\mu$ and $\tilde{m}^\mu$) in terms of null vectors k^μ and m^μ which are normalized with respect to the conformal metric:

$$m^\mu=\tilde{m}^\mu,\quad k^\mu=\Omega^{-2}\tilde{k}^\mu,\quad k^\mu m_\mu=-1.$$

We fix the extensions of k^μ and m^μ so that parallel propagation along the null rays from Σ to Σ^+ leads to nonsingular vectors at Σ^+. We re-express the physical space surface element $d\tilde{S}_{\mu\nu}$ and volume element $d\tilde{S}_\mu$ as

$$d\tilde{S}_{\mu\nu}=\tilde{k}_{[\mu}\tilde{m}_{\nu]}d\tilde{S}=\Omega^{-4}dS_{\mu\nu}=\Omega^{-4}k_{[\mu}m_{\nu]}dS$$

and

$$d\tilde{S}_\mu=\Omega^{-4}dS_\mu,$$

so that $dS_{\mu\nu}$ and dS_μ are nonsingular at J^+. When the

[17] P. G. Bergmann, Phys. Rev. 124, 274 (1961).

volume element lies along the outgoing null direction k_μ we also have

$$d\tilde{S}_\mu=\tilde{k}_\mu d\tilde{V}=\Omega^{-4}k_\mu dV.$$

The propagation law Eq. (5.8) now becomes

$$\xi^{(\mu;\nu)}k_\nu=\tfrac{1}{2}\xi^\rho{}_{;\rho}k^\mu-\Omega^{-1}\Omega_{,\rho}\xi^\rho k^\mu. \tag{8.11}$$

The linkage expression as given in Eq. (5.19) becomes

$$L_\xi(\Sigma)=-\oint[\xi^{\nu;\mu}k_\mu m_\nu-\Omega^{-1}(2\Omega^{;\mu}\xi^\nu k_{[\mu}m_{\nu]}-\Omega_{,\rho}\xi^\rho)]\Omega^{-2}dS, \tag{8.12}$$

and the local flux across an outgoing null hypersurface as given in Eq. (5.16) becomes

$$FdV=\tilde{F}d\tilde{V}=[-\xi^{(\mu;\nu)}k_{\mu;\nu}-\xi^\mu{}_{;\mu\nu}k^\nu+\Omega^{-1}\Omega_{,\nu}\xi^\nu k^\mu{}_{;\mu}+4k^\nu(\Omega^{-1}\Omega_{,\mu}\xi^\mu)_{;\nu}]\Omega^{-2}dV. \tag{8.13}$$

From Eq. (8.5) we see that the propagation law is nonsingular at J^+, and from Eq. (8.4) we see that it is automatically satisfied at J^+ by any asymptotic symmetry descriptor. Thus, the propagation law may be applied to asymptotic symmetry descriptors without inconsistency.

We now justify the reasonableness of our linkage definition from a mathematical point of view by proving that *the total linkage $L_Q(\Sigma^+)$ corresponding to any asymptotic symmetry is finite.* We carry out the proof in a conformal Bondi frame. We can always choose such a frame so that Σ^+ is described by u=constant, $r=0$. Consider the one-parameter family of sphere-like 2-spaces given by r=constant which converges to Σ^+ along the null hypersurface u=constant. Each member of the family intrinsically defines the null vectors

$$k_\mu=(-1,0,0,0), \tag{8.14a}$$

$$m_\mu=(-\tfrac{1}{2}g_{01}{}^2g^{11},g_{01},0,0). \tag{8.14b}$$

The propagation law becomes

$$\begin{aligned}&\xi^0{}_{,1}=0,\\ &\xi^A{}_{,1}=-g_{01}g^{AB}\xi^0{}_{,B},\\ &\xi^1=\tfrac{1}{2}r(\xi^C{}_{,C}-g_{01}g^{1A}\xi^0{}_{,A}),\end{aligned} \tag{8.15}$$

and uniquely determines ξ^μ in terms of its values at Σ^+ given in Eq. (8.9a). The linkage through a sphere of radius r becomes

$$L_\xi(r)=\oint\left[\xi^1{}_{;1}+\frac{1}{r}(g_{01}g^{11}\xi^0-2\xi^1)\right]r^{-2}dS,$$

where

$$dS=(4\pi)^{-1}dWd\varphi$$

is the fractional element of solid angle. Using Eq. (8.15) and the divergence theorem on the sphere, we also have

$$L_\xi(r)=\oint\left[\tfrac{1}{2}r^2(r^{-1}g_{01}g^{1A})_{,1A}\xi^0+r^{-1}g_{01}g^{11}\xi^0+\begin{Bmatrix}1\\1\rho\end{Bmatrix}\xi^\rho\right]r^{-2}dS.$$

This expression is of an indeterminate form for $r=0$, but the solution of the initial-value problem given in Sec. VII provides the necessary conditions for an application of l'Hospital's rule to determine a well-defined limit as $r\to 0$. In this way we find

$$L_\xi(\Sigma^+)=\oint_{\Sigma^+}[(M-\tfrac{1}{4}g^{1A}{}_{,11A})\xi^0+(g_{AB}N^B-\tfrac{1}{4}g_{01,11A}-\tfrac{1}{2}g_{AB,1}g^{1B}{}_{,11})f^A]dS. \tag{8.16}$$

Equation (8.16) would truly represent the total linkage for the system were its calculation independent of the particular family of closed 2-spaces used in the limiting process. This independence of the family chosen would follow if the local flux F across the outgoing null hypersurface were finite at J^+. The following theorem establishes an even stronger property:

The local flux across an outgoing null hypersurface corresponding to any asymptotic symmetry linkage vanishes at J^+.

To prove this we first calculate the local flux F, defined in Eq. (8.13) in terms of a conformal Bondi frame chosen such that the outgoing null hypersurface is given by u=constant. In Sec. V, we remarked that a descriptor field satisfying the propagation law Eq. (5.8) describes an infinitesimal coordinate transformation which preserves the Bondi coordinate conditions when k_μ is the normal to a coordinate surface u=constant. This statement has an analogy in conformal space language. A descriptor field satisfying the propagation law Eq. (8.11) describes an infinitesimal joint conformal and coordinate transformation which preserves the Bondi conformal-coordinate conditions Eqs. (6.19) when k_μ is the normal to a coordinate surface u=constant. When k_μ meets this condition, as in this proof, we may re-express the propagation law as

$$\bar{\delta}g^{00}=\bar{\delta}g^{0A}=g_{AB}\bar{\delta}g^{AB}=0, \tag{8.17}$$

where

$$\bar{\delta}g^{\mu\nu}=\bar{g}^{\mu\nu}(x)-g^{\mu\nu}(x)=2\xi^{(\mu;\nu)}-(2/r)\xi^1g^{\mu\nu} \tag{8.18}$$

describes the functional change in the conformal space metric associated with an infinitesimal change in conformal Bondi frames. Taking k_μ as in Eq. (8.14a) and using Eqs. (8.17) and (8.18) then gives in terms of the conformal Bondi frame

$$F=g^{01}[\tfrac{1}{4}g_{AB,1}\bar{\delta}g^{AB}+(g_{01}\bar{\delta}g^{01})_{,1}]/r^2. \tag{8.19}$$

Using Eq. (7.2) and the fact that $\bar{\delta}$ and coordinate differentiation commute, Eq. (8.19) becomes

$$F=\tfrac{1}{4}g^{01}(g_{AB,1}\bar{\delta}g^{AB}-rg_{AB,1}\bar{\delta}g^{AB}{}_{,1})/r^2. \qquad (8.20)$$

This expression for F is of an indeterminate form at $r=0$. Since g^{AB} is functionally an invariant at J^+ for all conformal Bondi frames, the numerator in Eq. (8.20) vanishes at J^+. Thus, we may apply l'Hospital's rule, which gives

$$[F]_{J^+}=\{[g_{AB,11}\bar{\delta}g^{AB}-r(g_{AB,1}\bar{\delta}g^{AB}{}_{,1})_{,1}]/8r\}_{J^+}.$$

Again this expression is indeterminate but allows an application of l'Hospital's rule, which gives

$$[F]_{J^+}=-\tfrac{1}{8}[g_{AB,1}\bar{\delta}g^{AB}{}_{,11}]_{J^+}.$$

This expression vanishes because of Eq. (7.8), yielding the desired result

$$[F]_{J^+}=0. \qquad (8.21)$$

This result is a direct consequence of the outgoing radiation conditions incorporated into the boundary conditions on the gravitational field at J^+. In the Penrose formalism, the differential and topological requirements on J^+ contain these outgoing radiation conditions. Equation (7.8) is one important consequence. Earlier formalisms based upon asymptotic series expansions used Sommerfeld-type conditions or peeling conditions in an equivalent way. F measures a local incoming flux in the sense that a purely outgoing flux would not give any contribution across an outgoing null hypersurface. From these considerations, it should not be surprising to find that F is at least finite at J^+. What does result, Eq. (8.21), is a much stronger condition. It has the following consequence. Let P_a be the energy-momentum linkages associated with the translational descriptors Y_a. Then Eq. (8.21) implies the asymptotic conservation law

$$P_a(\Sigma)=P_a(\Sigma^+)+O(r^2), \qquad (8.22)$$

where r measures the inverse luminosity distance from Σ to its image Σ^+ on J^+. The absence of an $O(r)$ dependence in Eq. (8.22) indicates that incoming fluxes of gravitational energy-momentum vanish more rapidly asymptotically than incoming fluxes of, say, electromagnetic energy-momentum which would give an $O(r)$ contribution to Eq. (8.22). In fact, even a static Coulomb field leads to an e^2r term in Eq. (8.22).[8] The outgoing radiation conditions thus impose even stronger asymptotic constraints on incoming gravitational fields than might be expected.

Given a coordinate system on J^+, we may label the asymptotic symmetry descriptors by means of the effects of the corresponding infinitesimal transformations. We shall adopt the labeling prescribed by Sachs.[4] This is based upon a coordinate system in which the metric satisfies Eq. (8.6). The descriptors then have the form at J^+ given in Eqs. (8.7) and (8.9a). We have already specified the translational descriptors $\xi_a{}^\mu$ by $f^A=0$ and $\alpha=Y_a$. In addition, six descriptors $\xi_{[ab]}{}^\mu$ with $\alpha=0$ and $f^A=f_{[ab]}{}^A$ describe the 6-parameter group of conformal transformations of a sphere which is isomorphic to the orthochronous homogeneous Lorentz group. The bivector labels $[ab]$ are chosen to correspond to Lorentz rotations in planes with the customary relationship to the polar axes. Finally, those descriptors $\xi_{lm}{}^\mu$ with $f^A=0$ form the supertranslation subgroup. They are labeled by expanding α in terms of spherical harmonics Y_{lm}. The components with $l\leq 1$ represent translations. Sachs[4] has shown that both the translations and supertranslations form invariant subgroups of the BMS group. The factor group of the supertranslation group is the conformal group. Thus, to each BMS transformation there corresponds a Lorentz rotation. We label the symmetry linkages in terms of the labels for the corresponding descriptors. The translational descriptors give rise to the energy-momentum P_a; the conformal descriptors to the angular-momentum bivector $L_{[ab]}$; and the supertranslational descriptors to the supermomenta P_{lm}.

This labeling may be carried out in any coordinate system satisfying Eq. (8.6). The set of all such coordinate systems defines a set of isometric observers on J^+, all mutually connected by BMS transformations. Each observer labels the asymptotic symmetries by the same prescription but with respect to his own frame. Thus, an arbitrary observer associated with, say, the $\bar{x}$ frame defines descriptors $\bar{\eta}_Q{}^\alpha$ at J^+, which in his own frame of reference are given by

$$\bar{\eta}_Q{}^{\bar{\alpha}}(\bar{x})=\eta_Q{}^{\bar{\alpha}}(\bar{x}).$$

Here the $\eta_Q{}^\alpha$ are a set of invariant functions, Q is a collective index for all the asymptotic symmetry labels, and the bar over α indicates that the descriptor components are being given in the $\bar{x}$ coordinates. Let the $\bar{x}$ frame differ infinitesimally from another asymptotically isometric frame x, so that

$$[\bar{x}^\mu]_{J^+}=x^\mu+\eta_Q{}^\mu(x)\epsilon^Q, \qquad (8.23)$$

where ϵ^Q is a set of infinitesimal parameters. We then have

$$\bar{\eta}_Q{}^\alpha(x)=\left[\frac{\partial x^\alpha}{\partial\bar{x}^\beta}\bar{\eta}_Q{}^{\bar{\beta}}(\bar{x})\right]_{J^+}=\left[\frac{\partial x^\alpha}{\partial\bar{x}^\beta}\eta_Q{}^{\bar{\beta}}(\bar{x})\right]_{J^+}.$$

To first order in ϵ this implies that

$$\begin{aligned}\bar{\eta}_Q{}^\alpha&=\eta_Q{}^\alpha(x)+(\eta_P{}^\beta\eta_Q{}^\alpha{}_{,\beta}-\eta_Q{}^\beta\eta_P{}^\alpha{}_{,\beta})\epsilon^P\\&=\eta_Q{}^\alpha(x)+C_{PQ}{}^R\eta_R{}^\alpha(x)\epsilon^P,\end{aligned}$$

where $C_{PQ}{}^R$ are the structure constants of the BMS group which have been evaluated by Sachs.[4]

Since the propagation law is linear, the propagated descriptor fields $\bar{\xi}_Q{}^\mu$ and $\xi_Q{}^\mu$ are similarly related by

$$\bar{\xi}_Q{}^\mu(x)=\xi_Q{}^\mu(x)+C_{PQ}{}^R\xi_R{}^\mu(x)\epsilon^P.$$

The $\xi_Q{}^\mu$ transform as vectors because the propagation law is covariant and the $\eta_Q{}^\mu$ transform as vectors at J^+.

As a result, the linkage expression is a scalar and may be evaluated in any allowed coordinate system. Furthermore, the linkage expression is linear in the descriptor field. Consequently,

$$\bar{L}_Q(\Sigma)=L_Q(\Sigma)+C_{PQ}{}^R L_R(\Sigma)\epsilon^P, \quad (8.24)$$

where $\bar{L}_Q$ is the linkage associated with the $\bar{\eta}_Q$ asymptotic symmetry intrinsic to an observer at J^+ in the $\bar{x}$ frame of Eq. (8.23). Hence, under transformations from one preferred observer at J^+ to another asymptotically isometric observer at J^+ the linkages transform among themselves as a representation (the adjoint representation[18]) of the BMS group. In particular, the energy-momentum linkages P_a and $\bar{P}_a$ associated with two such observers at J^+ are related by

$$\bar{P}_a=L_a{}^b P_b, \quad (8.25)$$

where $L_a{}^b$ is the Lorentz matrix associated with the BMS transformation connecting the two observers. Equation (8.25) states that the energy-momentum linkages transform as a Lorentz free vector when interpreted by a set of isometric observers on J^+. On the other hand, from Eq. (8.24) and Sachs' table of the structure constants[5] it follows that the angular-momentum linkages do not have their usual Minkowski space transformation properties due to the supertranslation freedom. The angular momenta mix with all of the supermomenta under a general BMS transformation.

Putting Sachs' canonical form for the descriptors into Eq. (8.16) gives explicit expressions for the various total asymptotic symmetry linkages in terms of the conformal Bondi metric. The one nonvanishing component of the translational descriptors at J^+ satisfies

$$Y_{a:BC}=\tfrac{1}{2}q_{BC}Y_a{}^{:D}{}_D. \quad (8.26)$$

Consequently, by using the tracelessness of $g_{AB,1}$ and the divergence theorem on the sphere, the total energy-momentum linking a sphere $\Sigma^+(u)$ on J^+ given by u=constant becomes

$$P_a(u)=\oint MY_a dS. \quad (8.27)$$

$M(u,x^A)$ has been called the mass aspect of the system because it is that part of the metric which determines the total energy.[1] The total supermomenta with $l>1$ are given by the slightly more complicated expression

$$P_{lm}(u)=\oint (M-\tfrac{1}{4}g^{1A}{}_{,11A})Y_{lm}dS.$$

The total angular-momentum bivector is given by

$$L_{[ab]}=\oint [\tfrac{1}{2}(M-\tfrac{1}{4}g^{1A}{}_{,11A})f_{[ab]}{}^A{}_{:A}u + (g_{AB}N^B-\tfrac{1}{4}g_{01,11A}-\tfrac{1}{2}g_{AB,1}g^{1B}{}_{,11})f_{[ab]}{}^A]dS.$$

[18] J. A. Schouten, *Ricci-Calculus* (Springer-Verlag, Berlin, 1954), p. 191.

For the spatial components $f_{[ab]}{}^A{}_{:A}$ vanishes, so that N^B plays the major role and may be appropriately called the angular-momentum aspect of the system.

The difference between the total linkages through two neighboring u=constant spheres on J^+ is found by differentiating Eq. (8.16) and inserting the time dependence of the mass and angular-momentum aspects as given in Eqs. (7.21) and (7.22). The resulting equation for the rate of radiation of energy and momentum out of the system is

$$\frac{dP_a(u)}{du}=-\oint |\dot{c}|^2 Y_a dS, \quad (8.28)$$

where Eq. (8.26) has been used to eliminate the additional divergence term. This immediately leads us to the important conclusion that the total energy of the system must *decrease* in the presence of outgoing radiation across J^+.[1]

IX. DISCUSSION

Although the most important results of this paper are of an invariant nature, the use of Bondi coordinates has played a major role both as a calculational aid and in supplying an intuitive picture of the properties of conformal space-time. To some extent the construction of a Bondi coordinate system is geometrical. Given a family of null hypersurfaces, Eqs. (2.2) for the ray labels and (2.6) for the luminosity distance take on a covariant form. While this is only as geometrically significant as the concept of a family of null hypersurfaces, it does lead to an intrinsic formulation of null-hypersurface data. Furthermore, Bondi coordinates are the natural coordinates for writing out the null hypersurface propagation law. On the other hand, the introduction of a luminosity distance in Eq. (2.6) in terms of derivatives of the ray labels destroys the differentiable structure of the manifold. As a result, it is awkward to describe the exact differentiable structure of J^+ in terms of a conformal Bondi frame.

The asymptotic symmetry linkages offer an invariant means of labeling the properties of finite regions of space. The choice of linkage expression and propagation law are fixed quite strongly by their desired transformation properties and by the insistence that they be meaningful for a single null hypersurface. This leaves only the freedom of extending the differential order incorporated in their construction. Any such modification must leave unaltered the result for the total energy momentum.

The application of these ideas to global questions will ultimately involve problems of outstanding current importance. To apply the linkages to the structure and dynamics of localized gravitational sources, a better understanding of the topology of null hypersurfaces in strong-field regions is necessary. The dual possibility of using the asymptotic symmetries of J^+ and J^- is related to the S-matrix problem for general relativity.

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APPENDIX A: BIANCHI IDENTITIES

Here we demonstrate how Bianchi identities group the field equations into the four sets, (4.4)–(4.7). We are given that the coordinate conditions,

$$g^{00}=g^{0A}=\det|g_{AB}|-r^4f(x^A)^2=0, \tag{A1}$$

and the field equations,

$$G_{1\mu}=0, \tag{4.4}$$

$$G_{AC}-\tfrac{1}{2}g_{AB}g^{CD}G_{CD}=0, \tag{4.5}$$

are satisfied globally. We write the Bianchi identities as

$$2[(-g)^{1/2}G_\alpha{}^\nu]_{,\nu}+(-g)^{1/2}G_{\mu\nu}g^{\mu\nu}{}_{,\alpha}=0. \tag{A2}$$

Starting with $\alpha=1$ in Eq. (A2), we obtain

$$G_{AB}g^{AB}{}_{,1}=0,$$

which is equivalent to

$$g^{AB}{}_{,1}(g_{AB}g^{CD}G_{CD})=0.$$

Since $g^{AB}{}_{,1}g_{AB}=-4/r\neq 0$, we conclude that

$$g^{CD}G_{CD}=0. \quad \text{Q.E.D.} \tag{4.7}$$

The remaining identities now have the form

$$[(-g)^{1/2}G_\alpha{}^\nu]_{,\nu}=0. \quad (\alpha=0,A). \tag{A3}$$

For $\alpha=A$, Eq. (A3) reduces to

$$g^{01}[r^2G_{0A}]_{,1}=0.$$

Requiring that $g^{01}\neq 0$ and $[G_{0A}]_\Gamma=0$ establishes that $G_{0A}=0$ globally Finally, for $\alpha=0$,

$$g^{01}[r^2G_{00}]_{,1}=0,$$

so that $[G_{00}]_\Gamma=0$ implies $G_{00}=0$ globally.

APPENDIX B: DYAD FORMALISM

In our discussion of the dynamical equations we found it convenient to introduce a dyad formalism. Here we discuss a particular form for t_A,

$$t_2=[\tfrac{1}{2}A(\sec\beta)\exp(z+i\omega]^{1/2},$$
$$-it_3=[\tfrac{1}{2}A(\sec\beta)\exp(-z+i\omega)]^{1/2}, \tag{B1}$$
$$z=\alpha+i\beta, \quad \bar{z}=\alpha-i\beta.$$

From Eq. (4.11), we obtain

$$g_{22}=Ae^\alpha\sec\beta, \quad g_{33}=Ae^{-\alpha}\sec\beta, \quad g_{23}=A\tan\beta, \quad \det|g_{AB}|=A^2. \tag{B2}$$

Coordinate condition (A1) determined the function

$$A^2=r^4f(x^A)^2.$$

From Eqs. (4.12), we have

$$t^2=-it_3/A, \quad t^3=it_2/A.$$

For reference, we list the following equations:

$$t^At^Bg_{AB,\mu}=22t^At_{A,\mu}=z_{,\mu}e^{i\omega}\sec\beta, \tag{B3}$$

$$2\bar{t}^At_{A,\mu}=(\ln r^2f)_{,\mu}+i[(\tan\beta)\alpha_{,\mu}+\omega_{,\mu}]. \tag{B4}$$

We partly restrict the arbitrary phase $\omega(x^\mu)$ by requiring that t^A undergo parallel transport along the null rays, i.e.,

$$t^A{}_{;1}=0. \tag{B5}$$

Using Eqs. (B3) and (B4), one readily finds that (B5) is equivalent to

$$\bar{t}^At_{A,1}=1/r, \tag{B6}$$

and

$$\omega_{,1}+(\tan\beta)\alpha_{,1}=0. \tag{B7}$$

Hence, ω is determined from α and β up to a function of integration ω_Γ. Equation (B6) reduces the dynamical equation (4.14) to

$$\dot{c}_{,1}=J, \tag{4.14a}$$

where

$$2\dot{c}=rg_{AB,0}t^At^B=r(\sec\beta)e^{i\omega}z_{,0}, \tag{4.15}$$

and

$$\dot{c}=[\dot{c}]_\Gamma+\int_\eta^r Jdr'. \tag{4.16a}$$