UNIPOLAR INDUCTION.[1]

By W. F. G. Swann.

Synopsis.

Unipolar Induction; Theory of the Uniform Motion of a System of Amperian Current Whirls.—The paper forms an inquiry as to the extent to which the so-called "Moving Line Theory" is the equivalent of the Maxwell-Lorentz theory. The latter gives the electric force on a fixed unit of charge in the form:

$$E = -\frac{1}{c}\frac{\partial U}{\partial t} - \text{grad}\,\psi,$$

where U is the Maxwellian vector-potential, and ψ is the electrostatic potential.

At first sight it might appear that ψ would be zero in the case of a *magnet which was uncharged when at rest.* It is shown, however, that the mere fact of the absence of resultant force upon a charge which accompanies a magnet in uniform rectilinear motion requires a rearrangement of electric density in the amperian whirls (or their equivalents) which constitute the molecular magnets; *and, this rearrangement is such as to endow the magnetic doublets with the properties of electric doublets as well,* the axes of corresponding magnetic and electric doublets being perpendicular to each other. These electric doublets result in a *finite value for ψ. It appears that, when ψ is included in this way, the Maxwell-Lorentz theory is the equivalent of the "Moving-Line Theory"* for the case of a single magnetic doublet in uniform motion.

Application to Rotating Magnetic Systems, and Relation to the "Moving Line Theory"; the Part Played by the Motional Intensity.—The above considerations may be extended to the case of a magnetic doublet (amperian current whirl) revolving about an axis parallel to its own and outside itself. In such application *we must, however, calculate the motion of the lines of magnetic induction as though they partook only of the translatory motion of the doublet in its orbit,* and not of its rotation about its own center. In the form in which the theory is usually applied to problems of rotation, the motion attributed to the magnetic lines is the same as if the lines from the elementary whirls were pinned down at the places where they pass through the axis of rotation of the system. In other words, the motion of the lines is calculated as though it partook of the rotational velocity of the *frame* of the whirl about its own center as well as of the translatory velocity of the whirl in its orbit. Bearing these considerations in mind, we may extend the idea to a system of amperian whirls, such as a magnet rotating in bulk.

If, in the case of a *conducting magnet,* we for the moment ignore the motional intensity which would act upon a unit of charge in the conductor as a result of its motion in its own magnetic field, the field due to the electric doublets would result in a distribution of electricity throughout the conductor of such a nature as to annul the electric field at all internal and external points. The *motional intensity* then superposes upon this condition an *electrostatic distribution* which is thus the *sole representative of the ultimate effect.* In general, this electrostatic distribution results in a potential other than zero at the axis of rotation, so that *if this axis is*

[1] Presented at the meeting of the American Physical Society, October 11, 1919.

earthed, an additional charge will come to the system, and this charge will also contribute to the electric field. In the "*Moving Line Theory*" *as ordinarily applied,* the electric intensity acting upon a unit of charge at rest turns out to be equal and opposite to the motional intensity acting upon the unit of charge as a result of its motion in company with the material of the rotating system, and it therefore results that, *on this theory, there is no electrostatic distribution in the rotating body.*

Discussion of Experimental Results Heretofore Obtained.—Recent experiments upon the subject of Unipolar Induction are discussed in terms of the conclusions arrived at in the paper. One of the main conclusions reached is to the effect that, in *all experiments where a null effect was obtained, the nature of the apparatus was such as to shield off any effect which might legitimately have been expected.*

Calculation of Effects for a Rotating Sphere.—The electrical field of a uniformly rotating, and uniformly magnetized sphere is calculated and discussed, under various conditions, *axis insulated, axis earthed, inductively magnetized, and permanently magnetized.*

Magnetic Doublet Rotating about an Axis Passing through its Center, and Perpendicular to its Own Axis.—It is shown that the electric field may be calculated for this case from the "Moving Line" law provided that the doublet is split into *two imaginary poles* and the lines for each are treated separately. It is necessary, however, to make the calculation on the basis that the *lines from the poles partake only of the linear velocity of the poles in their orbits,* and not of the rotational velocities about the centers of rotation. *In this form, the "Moving Line" law is applicable to a magnetic doublet moving in any arbitrary manner* so long as the velocities are not comparable with that of light.

INTRODUCTION.

CONSIDERING any magnetic system which, when at rest, is uncharged, the question primarily involved in the so-called problem of unipolar induction is this: Is it possible to calculate the electric field produced at an external point by uniform rotation, or for that matter by any kind of uniform motion, by supposing the lines which represent the magnetic induction B to be rigidly attached to the system, and by utilizing the expression

$$E = -\frac{1}{c}[V \cdot B] \tag{1}$$

where V is the velocity of the line perpendicular to its length, and c is the velocity of light.

Such a law is by itself *sufficient* to predict Faraday's law that the electromotive force integrated around a circuit shall be equal to the rate of change of induction through it; but, without further specification of the laws of electromagnetism, its *necessity* is not obvious. Most practical problems are concerned only with the electromotive force integrated around a circuit; and, for these, the relation (1) would lead to the correct result even though it might be erroneous as applied to obtain the field at a single point.

The class of experiments which have usually been associated with attempts to decide the validity of (1) are those in which a symmetrical system, such as a cylindrical bar magnet, is caused to rotate about its axis of magnetization, and devices are employed for the purpose of detecting any electrical field which may be generated in the vicinity of the magnet. Some of the most recent work on this matter has been done by E. H. Kennard,[1] S. J. Barnett,[2] and G. B. Pegram.[3] The symmetry of the problem is such that there is no rate of change of magnetic flux through any circuit, so that, even if the electric intensity were zero everywhere, Faraday's law would not be violated in this case.

A perusal of the literature on this subject suggests that many physicists have a feeling that there is a fundamental element of uncertainty as to what should happen in experiments of this kind; that there is, in fact, a question to which our electromagnetic scheme has no answer. This attitude may perhaps arise from a feeling that since Faraday's laws only expresses the field integrated round a circuit, the electromagnetic equations may not possess the power to dissect out the integral into its component elements. Such would indeed be the case were we entirely dependent upon Faraday's law; but, the electromagnetic theory involves another set of circuital relations besides Faraday's law, and these, combined with Faraday's law, enable the equations to be integrated so that the value of E is determined at each point. This immediately suggests the power of these equations to answer any problem as to the value of E at a point, provided that the problem is stated with sufficient definiteness. It is of interest to inquire therefore as to the extent to which the recognized electromagnetic theory has the power to provide an answer to questions of this kind.[4] It will conduce to clarity to first formulate the general bases which the electromagnetic theory provides for the discussion of the problem. It is desirable to do this in some detail since such confusion as arises comes often from a misinterpretation of the exact meaning of the symbols and definitions occurring in the theory.

[1] E. H. Kennard, Phil. Mag., 23, 937, 1912: 33, 179, 1917.

[2] S. J. Barnett, Phys. Rev., 35, 323, 1912.

[3] G. B. Pegram, Phys. Rev., 10, 595, 1917.

[4] I discussed certain aspects of this matter at the Washington meeting of the American Physical Society on April 21, 1916, in connection with a paper by Prof. G. F. Hull, on "The E.M.F. Generated by the Rotation of a Cylindrical Magnet about its Axis." Subsequently, at the meeting of the Physical Society in October, 1916, and in Phys. Rev., 10, 595–600 1917, Prof. G. B. Pegram expressed, quite independently, conclusions almost identical with those previously expressed by myself. A similar idea is also expressed on page 1,160 of a "Report on Electromagnetic Induction" by Prof. S. J. Barnett, presented at the joint meeting of the American Physical Society and of the Institute of Electrical Engineers, October 10, 1919. I mention this not of course as a matter of priority, but because, in what follows, I shall find it necessary to disagree to some extent as to the completeness of the conclusions drawn by Professors Pegram and Barnett, and incidentally therefore with those I had formerly expressed; for, although they represent an important part of the matter, they form by no means the whole story.

GENERAL FUNDAMENTALS CONCERNED.

With electric and magnetic quantities in Heavisidean electrostatic and electromagnetic units respectively, the two sets of circuital relations are, in the case of free æther:

$$\frac{1}{c}\left(\rho u + \frac{\partial E}{\partial t}\right) = \text{Curl } H \tag{2}$$

$$-\frac{1}{c}\frac{\partial H}{\partial t} = \text{Curl } E \tag{3}$$

and, in addition, we have:

$$\rho = \text{Div. } E. \tag{4}$$

An additional relation Div. $H = 0$ is simply added to exclude, from the problems we are interested in considering, all those in which the magnetic field arises from ordinary scalar potentials, *i.e.*, from distributions of hypothetical magnetic "charges."

H is the force on a fixed Heavisidean unit magnetic pole, and E the force on a fixed Heavisidean electrostatic unit of charge. It is important to observe that though E sometimes arises totally from electrostatic considerations and sometimes from the existence of changing magnetic fields as well, the relation (4) is universally true and is, in fact, the definition of ρ which, without some such definition, would have no meaning in the case of moving charges.

There are two forms in which (2) and (3) are customarily integrated. In the first form, which corresponds to that adopted by Maxwell, we have:

$$E = -\frac{1}{c}\frac{\partial U}{\partial t} - \text{Grad } \psi, \tag{5}$$

where

$$\psi = \frac{1}{4\pi}\int\int\int\frac{\rho}{r}d\tau \tag{6}$$

and

$$U = \frac{1}{4\pi c}\int\int\int\frac{(\rho u + \dot{E})}{r}d\tau. \tag{7}$$

Here ψ is the true electrostatic potential of the charges. It is the potential which the various volume elements of charge would exert at the time in question if, at that instant, they were suddenly brought to rest and pinned down in the positions which they happened to occupy. The vector potential U is not expressed entirely in terms of the motions of the charges, but involves also an integration of $\partial E/\partial t$ throughout space.

Div. U is zero at all points so that the density ρ, which is equal to Div. E is also minus the divergence of Grad ψ.

In the second or Lorentzian form, we have:

$$E = -\frac{1}{c}\frac{\partial A}{\partial t} - \text{Grad } \varphi, \tag{8}$$

where

$$\varphi = \frac{1}{4\pi}\int\int\int\frac{[\rho]}{r}d\tau \tag{9}$$

and

$$A = \frac{1}{4\pi c}\int\int\int\frac{[\rho u]}{r}d\tau. \tag{10}$$

Here, however, the square brackets are to be taken as indicating that, in seeking the value of φ or A at the time t, we must not put in the values of ρ and u in the various volume elements at the time t, but to each volume element we must assign the value which existed there at the time r/c earlier than t. Thus φ is no longer the electrostatic potential in the sense that ψ is. Div. A is not in general zero, so that the density is not given in general by Div. Grad. φ. The two formulas (5) and (8) are merely different ways of expressing the same quantity E. The Lorentzian form has in some cases the advantage in that it gives A and φ, and consequently E, explicitly in terms of the positions and velocities of the charges without invoking the knowledge of E at all points, as a direct determination of U would require. In both cases H is of course the curl of the corresponding vector potential.

E is the force on a unit charge fixed with respect to the observer; and, it is not always realized that neither the circuital relations (2) and (3), nor their analytical equivalents (5) or (8) have the power to tell us anything at all about the force on a charge which moves relatively to the observer. The force on a moving charge is $E + [V \cdot H]/c$, where the vectors refer to the parts not contributed by the charge itself;[1] but, in giving the name "Force" to this vector, we do not endow it with any properties which tell us in what way this so-called force influences the motions of the charges. The assumption which is made by Lorentz is that the electron moves in such a way that the force (as defined above) which it exerts on itself as a result of the motion is equal and opposite to the force which the external field exerts upon it. The expression for the former can be developed in terms of the acceleration and higher derivatives of the velocity of the electron. It turns out that the term proportional to the acceleration is the most important of all the terms

[1] The division of a field into two parts, that due to a certain charge, and that due to the remainder of the charges is a matter of definition. The field of a given charge is *defined* as the portion obtained by utilizing the parts of φ and A contributed to the integrals by the charge in question.

in the case of many types of motion; and, by giving the name "Mass" to the coefficient of the acceleration, the assumption of Lorentz becomes thrown into the familiar form "Force equals Mass multiplied by Acceleration." It is on account of the act of *forcing* the Lorentzian assumption into this form that we arrive at such apparently artificial concepts as "Longitudinal" and "Transverse" mass.

The assumption of Lorentz is not derivable from the circuital relations. Such theoretical basis as it has rests upon the fact that, neglecting certain objections which it is not the purpose of this paper to discuss, it becomes predicted by the same application of the Hamiltonian principle to the problem of the electromagnetic field as, with the proper choice of the Lagrangian function, brings out the Faraday law. In the case of a single coördinate, but only in this case, it can be deduced from the principle of the conservation of energy as might be expected since, for the case of a single coördinate, the Hamiltonian Principle is the analytical equivalent of the principle of the conservation of energy.

It is convenient to speak of the contribution $[V \cdot H]/c$ to the force acting upon an element of charge as the "Motional Intensity," so that the total force F acting upon the unity charge is the sum of the electric intensity E which would act upon the charge at rest, and the motional intensity. Thus, using (5) we have:

$$F = -\frac{1}{c}\frac{\partial U}{\partial t} - \text{Grad. } \psi + \frac{1}{c}[V \cdot H], \tag{11}$$

where U and ψ have the meanings defined in (6) and (7). In place of U and ψ we may of course use the Lorentzian potentials if we choose.

It must be distinctly understood that there is no ambiguity of meaning depending upon considerations as to whether the magnetic lines do or do not move. H is the force which would be experienced by a fixed magnetic pole; V is the velocity of the charge upon which we desire the force, and is measured relatively to the fixed observer. The u which occurs in the expression (7) for U refers, at each point of space, to the velocity of the electricity at that point. Further, in evaluating U, ψ and H, we must omit from our integrals the parts contributed by the charge whose motion we desire to study.

Motion of Molecular Magnet or Amperian Current Whirl.

It is fairly obvious, and may be seen rigorously by a glance at (5), (6) and (7), that the electric intensity at any point may be expressed as a sum of contributions from each element of the space. It is therefore of interest to begin by studying the effect of an infinitesimal circular amperian whirl (taking place around an infinitesimal anchor ring for example), and then discuss the field which will arise by the superposition of an infinite number of such whirls which together may be supposed to make up a magnetized body. To fix our ideas we shall suppose the whirl to be one of negative electricity; and, in order to avoid as far as the analysis will permit us, the complications arising from the portions of the field due to mere electrostatic distributions of electricity, we shall suppose that, on the top of our anchor-ring whirl, we superpose a uniform distribution of positive electricity which does not participate in the rotary motion of the whirl, but which is equal in total amount to the negative charge in the whirl.

The question which we wish to decide is whether, or under what conditions, the electric field due to the motion of such a whirl, and which is strictly given by (5) for example, may also be calculated, at any rate to a first approximation, from the moving magnetic line idea on the assumption that the lines are rigidly attached to the frame of the whirl. If such a method of calculation is possible then, by superposition of the effects, we may apply the moving line method to the motion of a magnet in bulk. In doing this, however, we must be careful to apply it to the motion of the field of each amperian whirl separately, and this will mean that in many cases, as for example in estimating the electric field at a point on the plane passing through the center of a rotating magnet and perpendicular to the axis of rotation, the lines contributed by the various elements of the rotating magnet will be moving in all sorts of directions, some indeed moving in diametrically opposite directions, for the portions of the surface of the magnet at the two ends of a diameter move in opposite directions. It is only in some such form as this that the "Moving Line Theory" could ever hope to represent the true state of affairs. Any other view would make the theory inconsistent with itself; for, if it is to apply to the magnet in bulk, it must manifestly apply to the constituent elements of which the magnet is composed.

As an introduction to our problem, we shall first consider a case of a whirl in uniform rectilinear motion.

Amperian Whirl in Uniform Rectilinear Motion.—Suppose that the axis of the amperian whirl is inclined to the axis of x, and that the rectilinear motion takes place parallel to the axis of x with velocity v. The field E is given by (5), and it might be thought at first sight that we should be justified in neglecting the term Grad. ψ, involving the purely electrostatic potential, on the ground that we superposed on our negative whirl a positive distribution which, as regards electrostatic potential, just cancelled the negative distribution when the whirl was without rectilinear velocity. Such an assumption would be unjustified, however, for it would lead us among other things to the relation:

$$E_x = -\frac{1}{c}\frac{\partial U_x}{\partial t},$$

so that, unless U_x were zero, there would be a component of E in the direction of motion, a conclusion which no one would be prepared to admit, and which would violate the principle of relativity to an extent measured by its whole value. For, the only reason that a moving observer fails to realize that the field which he measures is changed by the motion is that the term $[V \cdot H]/c$ in equation (11) just cancels the change in E produced by the motion of the system, and, the term $[V \cdot H]/c$ has obviously no component parallel to the direction of motion. That the vanishing of E_x does not depend upon the vanishing of U_x may readily be seen by putting E_x zero in (7), which then leaves us with a perfectly definite and finite expression for U_x.

In fact, if we are to have E_x zero, as relativity considerations require, we must admit the existence of a value of ψ determined by:

$$-\frac{1}{c}\frac{\partial U_x}{\partial t} - \frac{\partial \psi}{\partial x} = 0$$

or since

$$\frac{\partial}{\partial t} = -v\frac{\partial}{\partial x},$$

$$\frac{v}{c}\frac{\partial U_x}{\partial x} = \frac{\partial \psi}{\partial x}.$$

The arbitrary function of y and z which arises on integration of this equation must be zero as otherwise U_x or ψ would be finite for infinite values of x. Thus

$$\psi = \frac{v}{c}U_x \tag{12}$$

or, remembering that E_x is zero, and using (7) we have:

$$\psi = \frac{v}{4\pi c^2}\iiint \frac{(\rho_0 v - \rho_1 u_x)}{r}\,d\tau, \tag{13}$$

where, in terms of measurements by a fixed observer, ρ_0 is the density of the positive electricity, ρ_1 the density of the negative distribution in the moving system, and u_x the x component velocity of the negative electricity.

If i_x represents the x component of the current density as measured by a fixed observer, equation (13) shows that the rectilinear velocity has resulted in the creation of a resultant charge density given, at each point of the anchor-ring, by:

$$\Delta\rho = \frac{v\dot{i}_x}{c^2}. \tag{14}$$

Utilizing in (5) the value of ψ given by (12), and remembering that H = Curl U, we have:

$$E_y = -\frac{1}{c}\frac{\partial U_y}{\partial t} - \frac{v}{c}\frac{\partial U_x}{\partial y} = \frac{v}{c}\left(\frac{\partial U_y}{\partial x} - \frac{\partial U_x}{\partial y}\right),$$

$$E_z = -\frac{1}{c}\frac{\partial U_z}{\partial t} - \frac{v}{c}\frac{\partial U_x}{\partial z} = \frac{v}{c}\left(\frac{\partial U_z}{\partial x} - \frac{\partial U_x}{\partial z}\right),$$

so that:

$$E_x = 0, \qquad E_y = \frac{v}{c}H_z, \qquad E_z = -\frac{v}{c}H_y \tag{15}$$

or, in vector notation,

$$E = -\frac{[v\cdot H]}{c}. \tag{16}$$

Equation (16) is obviously the expression of the "Moving Line Theory" for the case of uniform rectilinear motion. In a sense, it is in more exact accord with the "Moving Line Theory" than are the usual statements of that theory; for, in (16) H represents the true value of the magnetic field as modified by the motion, and the result is true to any order of magnitude. As will be seen, moreover, from the analysis, the result is not limited to the particular amperian-whirl system which we have chosen as the basis of our discussion, but is true for example where the amperian whirls are replaced by electrons revolving about positive centers. The amperian whirl is however, the easiest system in terms of which to visualize the phenomena, and we shall continue to discuss matters in terms of it.

As may be seen from (14), ψ arises from a crowding together of the negative electricity on the side of the whirl on which the x component of the rotational velocity coincides with the rectilinear velocity of the whirl, and a thinning out of the negative density on the other side. In fact, the electrostatic potential corresponds, at large distances, to that of an electric doublet. The rearrangement of the density of the negative electricity is not cancelled by a similar rearrangement of the positive because, as shown by (14), this arrangement depends upon the velocity relative to the moving center as well as upon the velocity of the moving center itself.

A physical picture of the origin of ψ may readily be obtained. For suppose that our elementary whirl was originally at rest, as a whole, and without charge, and that it is now in rectilinear motion in a direction perpendicular to the plane of the paper, with its axis parallel to the plane of the paper, Fig. 1. Remembering that the magnetic lines pass upwards, for example, as they thread through the whirl and downwards as they return on the outside, everyone will admit that, at the surface of the anchor ring there will appear an electric field which will be in opposite directions at points A and B on the interior and exterior surface of the anchor ring. This of course means that there is a resultant electric flux from, and consequently a charge on, the part AB of the anchor ring. It is easy to see from the directions of the magnetic and electric forces that the sign of the charge at the place CD is opposite to that of the charge at AB, so that these two places contribute to the formation of an electric doublet, equilibrium being secured by the motional intensity.

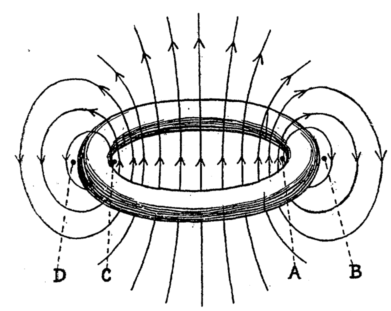

Fig. 1.

It thus appears that, even in the case of a magnetic doublet which is electrically neutral when at rest, the electric field resulting from the motion of the doublet depends partly upon an electrostatic potential, and is not determined entirely by the time rate of change of the vector potential. Further, by superposition of the contributions from the various elementary molecular magnets we shall find that this effect persists in the case of the rotation of a magnet in bulk. It is true, as we shall see later, that, in the case of a *conducting* magnet, the electrostatic potential causes a redistribution of the free electricity in the magnet, of such a nature as to annul its effect. The secondary electrostatic potential arising in this way is, however, of an origin quite different from that of the primary potential due to the electric doublets, as is also the potential of the electrostatic distribution which results from the motional intensity produced by the rotation of the conductor in its own magnetic field.

In the case of a symmetrical rotating magnetized system, the vector-potential is independent of the time; but, as we have seen above, this is not a sufficient criterion for the prediction of absence of electric field as a result of rotation. It is in this respect that I find it necessary to differ from the conclusions expressed by Professors Pegram and Barnett,[1] and incidentally therefore from those which I myself had expressed at an earlier date.

It may be remarked that the results which we have deduced as to the creation of an electric doublet in an amperian whirl as a result of imparting rectilinear motion to the whirl may be deduced by a direct application of the transformations of the theory of relativity, as may also the equivalence of the Lorentzian theory and the "Moving Line Theory" for this case.

Thus, if ρ' is the density at a point in any system, as measured by a fixed observer and the system be set in rectilinear motion with velocity v, ρ' will be the density as measured by an observer participating in the motion. But the density ρ as measured by a fixed observer will now be given by[2]:

$$\rho' = \beta\left(1 - \frac{vu_x}{c^2}\right)\rho$$

where u_x is the x component velocity of an element of the charge as measured by the fixed observer, and β is written for $(1 - v^2/c^2)^{-1/2}$. If ρ_0' and ρ_1' refer respectively to the densities of positive and negative electricity in our amperian whirl, for example, when without rectilinear velocity, while ρ_0 and ρ_1 refer to the corresponding quantities as measured by a fixed observer when the system has rectilinear velocity v, we have:

$$\rho_0' = \rho_1', \qquad \rho_0' = \frac{\rho_0}{\beta}, \qquad \rho_1' = \beta\left(1 - \frac{vu_x}{c^2}\right)\rho_1,$$

so that:

$$\Delta\rho = \rho_0 - \rho_1 = -\frac{vu_x}{c^2}\rho_1 + \frac{v^2}{c^2}\rho_0 = \frac{vi_x}{c^2},$$

which is the same result as that obtained in (14).

It must not be thought that we have *proved*, from the purely electromagnetic considerations contained in equations (5) to (7), that there is a redistribution of density in our amperian whirl as a result of its rectilinear motion. The theory of relativity is not deducible from electromagnetic considerations alone. What has been shown is that, if the theory of relativity is to hold, the density distribution must come about in the manner we have stated. When, in deducing (14), (15), and (16), we assumed that E_x was zero, we of course made an assumption which, for this purpose, was the equivalent of the theory of relativity. If we do not assume something such as the principle of relativity in addition to

[1] See note 5 of this paper.

[2] See for example, O. W. Richardson's Electron Theory of Matter, p. 307.

our electromagnetic scheme, we have no basis for forming a conclusion as to what happens to our whirl as a result of rectilinear motion. If we were to *assume* the negative and positive densities in our whirl to be unaltered at each point, so that one offset the other, as regards electrostatic field, we should not arrive at the "Moving Line Result." We should, moreover, predict the very astonishing conclusion that rectilinear motion results in the appearance of a component of E parallel to the line of motion, a result quite inconsistent with the theory of relativity.

In addition to the result (14), we may of course deduce also the results (15) by a direct application of the relativity transformations. For if symbols with and without dashes correspond respectively to quantities measured by a moving and by a fixed observer, the relativity transformations are:[8]

$$E_x' = E_x, \qquad E_y' = \beta\left(E_y - \frac{v}{c}H_z\right), \qquad E_z' = \beta\left(E_z + \frac{v}{c}H_y\right)$$

The symbols with dashes correspond also to the quantities which would have been measured by a fixed observer before the system was set into rectilinear velocity. The transformation equations thus tell us that if a system is devoid of electric field when without rectilinear motion, the impartation of uniform rectilinear motion will result in the appearance of an electric field given by:

$$E_x = 0, \qquad E_y = \frac{v}{c}H_z, \qquad E_z = -\frac{v}{c}H_y,$$

which are the same as equations (15).

This method is, of course, more general than that adopted above, and shows at once that the results are not limited to the particular case of an amperian whirl. The purpose of this paper is, however, to dissect out the physical processes at work in as clear a manner as is consistent with a rigorous treatment; and, in this respect, the more laborious development we have given may perhaps be more convincing than a direct appeal to the relativity transformations.

Moment of the Electric Doublet.—It has been shown that the part of the electric field of the moving amperian whirl determined by ψ is the same as would be produced by an electric doublet. To the extent that the magnetic field of the moving whirl is unaltered by the motion, *i.e.*, to a high approximation, the whirl is equivalent in its magnetic effect to a magnetic doublet; and, it is of interest to determine the relation between the moments M and N of the magnetic and electric doublets respectively.

For the value of U_x due to a magnetic doublet of moment M, and directional cosines l, m, n, we have[1]

$$U_x = -M\left(m\frac{\partial}{\partial z} - n\frac{\partial}{\partial y}\right)\cdot\frac{1}{r}$$

so that, from (12), we have:

$$\psi = -\frac{v}{c}M\left(m\frac{\partial}{\partial z} - n\frac{\partial}{\partial y}\right)\frac{1}{r} = \left(-\frac{v}{c}M_y\frac{\partial}{\partial z} + \frac{v}{c}M_z\frac{\partial}{\partial y}\right)\cdot\frac{1}{r},$$

which shows that ψ is the same as would result from an electric doublet of moment given by:

$$N_z = -\frac{v}{c}M_y, \qquad N_y = \frac{v}{c}M_z.$$

Thus, employing vector notation, we may write, quite generally:

$$N = -\frac{[v\cdot M]}{c}. \tag{17}$$

Rotation of American Whirl about its own Axis.—Rotation of an amperian whirl about its own axis of rotation can not give rise to any electric field; for, such rotation is simply equivalent to adding (algebraically), equal velocities to the positive and negative electricity in the whirl, so that the current density is unchanged. Neither on the basis of physical intuition, nor on the more exact basis of equations (5) to (7), should we be justified in expecting any electric field as a result of such a phenomenon. For, unless rotation of the whirl gives rise to a change in its constitution, such as to result in a finite value for ψ, the most that (5) could predict is:

$$E = -\frac{1}{c}\frac{\partial U}{\partial t},$$

which shows that E is zero in the steady state. Change of constitution of the whirl as a result of mere rotation, unlike that following from rectilinear motion, is not called for by any known fundamental considerations, and is, moreover, highly improbable from considerations of symmetry.

The above arguments may be applied, even with greater ease, to the case of the rotation of current-carrying solenoids, such as have occasionally been used in experiments on unipolar induction. No electrical effects are to be expected as a result of the rotation of such a solenoid about its own axis.

[1] See, for example, J. H. Jeans' Electricity and Magnetism, p. 394.

Application to the Case of Symmetrical Magnetized Systems Rotating about the Axis of Symmetry.

We may look upon the field due to a piece of magnetized material as made up of the fields of the innumerable magnetic doublets, or amperian whirls, of which it is composed. We may confine ourselves to the case where the axes of the elementary whirls are parallel to the axis of the rotating system as a whole (*i.e.*, to the axis of magnetization), since effects due to the motions of the components of the whirls perpendicular to the axis of magnetization must necessarily cancel out, since such components will be as often directed in one way as in the opposite way. Any doubts as to the validity of this reasoning may be removed by a subsequent reconsideration of the effects of these components in terms of the ideas to be developed in the remainder of the paper.

By way of introduction, let us first consider the case of a single amperian whirl which revolves about some axis outside itself, but parallel to its own axis. In the first place, the acceleration of the whirl as a whole towards the center of its orbit may result in a change in its actual constitution of a kind analogous to that which we have shown to result from uniform rectilinear motion. For rotation, however, we have no theory of relativity which will enable us to form a prediction for this case. Indeed, it is unlikely that there could be any theory which would predict the effects of rotation independently of the knowledge of the atomic structure of the particular material concerned. To realize the force of this contention, we have only to remember that a disc in uniform rotation will, in general, alter its dimensions by centrifugal force to an extent depending upon its elastic properties, *i.e.*, to an extent depending upon the molecular and atomic structure of the particular substance we happen to be dealing with; while, for uniform rectilinear motion, the theory of relativity predicts a contraction which is the same for all substances. If, however, we make the assumption that the distribution of electricity density in the whirl is altered by the *orbital* motion only to an extent determined by the *linear* velocity of the whirl we can proceed. Such an assumption would probably be readily accepted by most people as very reasonable; we must remember, however, that it is ultimately an *assumption*, and only to the extent that it is true will the solution of our problem be possible in the form in which we shall attempt it. It will appear later, however, that, as regards the external field due to a symmetrical, rotating, *conducting* magnet, there is no ambiguity arising from uncertainties as to the effect of rotation on the constitution of the whirl.

Even with the assumption cited above, the field of the whirl at any point, at any instant, will not in general be the same as that which would be calculated on the assumption that the whirl had been moving for an infinite time in a straight line with the constant velocity equal and parallel to the velocity which it has in its orbit at the instant concerned. Provided, however, that the velocity is not comparable with that of light, the assumption that the field may be calculated from the pure rectilinear velocity will be approximately justified. The assumption is, in fact, analogous to the so-called Quasi-Stationary Principle so frequently used in discussing the fields of moving electrons; and it involves corrections of an order of magnitude smaller than those in which we are interested here. It merely neglects the radiation field which is sent out by our amperian whirl as a result of the acceleration which it suffers in its orbit.[1]

If then we make this justifiable assumption, the results of the last section show that the electric field which the Maxwell-Lorentz theory predicts as a result of the motion of the whirl is exactly the same as that which we should calculate from the moving line theory provided that, in applying that theory, we exclude the part of the motion of the lines of force which would result from the rotation of the *frame* of the amperian

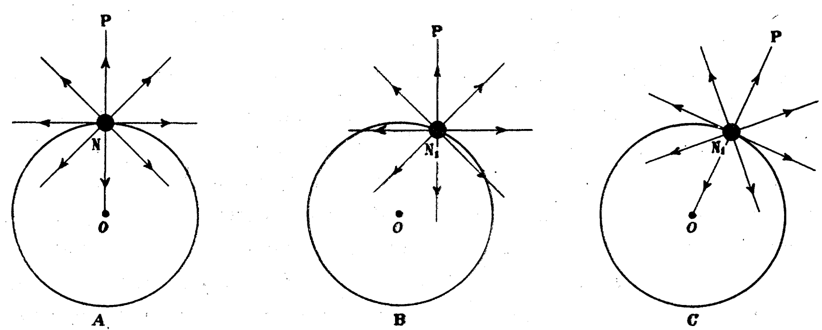

Fig. 2.

whirl about its *own* axis. In fact, in calculating the motion of the lines of force of our amperian whirl, with its axis parallel to the axis of rotation, we must not imagine the lines as attached rigidly to the frame of the whirl but rather, let us say, to a plane which passes through the center of the whirl and partakes of its linear velocity, but which remains parallel to itself during the rotation. Referring to Fig. 2, for example, where the axis of our whirl is perpendicular to the plane of the paper, where N represents the position of this axis, and O the axis of the *planetary* orbit

[1] In any case, when we consider the superposition of effects due to all the whirls in a symmetrical rotating body, it is easy to show that terms of this kind would balance out.

of the whirl, we see that, if the "Moving Line Theory" is to agree with the Maxwellian theory, then, on passing from the position N to the position N_1 we must calculate the velocity of a line of force NP as though it had assumed the orientation N_1P in diagram B, and not as though it had assumed the orientation N_1P in diagram C. A calculation on the latter basis would correspond to assuming the magnetic lines to partake of the rotational velocity of the frame of the whirl; and, it is in this respect that the "Moving Line Theory" as it is usually applied to rotational problems differs from the theory in the form in which it ought to be applied.

Thus, let us consider the case of a symmetrical system rotating with uniform angular velocity about its axis of magnetization; and, in order to avoid complications which will be presently discussed and elucidated, let us suppose that our system is non-conducting, and of unit specific inductive capacity. A convenient system to think of is two cylindrical bar magnets with axes in line, with opposite poles facing each other, and separated by a space. Since the effects of the individual amperian whirls are additive, the field due to each may be calculated on the basis of the "Moving Line Theory" as properly applied, and the resultant may then be obtained by summation. We cannot apply the "Moving Line Theory" to the resultant magnetic field, however, as is usually done when the line of force passing through the axis of rotation is supposed to be stationary, and the other lines of force are supposed to revolve around it with constant angular velocity. Such an application results in just what we should obtain were we to suppose that, in the case of each whirl, the particular line of force which happened to pass through the axis of rotation of the system was pinned down at the axis. In other words, the usual application of the theory corresponds to assuming the lines of force of the amperian whirls to partake of the *rotational* velocity of the *frame* of the whirl about its own axis, as well as of its rectilinear velocity in its orbit; and, as we have seen, such an assumption would be contrary to the Maxwell-Lorentzian requirements, and indeed to physical intuition as well.

Referring to the two cylindrical magnets with axes in line and poles facing, as cited above, we see that the ordinary application of the "Moving Line Theory" would result in predicting, for the space between the poles, an electric field radial to the axis of rotation. The divergence of this field would not be zero, and we should be driven to the impossible conclusion that there was a creation of charge density in the space between the magnets, a conclusion to which a proper application of the theory could not possibly lead, since the proper theory makes Div. E zero at all points except at the whirls themselves.

In a symmetrical uniformly rotating system, the vector potential does not change with the time, so the second member of the right-hand side of (5) is the only term which contributes to E. As we have seen in the preceding section, the effect of this term is the same as we should have if each amperian whirl were replaced by an electric doublet whose axis was perpendicular to the direction of motion. In the case under discussion, the axes of the doublets would be radial; and, if the system is uniformly magnetized, the moments of the doublets will be proportional to the distance from the axis of rotation.

By Poisson's mathematical theorem relating to distributions of doublets; the effect of such an assemblage of doublets is equivalent to that of a fictitious charge distributed throughout the volume of the magnets, and an equal fictitious charge distributed over the surface, the density of the fictitious volume charge being equal, at each point, to minus the divergence of the polarization (moment per unit volume of the doublets) and the density of the surface charge being equal, at each point, to the normal component of the polarization at that point. The field as determined by these fictitious charge distributions is thus the analytical equivalent of the field which would be obtained by a proper application of the "Moving Line Theory" to a symmetrical, insulating, and uniformly rotating magnet, of unit specific inductive capacity.

Suppose now we extend these ideas to the case of a conducting system. If for the moment we ignore the last term in (11), which corresponds to the motional intensity brought into play on the electrons of the conducting magnet as a result of its motion in its own magnetic field, we are left with the field due to the electric doublets referred to above, and it will be necessary for a redistribution of electricity to take place in the conducting magnet in such a way as to annul the field therein produced by the electric doublets. At a point within the substance, the field due to a distribution of electric doublets of this kind is composed of two parts. We have a part E_1, represented by the field which would be measured in an infinitesimal cavity whose length is parallel to the polarization and whose diameter is infinitesimal compared with its length. Then, we have a part E_2, which depends upon the local effect of the doublets in the vicinity of the point concerned. The part E_1 is the part which would be given by the fictitious charges: it is the part whose line integral over a path comprising an infinite number of molecules is equal, in the limit, to the actual difference in potential between the starting and ending points, provided that neither of these points is chosen at a distance from one of the individual doublets small compared with the average distance between the doublets. The part E_2 fluctuates in magnitude, and even in sign, from molecule to molecule. At any point, it is proportional to the polarization (moment per unit volume), the factor of proportionality being a function of the position of the point. Its average value along any line long compared with molecular dimensions is, however, zero. The field E_1 determined by the fictitious charges is consequently the only part of the total field which is operative in causing finite movements of the free electrons of the conducting material. In order to balance this part, it is necessary for a real separation of electricity to take place in the material in such a way that the real volume and surface densities just annul the fictitious volume and surface densities at each point of the material. Such a redistribution of electricity will, however, result in complete cancellation of the external field due to the fictitious charges, so that, as a result of these influences, there will be no field at any point. This idea may readily be extended to the field due to any type of re-arrangement of electricity in the amperian whirl; so that, any uncertainty as to the influence of orbital rotation in altering the constitution of the whirl is without effect on the ultimate conclusion as to the absence of any external field resulting from the combined actions of these circumstances.

If we now introduce the last term of (11), representing the motional intensity, it will result in an additional distribution of electricity, such as to just cancel its effect within the material. At a point within the material where the actual magnetic field is H, the motional intensity is given by $[v \cdot H]/c$. Now the actual value of H in the material varies rapidly in direction and magnitude in the vicinities of the amperian whirls. Nevertheless, the true H, unlike the ordinarily defined H as measured in a filamental cavity, is a solenoidal vector; and, since B, the induction, is a solenoidal vector which is equal to H outside the material, its average value normal to any element of surface of dimensions large compared with the dimensions of the amperian whirls, is the same as the average value of the normal component of the true H taken over that element of surface. Moreover, this is true in whatever way the element of surface may be made to bend about between the amperian whirls. The effective value of the motional intensity is thus $[v \cdot B]/c$; and, the electrostatic distribution which results will consequently be of a nature such as to produce an electric intensity—$[v \cdot B]/c$.

It is of interest to inquire the extent to which the above considerations must be modified in case the rotating material has specific inductive capacity greater than unity. At first sight we might argue that the state we have predicted above is one which results in there being no force on an element of electricity moving with the material and that, in consequence, there will be no *additional* polarization of the material as a result of its specific inductive capacity. Such an argument would not be justifiable, however; for, while the effect of the field E_2 representing the local effect of the electric doublets averages out when integrated over a random path in the material, it does not necessarily average out when applied at an electron in the atom itself; for, there is a definite relationship in position between the electrons of the atoms and the electric doublets which figure in the determination of E_2. Additional polarization of the medium might therefore result, even though the *average* field over any finite region of the material were zero. When everything is taken into account, however, the net result of all these considerations is this: Electric doublets of the type we have discussed are created by the motions of the amperian whirls, and motional intensity is produced as a result of the motion of the material in the magnetic field. Redistribution of charge and possibly additional polarization will take place. The result of the charge distribution, and of the two possible types of polarization (electric doublet separation) must be such as to produce within the material a resultant field equal and opposite to the motional intensity $[v \cdot B]/c$. The electric intensity within the material is thus determined partly by a real charge distribution, and partly by fictitious charge distribution; but, as regards the calculation of the fields at points outside the material, it matters not whether the charges are real, fictitious, or both. The net effect of all considerations is that the external electric field is determined by that system of volume and surface charge distributions which would give, within the material, a field—$[v \cdot B]/c$ at each point.

So far we have supposed that the rotating system is insulated, although the substance of which it is composed may be a conductor. In the case where the rotating system is not only conducting but is also kept at zero potential along its axis, a further consideration becomes involved. For the motional intensity results in an electrostatic distribution which does not of itself result in the axis of rotation being at zero potential. If the axis is to be kept at zero potential, it will be necessary for an additional charge to come to the rotating system. This additional charge will distribute itself in the same manner as it would distribute itself on a non-magnetic system of the same shape, and in a manner sensibly the same as on a similar system devoid of rotation. If we earth some point of the system other than the axis of rotation, the charge which will come to the system on this account will be such as to alter the potential of the whole system by an amount necessary to bring the point which we have earthed to zero potential.

It is important to observe that, in line with the above developments, such electric fields as are produced in the vicinity of rotating magnetic systems result entirely from electrostatic distributions, so that *there would be no electric field in any space from which the magnetic system was shielded by an earthed conducting shield which did not participate in the rotation.* In experiments on this subject, it has been customary to employ shields on the general principle of shielding off extraneous electrostatic effects, so that, quite apart from all other considerations, one reason for the absence of any measured effects is obvious.

As ordinarily applied, the "Moving Line Theory" predicts a field due to the motion of the magnetic lines inside and outside the magnetized body, and the nature of this application is such as make the field inside the substance just equal in amount and opposite in sign to the motional intensity so that, in this form, the theory predicts no tendency for the formation of an electrostatic distribution of any kind within the magnetic system. Outside the magnetic substance the theory, as ordinarily applied, predicts an electric field of the form—$[v \cdot H]/c$, where H is the resultant magnetic intensity at the point, and v the velocity which the point would have if rigidly attached to the rotating system. Moreover, this electric field is supposed to be of such an origin that it would not be affected by the presence of electrostatic screens.

Summary of Conclusions.—It will thus be seen that the complete story of the possible origins of electric fields in systems such as we have discussed is one involving a number of considerations; and, it may be well to review these considerations as follows:

(A) The field at a point external to the rotating system is given by (5); and, in the case of a symmetrical system rotating with uniform angular velocity, U is independent of the time, so that Grad. ψ represents the only contribution to the electric field E in this case.

(B) There is, of course, no electric field as a result of uniform rotation about the magnetic axis, of a current-carrying solenoid which contains no magnetizable material participating in the rotation.

(C) It has been shown that an amperian current whirl will experience a rearrangement of its charge density when set into rectilinear motion, even thought, when at rest, the positive and negative charge densities compensate in such a way as to result in no electrostatic potential function for the whirl. The rearrangement of charge density as a result of the rectilinear motion is of such a kind as to give rise to an electrostatic potential which is equivalent to that of an electric doublet with its axis perpendicular to the magnetic axis of the whirl.

(D) In the case of an insulating, uniformly rotating system of unit specific inductive capacity, magnetized parallel to the axis of rotation, the whole electric fields at points external to the magnetized system is that due to the electric doublets referred to under (C). It is equivalent to the field which would be produced by a distribution of electricity of one sign throughout the volume, and an equal distribution of electricity of opposite sign over the surface of the magnetized material.

Moreover, in the case of a rotating magnetic system, the contribution of these electric doublets to the field is the true representative of the "Moving Line Theory" when that theory is properly applied to the space outside the magnetized material.

(E) The "Moving Line Theory" as usually applied to the space outside the magnetized material differs from the form in which it ought to be applied to be consistent with the equations of Maxwell and Lorentz. In the theory as properly applied, after resolving each amperian whirl to an axis parallel to the axis of rotation, the motions of the magnetic lines of the individual whirls must be treated separately; and, in calculating the motion, the magnetic lines must be pictured as partaking of only the *rectilinear* velocity of the whirl in its orbit, and not of the *rotational* velocity of the frame of the whirl about its own axis. The usual application of the theory attempts to deal with the velocity of the resultant magnetic field at a point; and, in so doing, it does the equivalent of supposing the magnetic lines from the individual whirls to be pinned down at the places where they pass through the axis of rotation of the system. In other words, the motions of the magnetic lines become calculated as though they partook of the rotational velocities of the frames of the whirls about their own axes, as well as of the rectilinear velocities of the whirls in their orbits.

(F) If the rotating system referred to in (D) is conducting instead of insulating; and, if, for the moment, we ignore the motional intensity represented by the last term of (11), it follows that the material will experience a charge distribution of such a nature as to completely annul, at all external points, the fields of the doublets referred to under (C), so that there will be no external field. If now we introduce the motional intensity, it will produce, in the conducting system, an additional distribution of electricity whose field inside the material just balances the motional intensity as represented by $[v \cdot B]/c$.

The field due to the charge distribution brought about by the motional intensity is the sole contribution to the electric field at external points. It further turns out that the magnitude of the external electric field is unaltered by the existence of a specific inductive capacity greater than unity in the material.

(*G*) The system referred to under (*F*) though conducting is nevertheless supposed to be insulated. The electrostatic distribution resulting from the motional intensity will be such as to cause a variation of electrostatic potential throughout the rotating body; and, the potential of the axis of rotation will not be zero. If this axis is subsequently earthed, an additional charge will come to the system. This charge will be distributed in the same way as it would be distributed on a non-magnetic body of the same size and shape; and, indeed, to a high order of approximation, its distribution will be the same as on the conductor at rest. Its total amount will be equal to minus the product of the capacity of the body and the potential of the axis before it was earthed. If some point other than the axis of rotation is earthed, the charge which comes to the body will be that necessary to bring this point to zero potential.

The case of practical importance is, of course, that in which some point of the system (usually the axis of rotation) is earthed; and, we thus see that, in this case, the external field is entirely determined by the electrostatic distributions resulting from (*a*) the direct action of the motional intensity as calculated from the induction as ordinarily defined for a point within the material, and (*b*), the charge which comes to the system as the result of a point on it being earthed.

(*H*) It is to be observed that the distribution resulting from (*a*) under (*G*) is of a type comprising a volume distribution and an equal surface distribution. These two distributions do not, in general, cancel each other's effects as regards the field at an external point. As regards alteration of *potential* of any fixed conductor which completely surrounds the rotating body, these two charges, owing to their equality, do cancel each other's influence, however; for the potential produced on a closed conductor by a point charge within it is independent of the location of the point charge. The charge resulting from (*b*) under (*G*) is thus the only agency active in causing alteration of potential of such an inclosing conductor.

(*I*) It is to be observed that all contributions to the field in the case of a symmetrical rotating system are of an electrostatic nature; and, there will consequently be no electric field in any space from which the magnetic system is completely shielded by an earthed conducting shield not participating in the rotation.

Discussion of Experimental Results Heretofore Obtained.

In line with what has been written above, we should expect that, in the case of a rotating magnetized system, there would be an external electric field of the type described in sections (*F*) and (*G*) of the summary given above, and having its detailed origin in the considerations summarized in (*A*) to (*G*).

One of the first of the modern experiments on this matter was made by E. H. Kennard.[1] The essentials of this experiment are as follows:

A steel bar *B*, Fig. 3, was surrounded by a current-carrying solenoid *S*, in such a way that the bar could be rotated at high speed. The solenoid was fixed; it was surrounded by an earthed metal sheath *K*, and concentric with this was an outer cylinder *C*. The cylinders *K* and *C* formed the two plates of a condenser, and were connected to the quadrants of an electrometer. No effect was obtained as a result of the rotation of the magnetic system.

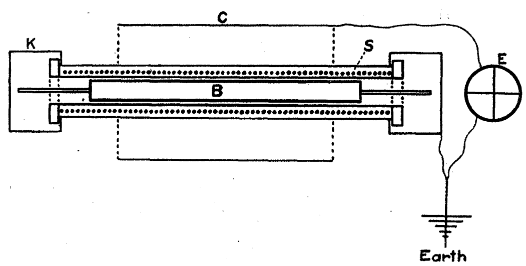

Fig. 3.

It will be observed that the magnetic system was surrounded by a stationary earthed conductor, so that, in accordance with (*I*) in the summary above, no external electrical field could possibly arise since all possible electrical fields are of a type derivable from an electrostatic potential. Indeed, the earthed solenoid itself would have been sufficient to provide a shield even if the inner cylinder had been absent.

Apart from these considerations, however, even if the solenoid had rotated with the bar, and the shield *K* had been absent, there would have been no effect provided that the bar was insulated; for, the sole origin of a field in such a case would have to be sought in the surface and volume distributions of charge referred to under (*F*) above, and these, being equal and uniformly distributed as regards the cylindrical magnet, would practically cancel each other's effects at all points outside. The cancellation would be complete for an infinitely long bar. Even if the rotating body had not been in the form of a cylinder, however, there would have been no alteration in the potential of the outer member of the condenser, even though the earthed shields had been absent, provided that the rotating body itself was insulated. For, the total volume and surface charges being equal, there would, in accordance

[1] "Unipolar Induction," Phil. Mag., S. 6, 23, 937, 1912.

with (H) above, be no alteration of potential of any cylinder such as C, completely surrounding the rotating body.

If, however, the solenoid rotated with the bar, K were eliminated, and the axis of the rotating system were earthed, there would exist the field discussed under (G), and produced by the charge which comes to the system to maintain the potential of the axis zero.

In a later experiment by S. J. Barnett,[1] the two armatures of the condenser were surrounded by a coaxial solenoid which could be rotated about its axis. The outer cylinder was earthed, and the inner cylinder could be insulated or connected to the outer cylinder at will. It was found that rotation of the solenoid produced no effect. This is what we should expect in accordance with (B) and the arguments in the earlier portion of this paper, upon which this conclusion is based.

Barnett performed another experiment in which the magnetic field was produced by two electromagnets which were arranged so that they could be rotated about their axes of magnetization, which were in line. The condenser system was placed between the two electromagnets so that its axis was also the axis of the magnets. The ends of the outer cylinder were closed with brass caps so that the outer cylinder and these brass caps formed a complete shield around the inner cylinder, which could be insulated or earthed. In accordance therefore with (I), and quite apart from all other considerations, we should expect no alteration of potential of the inner cylinder as a result of the rotation of the magnets; and, no alteration of potential was obtained. In the absence of the brass caps, small effects should, theoretically, be obtained in accordance with the principles summarized under (F) and (G); but, they would have no simple relation to the effects as calculated by the ordinary application of the "Moving Line Theory."

It is perhaps worth while to emphasize the fact that we are not denying the possibility of an effect on the general grounds of supposing that a closed earthed shield will always prevent external actions from creating a field within. Such a conclusion would not be justified. The essentials of the argument are contained in the fact that such fields as are to be expected in experiments of this kind are fields determined by electrostatic potentials, in accordance with the views summarized in sections (A) to (I) above.

Barnett's experiment with a rotating solenoid, and without iron, was repeated by Kennard[2] with a modification which permitted of the rotation of the condenser system with the solenoid. A similar experiment

[1] PHYS. REV., 35, 323, 1912.

[2] Phil. Mag., 33, 179, 1917.

has also been performed by G. B. Pegram.[1] When the condenser system was stationary, no effect was of course observed. When the solenoid and condenser were rotated together, the potential difference set up between the condenser armatures, when connected together, was simply that calculated from the motional intensity produced in the connecting wire, as a result of its motion in the magnetic field. This potential difference was of course measured by an electrometer, the armatures being separated after the rotation was started, and the system being allowed to come to rest before readings were taken. Concerning this part of the effect there is presumably no divergence of opinion. It may be of interest to observe, however, that it would be obtained even though the wire connecting the armatures were outside the field. For, imagine the most general case as depicted in Fig. 4, where the connecting wire is

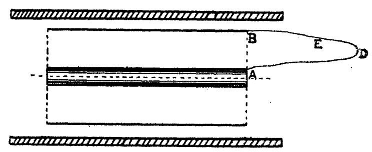

Fig. 4.

partly within the solenoid and partly outside. If H_n is the component of the field measured in the direction of the outwardly drawn normal to the surface (say the surface S) traced out by the wire $ADEB$, r the distance of an element of the wire from the axis of rotation, and ω the angular velocity, we have, for the potential difference between A and B as measured along this wire,

$$V = \frac{1}{c} \int (\omega r) H_n ds,$$

ds being an element of the path $ADEB$, and the integral being taken along the whole path. Now, $\int 2\pi r H_n ds$ is simply the total flux of magnetic force through the surface S, and is the same, therefore, as the magnetic flux through the annulus between A and B. Hence, if a and b are the radii of the two armatures, and H the magnetic field, which is uniform in the solenoid although it is not uniform all over the wire $ADEB$, we have:

$$V = \frac{H\omega}{c} \int_a^b r dr = \frac{H\omega}{2c}(b^2 - a^2),$$

which is the same as the value which would have been obtained if the connecting wire had gone straight from A to B. Uniformity of the field

[1] PHYS. REV., 10, 591, 1917.

over the path is not necessary; the only essential is that if there is any part of the wire which does not revolve, it shall be situated in a place where the magnetic field is negligibly small. We see that, even in the limiting case, where the wires pass out of the solenoid parallel to the axis, and are connected together at some remote point outside, the full value of V may be expected.

An experiment seeking to detect a field of electromagnetic induction due to the uniform *rectilinear* motion of a magnet has been recently performed by Barnett.[1]

It is obvious that, since the electric field of a simple amperian whirl moving with uniform rectilinear velocity obeys the "Moving Line Law," the resultant electric field of any magnet whose parts all move with the same uniform rectilinear velocity will be that calculated from the "Moving Line Law" as applied to the *resultant* magnetic field. The problem is thus much simpler than that of rotation. Moreover, it involves no extra field resulting from electrostatic distributions brought about in the magnet by the motional intensity; for, in the magnet itself, the forces on an element of charge, as a result of the moving line field, is just cancelled by the motional intensity, so that there is no electrostatic distribution due to this cause. If there are any fixed magnets in the system, they of course contribute nothing to the effect.

In Barnett's experiment, the attempt is made to charge a condenser by causing an electromagnet to swing past it, the electromagnet constituting the bob of a large pendulum. A fixed magnet also figured in the experiment for a reason which we need not enter into here; for, on the views expounded above, it produces no electrical effect whatever. No charging effect was found in the experiment.

The condenser and insulated system were shielded by an earthed case, to protect them from electrostatic effects. We shall show that the charge distribution set up in the case by the field of the moving lines produces, at all points within the case, a field just equal and opposite to the field which the moving lines produce there. No further comment is there necessary to show that no charging effect is to be expected in an experiment of this kind.

We cannot immediately import the ideas we have invoked in the case of uniform rotation to explain a shielding effect of the type referred to, because the vector potential is not independent of the time in the case of rectilinear motion; so that, it is not immediately obvious that the whole of the field E as given by (5) is derivable from an electrostatic potential.

[1] PHYS. REV., 12, 95, 1918.

In the direction parallel to the line of motion, the poles in Barnett's experiment were 33 cms. long. The corresponding dimension of the shielding case was 21 cms.; so that, since the field was sought at the instant during the swing of the magnet when the middle of the poles passed the condenser, and presumably therefore, when the 21 cms. of the case fell well within the poles, the problem may be treated as a two-dimensional one in the yz plane perpendicular to the line of motion of the magnet.

The "Moving Line Theory" gives, for the components E_y and E_z of the electric field, in terms of the components H_y and H_z of the magnetic field:

$$E_y = \frac{v}{c} H_z, \qquad E_z = -\frac{v}{c} H_y.$$

Now the magnetic field H is derivable from a potential Ω; and, as Ω is a solution of LaPlace's equation outside the magnet, we know, from the theory of functions of a complex variable, that it is always possible to find a potential Φ, itself a solution of LaPlace's equation in the region where Ω is a solution, and such that:

$$\frac{\partial \Phi}{\partial y} = \frac{v}{c} \cdot \frac{\partial \Omega}{\partial z}, \qquad \frac{\partial \Phi}{\partial z} = -\frac{v}{c} \cdot \frac{\partial \Omega}{\partial y}$$

Φ is, in fact, v/c times the function conjugate to Ω. Thus, we have:

$$E_y = -\frac{v}{c}\frac{\partial \Phi}{\partial y}, \qquad E_z = -\frac{v}{c}\frac{\partial \Phi}{\partial z},$$

showing that E is derivable from a potential, the apparent charges which give rise to it being situated *outside* the region in which Ω satisfies LaPlace's equation, and hence outside the shield, in the present instance. This is all we need in order to say that, under the influence of the potential $v\Phi/c$, the case will acquire a charge distribution such as to just annul the field which $v\Phi/c$ itself gives at all points within the case.[1]

An exactly analogous argument would show that, if the magnet were at rest, and the remainder of the apparatus were in uniform rectilinear motion, there would be complete shielding as regards the effect of the motional intensity on a charge which was within the shield, and moved in company with it.

A simpler, but interesting example of this type of action is afforded by the case of the motion of a closed conductor in a uniform magnetic field. The motional intensity acting on a charge within the conductor and moving with it would be uniform. In order to balance the effect

[1] This demonstration was presented by the writer at the meeting of the American Physical Society at St. Louis, December, 1919.

of this motional intensity on the surface, the conductor would experience a charge distribution; and, we know that the charge distribution which will produce, over the surface of the conductor, equilibrium under the action of the uniform field of force, will also produce complete cancellation of that field inside the conductor.

Solution for a Typical Case of Uniform Rotation.

The rotation of a permanently and uniformly magnetized conducting sphere, whose axis of rotation is earthed, and whose direction of rotation is clockwise[1] *as viewed by an observer looking in the direction of the magnetization.*

The field will be represented entirely by the fields of the electrostatic distributions referred to under (F) and (G) in the summary given earlier in the paper. Let us first consider the results which would be obtained if the axis of rotation were insulated.

If r and θ are the polar coördinates of a point in the sphere, B the magnetic induction at a point inside the sphere, and ω the angular velocity, then, on account of the motional intensity, a unit of charge moving with the sphere will experience a force perpendicular to the axis of rotation; and, as shown in the earlier part of this paper, the magnitude of this force when measured outwardly from the axis of rotation is $B\omega r \operatorname{Sin} \theta/c$. Consequently, this represents the field which must be balanced by the electrostatic distribution. The electrostatic distribution is of the same type as that necessary to balance the effect of centrifugal force on the free electrons of a rotating sphere, since the magnitude of this force would be, for an electron of mass m, $mr\omega^2 \operatorname{Sin} \theta$. In a former paper,[2] I have given the solution of the latter problem; and, in order to adapt it to the present case, it is only necessary to replace $m\omega^2/e$ by $B\omega/c$, and then change the sign, since, in the paper referred to, the effect of centrifugal force on a *negative* electron was under consideration. Making this change, we find from equation (24) of the paper referred to, that the potential V is, for points outside the sphere,

$$V = -\frac{B\omega a^5}{3cr^3}\left(1 - \frac{3}{2}\cdot \operatorname{Sin}^2 \theta\right). \tag{18}$$

At the surface of the sphere and on the equator, the electric field is

$$-\left(\frac{\partial V}{\partial r}\right)_a = \frac{1}{2}\cdot\frac{Bwa}{c}. \tag{19}$$

[1] And hence clockwise as viewed in the direction of the induction.

[2] Terr. Mag., 22, 162, 1917. The solution given in this paper is carried out in electromagnetic units; but, it is true as it stands when the quantities are in Heavisidean units.

If I is the intensity of magnetization, the demagnetizing field H_1 within the sphere is $I/3$, and the induction is $I - H_1$, *i.e.*, $2H_1$. Hence, since the tangential fields just inside and just outside of the surface are equal, the result given by (19) is the same as would be given by the ordinary application of the "Moving Line Theory." The forms of variation of the field over the surface of the sphere are quite different for the erroneous and correct method of calculation, however; and, at places where $\text{Sin}^2\,\theta$ is less than 2/3, even the signs are opposed. Equation (18) shows that the points on the sphere corresponding to the axis of rotation will have potential $-B\omega a^2/3c$. If the axis of rotation is now earthed, it will be necessary for the sphere to take a uniformly distributed charge q given by:

$$\frac{q}{4\pi a} = \frac{B\omega a^2}{3c}. \tag{20}$$

This will result in a contribution $B\omega a^3/3cr$ to the potential at external points, so that the complete expression for V is now:

$$V = -\frac{B\omega a^2}{3c}\left[\frac{a^3}{r^3}\left(1 - \frac{3}{2}\cdot\text{Sin}^2\,\theta\right) - \frac{a}{r}\right]. \tag{21}$$

At a point on the surface of the sphere, the component of the field in the direction of the outwardly drawn normal is:

$$-\left(\frac{\partial V}{\partial r}\right)_a = -\frac{B\omega a}{c}\left(\frac{2}{3} - \frac{3}{2}\cdot\text{Sin}^2\,\theta\right). \tag{22}$$

On the equator this is $5B\omega a/6c$, and is just 5/3 times the field which we should have calculated from the "Moving Line Theory" as ordinarily applied, if we had ignored every effect other than the field due to the so-called motion of the lines of force. The ordinary application of the theory would not lead to the same type of distribution of field over the sphere as that given by (22).

In the case where the sphere is insulated, the total volume and surface distributions of charge are equal; so that, if we were to surround the sphere with a fixed insulating conducting sphere of radius b, there would be no change of potential of the outer sphere as a result of rotation of the inner sphere; although there would, of course, be rearrangement of charge over its surface. When the axis of rotation is earthed, however, the excess charge q, given by (20), which comes to the inner sphere, will produce a potential

$$W = \frac{B\omega a^3}{3cb} \tag{23}$$

in the outer sphere, since the potential of the latter is independent of the distribution of charge within it. If W and B are in ordinary electrostatic and electromagnetic units respectively, the equation will remain unchanged. If the outer sphere were connected to an electrometer, we should have to replace b by the capacity of the whole insulated system, including the electrometer.

We have seen that the induction within a uniformly magnetized sphere is $2H_1$, where H_1 is the demagnetizing field within the sphere. It is, of course, impossible to magnetize a piece of steel to such an extent that the accompanying demagnetizing field which it produces within itself is greater than, or even as great as a fixed limit known as the "Coercivity" of the substance. Even for tungsten steel, the coercivity is only $52 \cdot 6$; so that 105 certainly represents a value of B greater than that actually attainable. If the inner sphere had a radius of 5 cms., and were rotated at a speed of 100 revolutions per second; and, if the outer sphere were connected to an electrometer such that the total capacity was 50 cms., we find by using (23), that the alteration of potential of the electrometer would certainly be less than 18×10^{-7} E.S.U., *i.e.*, $5 \cdot 4 \times 10^{-4}$ volt.

The calculation for the case of an iron sphere inductively magnetized by an external magnetic field is the same as that for the permanently magnetized sphere cited above. B represents the resultant induction within the sphere, and is related to the external field H by the expression:[1]

$$B = \frac{3\mu}{\mu + 2} \cdot H,$$

so that if μ is of the order 2000 C.G.S. unit, as in the case of soft iron, B is practically three times the external field. For a case where H is 2,000 C.G.S. units, the alteration in potential of the insulated sphere system cited in connection with the permanently magnetized sphere will be 3×10^{-2} volt. If the sphere were of non-magnetic material, such as copper, the alteration in potential would be one third of this amount. The influence of permeability would, of course, be greater in the case of elongated bodies such as ellipsoids.

No special interest would attach to measurements made upon a copper sphere, for, presumably, nobody would doubt that, under the influence of the motional intensity, a copper sphere would experience the sort of charge distribution, etc., we have discussed. Perhaps a greater interest lies in the value for the iron sphere rotating in an external magnetic field; for we may here look upon the effect as composed of two parts, the part corresponding to the rotation in the external field, which part is the same as for a copper sphere, and the part due to the rotation in the

[1] See for example J. J. Thomson's Elements of Electricity and Magnetism, p. 260.

field of induction resulting from the magnetization of the sphere itself. The existence of the latter part as would be shown by the increase of the effect on subsituting an iron sphere for a copper sphere, is perhaps of some interest, as it arises from the magnetic induction of a rotating body; and, to one who thinks in terms of moving magnetic fields, there might be some doubts concerning it, although, from the standpoint of the development given in this paper, the reality of its existence seems perfectly clear.

The Rotation of a Magnetic Doublet about an Axis Passing through its Center and Perpendicular to its own Axis.[1]

While this problem is not primarily involved in the discussion of the usual experiments on electromagnetic induction and relative motion, it is of considerable interest. With the doublet considered as the representative of an amperian whirl, the true solution for the field is, of course, that corresponding to equation (5).

Without some theory for rotation, analogous to the theory of relativity for rectilinear motion, we are unable to say what would happen to the constitution of the amperian whirl as a result of the rotation; we are unable to predict what ψ would be. If, however, we assume that the constitution of the whirl is not altered by the rotation, the field will be given by the first term of the right-hand side of (5).

The vector potential of a magnetic doublet of moment μ_x, with its axis parallel to the axis of x is given by:[2]

$$U_x = 0, \qquad U_y = \mu_x \frac{\partial}{\partial z}\left(\frac{1}{r}\right), \qquad U_z = -\mu_x \frac{\partial}{\partial y}\left(\frac{1}{r}\right)$$

with corresponding expressions for the doublets parallel to the other axes.

Without sacrifice of generality, we may take the y axis as the axis of rotation, and the z axis as the axis of the doublet at some particular instant. If we assume that the magnetic field of the doublet is, at each instant, the same as if the doublet were at rest, we shall merely neglect effects of a small order of magnitude; and, in this case, the change of moment per second corresponds to the creation of moment along the x axis at the rate $m_0 h\omega$, that is $\mu\omega$ units per second, h being the length of the doublet, m_0 its magnetic pole strength, and μ its magnetic moment. Hence, since the components of the electric field are equal to minus $1/c$ times the rates of increase of the corresponding components of the vector potential, we have for the field components:

$$E_x = 0, \qquad E_y = -\frac{\omega\mu}{c} \cdot \frac{\partial}{\partial z}\left(\frac{1}{r}\right), \qquad E_z = \frac{\omega\mu}{c} \cdot \frac{\partial}{\partial y}\left(\frac{1}{r}\right). \qquad (24)$$

[1] A paper on this section was presented by the writer at a meeting of the American Physical Society at Chicago, November 29, 1919.

[2] See, for example, J. H. Jeans's Electricity and Magnetism, page 394.

Now suppose we calculate the electric field which would be given by the application of the "Moving Line Theory" to the problem on the understanding that the magnetic lines from each pole are to be treated separately, and are to partake of the rectilinear velocity of the pole in its orbit, but not of any rotary motion. At the instant when the axis of the doublet is parallel to the z axis, the "Moving Line Theory" as applied to one of the poles gives o, $\omega h H_z/2c$, $-\omega h H_y/2c$ for the components of the electric field, $\omega h/2$ being the velocity of the pole, and H_y and H_z being the y and z magnetic field components at the point at which the electric field is sought, in so far as they are determined by the single pole under consideration. Treating the effects of the two poles as additive as regards the electric fields, putting l, m, n for the directional cosines of the radius vector, and observing that, for a single pole,

$$H_y = \frac{m_0 m}{r^2}, \qquad H_z = \frac{m_0 n}{r^2}$$

we have, from the "Moving Line Theory" in this form:

$$E_x = 0, \qquad E_y = \frac{\omega h}{c}\left(\frac{m_0 n}{r^2}\right), \qquad E_z = -\frac{\omega h}{c}\left(\frac{m_0 m}{r^2}\right),$$

.e.,

$$E_x = 0, \qquad E_y = -\frac{\omega\mu}{c}\cdot\frac{\partial}{\partial z}\left(\frac{1}{r}\right), \qquad E_z = \frac{\omega\mu}{c}\cdot\frac{\partial}{\partial y}\left(\frac{1}{r}\right),$$

which agree with the results obtained in (24) from the Lorentzian theory. Although we have utilized the idea of a pole-distance in the above calculation, the results naturally do not involve this pole-distance explicitly, but depend only upon the moment of the doublet.

We thus see that the ideas controlling the application of the "Moving Line Theory" to this case are an extension of, and bear a close analogy to those applying to the case where the doublet revolves about some center outside itself. If, for the moment, we utilize Fig. 2 for a purpose different from that for which it functioned before, and suppose N to represent one of the magnetic poles of the doublet, and O the center of the doublet, we see that, on applying the "Moving Line Theory" to this case, on passing from the position N to the position N_1, we must calculate the velocity of a magnetic line NP as though it had assumed the orientation N_1P in diagram B, and not as though it had assumed the orientation N_1P in diagram C. In fact, while we must imagine the magnetic lines to be rigidly attached to the pole, we must suppose that the latter does not rotate about an axis within itself as it revolves about the center of the doublet. Any calculation founded upon the idea of the resultant lines of force of the doublet turning with the doublet as though rigidly attached to it would be the equivalent of supposing the lines due to the individual poles to partake of the rotary motion referred to above, as well as of the rectilinear motion. Such a view would lead to absurdities from the outset; for, it would obviously predict infinite velocities for the lines at infinite distances from the doublet. It would be analogous to what we should be driven to if, in the case of an electron revolving in an orbit, we were to assume the tubes of electric force to be rigidly attached to the electron, while the latter partook of the kind of motion it would experience if stuck on to a spoke emanating from the center of the orbit. In this case, even the importation of the ideas of finite rate of propagation of effects could do no better than leave us with the picture of the tubes of force winding themselves up into spirals as the electron continued to describe its orbit over and over again. A proper calculation of the effects due to a revolving electron simply represents the tubes as moving backwards and forwards as the electron passes from one extremity of a diameter to the other; and, it is never necessary to suppose the tubes to have a velocity greater than that of the electron itself. A similar remark applies in the case of the rotating magnetic doublet.

It may be of interest to remark that, if we calculate the electric field due to a rotating doublet, at a point on a prolongation of its axis, the erroneous method of calculation, which treats the resultant magnetic lines as moving rigidly with the doublet, gives just twice the value resulting from the correct method of calculation.

In conclusion it may be observed that, to the extent that the magnetic field of a moving amperian whirl may be considered as equivalent to that of a magnetic doublet, the electrical field resulting from the most general type of motion of the whirl, and hence of a non-conducting magnet (of unit specific inductive capacity) in bulk, may be calculated from the "Moving Line Theory" as applied individually to the magnetic lines from the separate representative magnetic poles, provided that the velocities of the lines are calculated from the rectilinear velocities of the poles, and not in a manner such as to make them partake of any rotary velocity of the poles about their own centers. In the case of a conducting magnet, the portion of the above field derivable from a potential will become cancelled by the distribution of charge which it sets up in the magnet itself, as stated under (*F*) of the summary given earlier in the paper. We shall be left with the part $-\frac{1}{c}\cdot\frac{\partial U}{\partial t}$; and, superposed upon this, we shall have the fields of the electrostatic distributions produced by this part, and by the motional intensity resulting from the motion of the magnet in its own magnetic field, and discussed under (*F*), (*G*) and (*H*) of the summary above referred to.

The ideas developed in this paper arose during the process of writing up an account of some experimental work which will be published later, and for which funds were provided by the executive committee of the graduate school of the University of Minnesota. The writer desires to take this opportunity of expressing his thanks to the committee.

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