Normalization of and Finite-Range Effects in (^{3}He,d) and (t,d) Reactions*

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The effects of including a realistic wave function for the ^{3}He ion on the predictions of the distorted-wave theory for stripping and pickup reactions are studied in some detail. On the assumption that the ^{3}He ion and triton are similar, the results are carried over to reactions involving the triton. Good agreement with experiment is found.

I. INTRODUCTION

As is well known, the (d,p) stripping and (p,d) pickup reactions have yielded important information about neutron single-particle and hole states. The utility of this reaction has been enhanced by the development of the distorted-wave theory,[1,2] which, in most cases, accurately predicts shapes of angular distributions and the energy and Q-dependence of the reactions. Recent tests of the validity of the distorted-wave Born approximation (DWBA) have also indicated that the absolute cross section is reasonably well predicted.[3]

Until recently, rather little was known about proton particle and hole states. These states can be reached via the (d,n) and (n,d) reactions, but these experiments are difficult. Alternative ways of studying these states are with the (^{3}He,d) and (d,^{3}He) reactions. Spectroscopy with these reactions has been limited, either because of poor energy resolution or low bombarding energy. These difficulties have now been largely overcome, and extensive studies of both the stripping and pickup reactions have been reported,[4] and are in progress.

Some discussion of these reactions and the application of the distorted-wave theory has been reported.[4] In general, the theory has been quite successful in predicting the shapes of angular distributions but, since little is known about the wave functions of the three-nucleon system, the theory has been essentially unnormalized, and extraction of spectroscopic information has relied on empirical normalization.

Recent studies of photodisintegration of the ^{3}He ion[5] into a proton and deuteron have been "quantitatively" explained by assuming an Irving-Gunn[6] wave function for the fully space-symmetric S state of the ^{3}He ion. A further success of this wave function is the explanation of the electron-proton coincidence cross section—if one assumes that the inelastic scattering of electrons by ^{3}He ions is dominated by the breakup into a proton and deuteron.[7] The single-parameter wave function used in these analyses also accounts for the Coulomb energy difference between the triton and the ^{3}He ion, the over-all form factor of the ^{3}He ion for $q^2<5$ F^{-2}, and, crudely, the ^{3}He size.[5]

These successes suggest that this wave function be used to estimate the absolute cross section predicted by the distorted waves theory for the (^{3}He,d) and (d,^{3}He) reactions. Since the triton is expected to be similar to the ^{3}He ion, we extend our estimates to include (d,t) and (t,d) reactions.

In succeeding paragraphs we consider the finite range corrections to the theory implied by the choice of the Irving-Gunn wave function, and compare our predictions with experiment. Finally, we summarize our findings.

II. THEORY

The transition amplitude for the reaction $A(a,d)B$ is written

$$T_{\mathrm{DW}}=J\int d\mathbf{r}_a\int d\mathbf{r}_d\chi_f^{*(-)}(\mathbf{k}_f,\mathbf{r}_d)\times\langle B,d|V_{\mathrm{eff}}|A,a\rangle\chi_i^{(+)}(\mathbf{k}_i,\mathbf{r}_a). \quad (1)$$

Here $\mathbf{r}_a$ is the relative coordinate between a (the ^{3}He ion or the triton) and the target A, and $\mathbf{r}_d$ is the relative coordinate between the deuteron and the residual nucleus B. J is the Jacobian of the transformation to these relative variables, and χ_a and χ_d are the distorted waves, usually treated in optical-model approximation.

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[1] J. Horowitz and A. M. L. Messiah, J. Phys. Radium **14**, 695 (1953); **14**, 731 (1953); W. Tobocman, Phys. Rev. **94**, 1655 (1954).

[2] W. Tobocman, *Theory of Direct Nuclear Reactions* (Oxford University Press, New York, 1961); N. Austern, in *Fast Neutron Physics*, edited by J. B. Marion and J. L. Fowler (Interscience Publishers, Inc., New York, 1963), Vol. II; R. Huby, M. Y. Rafai, and G. R. Satchler, Nucl. Phys. **9**, 94 (1958); G. R. Satchler and W. Tobocman, Phys. Rev. **118**, 1566 (1960); L. C. Biedenharn and G. R. Satchler, Helv. Phys. Acta. Suppl. **6**, 372 (1960); G. R. Satchler, Nucl. Phys. **18**, 110 (1960); L. J. B. Goldfarb and R. C. Johnson, *ibid.* **18**, 353 (1960); **21**, 462 (1960); B. Buck and P. E. Hodgson, Phil. Mag. **6**, 1371 (1961); D. Robson, Nucl. Phys. **22**, 34 (1961); **22**, 47 (1961); R. C. Johnson, *ibid.* **35**, 654 (1962); R. H. Bassel, R. M. Drisko, and G. R. Satchler, Oak Ridge National Laboratory Report No. ORNL-3240 (unpublished).

[3] L. L. Lee, Jr., J. P. Schiffer, B. Zeidman, G. R. Satchler, R. M. Drisko, and R. H. Bassel, Phys. Rev. **136**, B971 (1964).

[4] J. L. Yntema and G. R. Satchler, Phys. Rev. **134**, B976 (1964); D. D. Armstrong and A. G. Blair, *ibid.* **140**, B1226 (1965), and references therein; J. C. Hiebert, E. Newman, and R. H. Bassel, Phys. Letters **9**, 160 (1965); J. R. Erskine, A. Marinov, and J. P. Schiffer, Phys. Rev. **142**, 633 (1966).

[5] B. L. Berman, L. J. Koester, and J. H. Smith, Phys. Rev. **133**, B117 (1964).

[6] J. C. Gunn and J. Irving, Phil. Mag. **42**, 1353 (1951).

[7] T. A. Griffy and R. J. Oakes, Phys. Rev. **135**, B1161 (1964).

A standard formulation of the amplitude, via the Gell-Mann–Goldberger transformation, yields

$$V_{\text{eff}} = V_f - U_f = V_{dx} + V_{dA} - U_f, \qquad (2)$$

where V_f is the interaction between the outgoing deuteron and the residual nucleus B, and U_f is the distorting potential which "satisfies" elastic scattering in the final channel.

V_{eff} is usually chosen to be the interaction between the outgoing particle and the transferred nucleon (x), i.e.,

$$V_{\text{eff}} \simeq V_{dx}, \qquad (3)$$

arguing that there is considerable cancellation between the remaining terms. While there is little formal justification for this approximation, its empirical success in describing the (d,p) reaction[3] suggests that it be tried for the reactions under consideration here.

With this approximation, the "nuclear" matrix element can be written

$$\langle B,d|V_{dx}|A,a\rangle = \langle B|A\rangle\langle d|V_{dx}|a\rangle. \qquad (4)$$

The overlap $\langle B|A\rangle$ of residual and target nuclear wave functions has been discussed by several authors.[8] Here we assume its properties are known and turn to the evaluation of the remaining factor,

$$\langle d|V_{dx}|a\rangle = \nu^{1/2}c\int d\rho\ \varphi_d(\rho)V_{dx}\varphi(r,\rho) = \nu^{1/2}cD(r), \qquad (5)$$

where ρ is the internal coordinate of the deuteron and r is the coordinate of the nucleon relative to the deuteron center of mass. The factor ν arises from antisymmetrization and is equal to the number of equivalent nucleons x in the three-nucleon system, and c is a coefficient of fractional parentage.

For the decomposition of the three-nucleon system into the triplet deuteron and captured nucleon, the product $\nu^{1/2}c = \sqrt{\tfrac{3}{2}}$.

It is convenient to factor the integral in (5) into an amplitude D_0 and a normalized function $f(r)$, with

$$D_0 = \int d\mathbf{r}\int d\boldsymbol{\rho}\ \varphi_d(\rho)V_{dx}\varphi(r,\rho),$$

and

$$f(r) = (1/D_0)\int d\boldsymbol{\rho}\ \varphi_d(\rho)V_{dx}\varphi(r,\rho). \qquad (6)$$

The distorted-wave cross section is then proportional to the factor

$$N = \tfrac{3}{2}D_0^2. \qquad (7)$$

[8] M. H. Macfarlane and J. B. French, Rev. Mod. Phys. **32**, 567 (1960); J. B. French, in *Nuclear Spectroscopy*, edited by F. Ajzenberg-Selove (Academic Press Inc., New York, 1960), Part b.

III. ANALYSIS

Our central task is to evaluate the integral in Eq. (5). For this purpose we make use of the Hulthén wave function to describe the s state of the deuteron, i.e.,

$$\varphi_d(\rho) = \frac{B}{(4\pi)^{1/2}}\left[\frac{\exp(-\alpha\rho)-\exp(-\beta\rho)}{\rho}\right], \qquad (8)$$

with

$$\alpha = 0.231\ \text{F}^{-1}, \quad \beta = 1.438\ \text{F}^{-1},$$

and

$$B = [2\alpha\beta(\alpha+\beta)]^{1/2}/(\beta-\alpha). \qquad (9)$$

For the ^{3}He ion and the triton, we take the Irving-Gunn wave function which has the form

$$\varphi(r_{12},r_{13},r_{23}) = A\ \exp(-\delta R/2)/R, \qquad (10)$$

with normalization

$$A = 3^{1/4}\delta^2/(2\pi^3)^{1/2}. \qquad (11)$$

Here

$$R^2 = \sum_{i<j}(r_{ij})^2 = 2r^2 + \tfrac{3}{2}\rho^2. \qquad (12)$$

For the size parameter we use $\delta = 0.768$ F^{-1}, the value used in Refs. 5 and 7.

We have chosen as a model interaction the force between the transferred nucleon and the deuteron. Therefore, for (t,d) and (d,t) reactions, we use the nuclear force between the neutron and the deuteron. For the reactions involving the ^{3}He ion it is not clear whether the matrix element we evaluate should not also include contributions from the Coulomb interaction between the proton and the deuteron or whether the Coulomb interaction should be grouped with the other neglected interactions. Arguments similar to these which lead to V_{dx} as effective interaction indicate the latter choice. Since there is some discussion of this point in the literature,[4] we shall calculate with both alternatives. We show below that this procedure also gives a check of the consistency of our theory.

Our approximations then are:

(a) $$V_{pd} = V_{nd} + V_{\text{Coulomb}};$$

and

(b) $$V_{pd} = V_{nd},$$

where V_{nd} is the nuclear interaction between the neutron and the deuteron.

We show in the Appendix that these assumptions lead to the following expressions for the amplitude D_0:

$$D_0(t,d) = -\frac{\hbar^2}{2\mu}\gamma_t^2\int d\mathbf{r}\int d\boldsymbol{\rho}\ \varphi_d(\rho)\phi(r,\rho), \qquad (13)$$

$$D_0^a(^3\text{He},d) = -\frac{\hbar^2}{2\mu}\gamma_3^2\int d\mathbf{r}\int d\boldsymbol{\rho}\ \varphi_d(\rho)\phi(r,\rho), \qquad (14)$$

and

$$D_0{}^b = D_0{}^a - \int d\mathbf{r} \int d\boldsymbol{\rho}\, \phi_d(\rho) V_{\text{Coulomb}} \phi(r,\rho) \,. \quad (15)$$

In the equations above, μ is the reduced mass of the nucleon and $(\hbar^2/2\mu)\gamma_t{}^2$ and $(\hbar^2/2\mu)\gamma_3{}^2$ are the separation energies of the neutron from the triton and the proton from the ^{3}He ion, respectively.

Details of the evaluation of these integrals are given in the Appendix. Here we quote the numerical results:

$$\begin{aligned} D_0(t,d) &= -183.6 \text{ (MeV F}^{3/2}) \,, \\ D_0{}^a({}^3\text{He},d) &= -161 \text{ (MeV F}^{3/2}) \,, \\ D_0{}^b({}^3\text{He},d) &= -172.8 \text{ (MeV F}^{3/2}) \,. \end{aligned} \quad (16)$$

In terms of the reduced cross section $\sigma_{\text{DW}}(\theta)$ [9] discussed by Satchler,[10] and the spectroscopic factor S discussed by Macfarlane and French,[8] our estimates of D_0 yield, for the stripping reactions:

$$\begin{aligned} \frac{d\sigma}{d\Omega}(t,d) &= 5.06\left(\frac{2J_f+1}{2J_i+1}\right) S\sigma_{\text{DW}}(\theta) \,, \\ \frac{d\sigma^a}{d\Omega}({}^3\text{He},d) &= 3.84\left(\frac{2J_f+1}{2J_i+1}\right) S\sigma_{\text{DW}}(\theta) \,, \\ \frac{d\sigma^b}{d\Omega}({}^3\text{He},d) &= 4.42\left(\frac{2J_f+1}{2J_i+1}\right) S\sigma_{\text{DW}}(\theta) \,. \end{aligned} \quad (17)$$

And, for the inverse reactions:

$$\begin{aligned} (d\sigma/d\Omega)(d,t) &= 3.33 S\sigma_{\text{DW}}(\theta) \,, \\ (d\sigma^a/d\Omega)(d,{}^3\text{He}) &= 2.56 S\sigma_{\text{DW}}(\theta) \,, \\ (d\sigma^b/d\Omega)(d,{}^3\text{He}) &= 2.95 S\sigma_{\text{DW}}(\theta) \,. \end{aligned} \quad (18)$$

If the isotopic-spin formalism is used, S should be replaced by C^2S, where C^2 is the square of the isospin Clebsch-Gordan coupling coefficient.

Most distorted wave calculations have used the zero-range approximation in order to reduce the complexity of T_{DW}. More specifically, the range func-function of Eq. (6) is taken as a delta function.

$$f(r) = \delta(r) \,, \quad (19)$$

and the six-dimensional integral is thereby reduced to a three-dimensional integral.

Recently, Austern *et al.*[11] have developed techniques which allow the matrix element to be evaluated exactly. The finite range effects have been shown to be important,[12] in the sense that this modification partially obviates the use of a radial cutoff on the matrix element. In addition the magnitude of the predicted cross section is different from the magnitude predicted in zero-range approximation—this difference is, of course, reflected in the deduced spectroscopic factor.

Our expressions for $f(r)$ can be evaluated numerically and the results treated within the formalism of Austern *et al.*[11] Rather than undertake this task, we use the fact that ^{3}He ions and tritons,[13–16] and deuterons[17,18] are strongly absorbed particles and mainly sample the nuclear surface. In addition, for ^{3}He ions the Coulomb barrier aids in confining reactions to the surface.

For practical calculations then it should be reasonable to use an approximation suggested by Drisko and Satchler[18] and demonstrated to be successful for deuteron stripping. These authors note that, if low-momentum components dominate the reaction, a Gaussian range function is an adequate approximation to the more complicated range function $f(r)$, that is,

$$f(r) \sim \exp(-r^2/R^2) \,. \quad (20)$$

In the above expression, R is chosen to give the correct small-momentum components of $f(r)$.

The presciption to find R is straightforward and consists of expanding the Fourier transforms of $f(r)$ and the Gaussian and comparing to order k^2.

As in our evaluation of D_0, there is a unique prescription for the range for the (t,d) and (d,t) reactions. But for the $(^3\text{He},d)$ and $(d,^3\text{He})$ reactions we find two expressions for the range according to our approximations a and b discussed above. The expressions are quite lengthy and are given as Eqs. (A12), (A13), and (A14), in the Appendix. Numerical evaluation gives

$$\begin{aligned} R(t,d) &= R(d,t) = 1.69 \text{ F} \,, \\ R_a({}^3\text{He},d) &= R_a(d,{}^3\text{He}) = 0.268 \text{ F} \,, \\ R_b({}^3\text{He},d) &= R_b(d,{}^3\text{He}) = 1.54 \text{ F} \,. \end{aligned} \quad (21)$$

Our result for the range and amplitude in approximation b for the $(^3\text{He},d)$ case are comparable to but about 10% smaller than the equivalent quantities for the (t,d) reaction. Since we are using the same wave function for the ^{3}He ion and the triton, they should be identical. The discrepancy, due to the fact that the Irving-Gunn wave function is too singular for small values of ρ, then gives a lower bound on the accuracy of the calculation.

The smallness of the range in approximation a arises from Coulomb repulsion. This can be understood by noting that the nuclear and Coulomb contributions are opposite in sign and that their sum results in a function which falls off more rapidly than the function from the nuclear force alone.

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[9] $\sigma_{\text{DW}}(\theta)$ corresponds to $\sigma_{lsj}(\theta)$ of the code SALLY.

[10] G. R. Satchler, Nucl. Phys. **55**, 1 (1964).

[11] N. Austern, R. M. Drisko, E. C. Halbert, and G. R. Satchler, Phys. Rev. **133**, B3 (1964).

[12] R. M. Drisko and G. R. Satchler, Phys. Letters **9**, 342 (1964).

[13] J. L. Yntema, B. Zeidman, and R. H. Bassel, Phys. Letters **11**, 302 (1964).

[14] P. E. Hodgson, *The Optical Model of Elastic Scattering* (The Clarendon Press, Ltd., Oxford, 1963).

[15] D. D. Armstrong, A. G. Blair, and R. H. Bassel (to be published).

[16] R. N. Glover and A. D. W. Jones, Phys. Letters **16**, 69 (1965).

[17] E. C. Halbert, Nucl. Phys. **50**, 353 (1964).

[18] C. Perey and F. Perey, Phys. Rev. **132**, 755 (1963).

To the extent that the Irving-Gunn wave function is realistic, and that low-momentum components of $f(r)$ dominate, approximation a indicates that the zero-range approximation will be good for the (^{3}He,d) reaction.

It is difficult to argue, on theoretical grounds, which approximation is more satisfactory. At this stage of the distorted waves theory, we prefer to let the choice rest on comparison with experiment.

IV. APPLICATION

A test of our estimates of D_0 and the range can be made by comparing the predictions of the theory with an experiment involving a closed-shell nucleus. Here we might expect spectroscopic factors to be reasonably predicted by the shell model. We recognize, however, that there may be particle-hole correlations in the wave function of the "closed"-shell nucleus and that the shell model thus may overestimate the spectroscopic factor.

Our choice of test case is also restricted to an experiment in which the optical-model parameters for the incident and final channels are known or can be reasonably inferred from analyses of scattering from neighboring nuclei.

Of the available experiments, the measurements of Blair and Armstrong[19] on the ^{48}Ca(^{3}He,d)^{49}Sc ground-state reaction seem a likely choice. Two-particle, two-hole ground-state correlations are presumably small since these authors find only a weak $l=2$ transition in the experiments. Thus the spectroscopic factor for the ground-state transition might be expected to be close to the shell-model prediction, $C^2S=1$. Blair and Armstrong have measured the elastic scattering of ^{3}He ions from ^{48}Ca, and this data has been fitted with the optical model.[15] Since no data exist for deuteron scattering from the unstable nucleus ^{49}Sc, we make use of the optical-model analysis of Yntema and Satchler[4] for the neighboring nucleus ^{49}Ti.

The optical model used has the form

$$U(r)=-V_0(1+e^x)^{-1} -i\left\{W_0(1+e^{x'})^{-1}+4W_D\frac{e^{x'2}}{(1+e^{x'})^2}\right\}+U_c(r)\,, \quad (22)$$

with

$$x=(r-r_0A^{1/3})/a;\quad x'=(r-r_0'A^{1/3})/a';$$
$$U_c(r)=ZZ'e^2/r \quad \text{for} \quad r\geq R_c$$
$$=(ZZ'e^2/2R_c^2)(3-r^2/R_c^2) \quad \text{for} \quad r\leq R_c\,,$$

and $R_c=r_cA^{1/3}$.

The parameters are listed in Table I.

The final ingredient in our calculation is the wave function of the transferred proton implied by the nuclear overlap $\langle B|A\rangle$. We treat this as an eigenfunction

TABLE I. Optical-model and bound-state well parameters.

Reaction	V_0 (MeV)	r_0 (F)	a (F)	r_c (F)	V_{s0} (MeV)	W_0 (MeV)	W_D (MeV)	r_0' (F)	a' (F)
^{3}He+^{48}Ca	139	1.08	0.8	1.4	0	12.3	0	1.743	0.721
d+^{49}Sc	112	0.974	0.912	1.3	6	0	18.3	1.439	0.6
p+^{48}Ca	64.3	1.2	0.65	1.25	8.9	···	···	···	···

of a Woods-Saxon well with eigenvalue equal to the separation energy. For the case of a closed-shell core, this well can be interpreted as the shell-model well, and the procedure seems justified. However, the input parameters (r_0,a) of this well are poorly defined. Usually, these parameters are chosen to correspond to the parameters of the real well found for the scattering of nucleons from the appropriate target. Because of ambiguities in the optical potential, and the possibility that these parameters may be energy- and mass-dependent, several sets of values for nucleon scattering have been found. We use here a set, listed in Table I, due to Satchler,[3] which describe proton scattering from ^{40}Ar and ^{42}Ca. We should stress that the magnitude (but not the shape) of the predicted cross section is dependent on the parameters of this well, and this introduces considerable uncertainty into our determinations, perhaps as much as 15%.

It has been suggested that the wave functions used in distorted-wave calculations should be eigenfunctions of of nonlocal wells. Accordingly, we have studied both local and nonlocal wave functions, nonlocal corrections to the local wave functions being made in the local energy approximation of Perey and Saxon.[20]

Briefly, this correction amounts to multiplying the local wave functions by a damping factor.

$$F(r)=C[1-(M\beta^2/2\hbar^2)U_L(r)]^{-1/2}. \quad (23)$$

In the above formula, β is the nonlocality range, M the reduced mass of the particle, and $U_L(r)$ the local potential. In principle, this correction is applied to both the scattered functions, and the bound-state wave function, although, as we discuss below, β is not well known for the three functions considered here. The constant has the value $C=1$ for the scattered wave functions, since the nonlocal wave function is identical to the local wave function asymptotically. For bound particles C exceeds unity in order that the nonlocal wave function be normalized.

Perey and Buck[21] have shown that $\beta_n=0.85$ F accounts for the energy dependence of the optical potential for nucleons. However, some of this dependence may be intrinsic and not due to nonlocality, so that this value is probably an upper limit. The energy dependence of the shell-model potential is not well known. The $(p,2p)$ and $(e,e'p)$ experiments suggest that

[19] D. D. Armstrong and A. G. Blair, Phys. Letters **10**, 204 (1964).

[20] F. G. Perey and D. Saxon, Phys. Letters **10**, 107 (1964); and to be published.

[21] F. G. Perey and B. Buck, Nucl. Phys. **32**, 353 (1963).

the well is nonlocal for deeply bound particles to about the same extent as the optical potential for nucleons. For particles near the top of the Fermi sea there is evidence that the energy dependence is smaller than for deeply bound orbitals or continuum wave functions.[22] Nevertheless, in the absence of more detailed knowledge, we take $\beta=0.85$ F for the nonlocality of the shell-model well. Assuming that the energy dependence of the deuteron and ^{3}He wells is due entirely to nonlocality yields $\beta_d\simeq0.54$ F, and $\beta_{^3\text{He}}\simeq0.2$ F, which values we adopt for our analysis.

Calculations were made using the Gaussian range function with the range prescribed by approximation *b*, and in zero-range approximation, to test which form of the theory is more applicable. The results are compared with the data in Fig. 1. Curve (a) shows the prediction assuming local wave functions in zero-range approximation. While the position of the main peak is accurately given, the theory grossly overestimates the large-angle cross section. Curve (b) is the theoretical result when nonlocal corrections are added to the theory but still using the zero-range approximation. Because the bound orbital is more surface localized and the interior contribution suppressed, low-momentum components are emphasized, with consequent reduction of the large-angle cross section. Curve (c) shows the result incorporating the finite-range modification but neglecting the nonlocal corrections. Again, high-momentum components are suppressed, low-momentum components enhanced, and the shape of the theoretical curve improved at large angles. Finally, curve (d) is the result when both finite range and nonlocal modifications are included. Since both corrections go in the same way, there is a further lowering of the predicted cross section at back angles.

While none of the theoretical predictions is in exact agreement with the experimental data, the two finite range predictions, curves (c) and (d), are reasonably satisfactory, with curve (c) perhaps to be preferred. This result may, of course, be fortuitous. It does suggest, however, that the finite range form of the theory is superior.

The predicted values of C^2S extracted from the data with these four forms of the theory are listed in Table II. If the interaction V_{dp} is a sum of nuclear and Coulomb interactions, the zero-range and finite range calculations

TABLE II. Spectroscopic factors.

Mode	$C^2S(a)$	$C^2S(b)$
Local, zero range	1.04	0.92
Nonlocal, zero range	0.87	0.8
Local, finite range	···	0.77
Nonlocal, finite range	···	0.66

[22] G. E. Brown, J. H. Gunn, and P. Gould, Nucl. Phys. 46, 598 (1963); and G. R. Satchler (private communication).

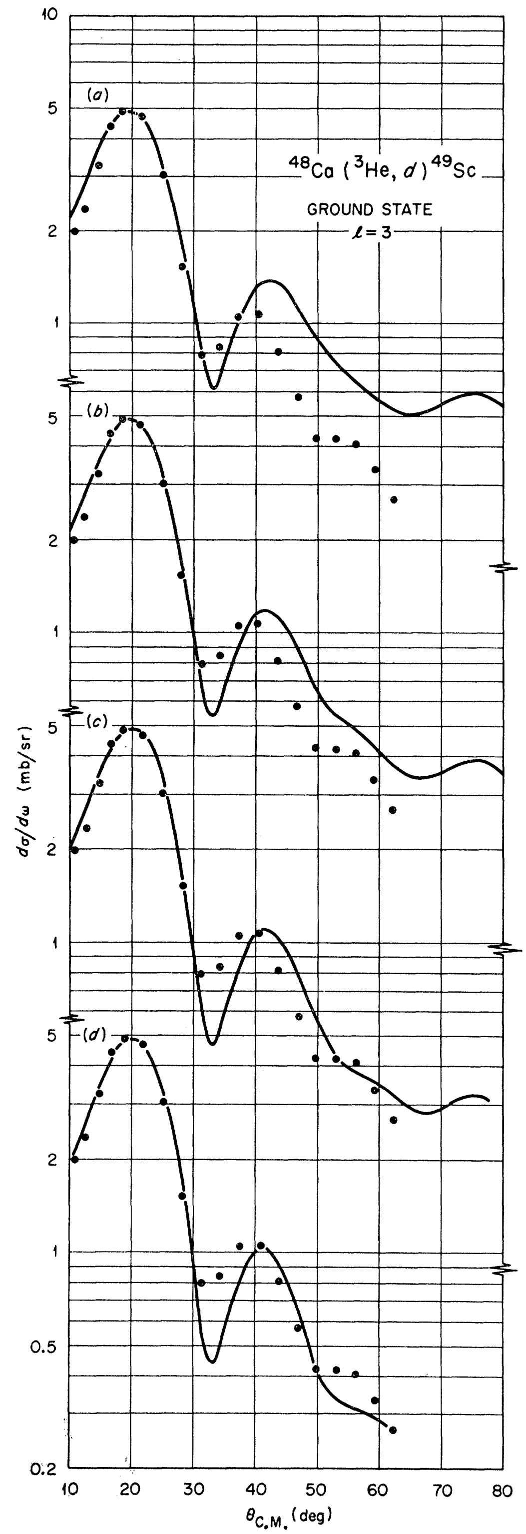

FIG. 1. Distorted-wave predictions compared with the data. The four curves are discussed in the text.

are essentially equivalent, as discussed before; the results for such an interaction are listed in Column I. The local estimate is only slightly in excess of the shell-model prediction, while the nonlocal estimate is some 20% smaller.

The interpretation of the interaction as purely nuclear leads to the estimates tabulated in column II, where for completeness we have included the results based on the zero-range approximation. The most detailed form of the theory underestimates the shell model C^2S by some 34%. But if the nonlocal corrections are dropped, agreement to 20% is again achieved.

Alternative Wave Functions for the Three-Nucleon System

In preceding paragraphs we have computed the normalization of the (^{3}He,d) reaction theory using an Irving-Gunn wave function for the ^{3}He ion. As we have shown, the results are in fair agreement with experiment. Before assigning significance to this "success" it is important to repeat our calculations using other three-body wave functions which have been shown to describe features of the three-body system.

The wave functions we shall consider are the Gaussian and Irving wave functions used by Schiff[23] in his analysis of elastic e-^{3}He and e-triton scattering.

The spatially symmetric Gaussian wave function is

$$\varphi(r,\rho)=A_G \exp[-\delta_G{}^2R^2/2],$$

where the normalization constant is $A_G=3^{3/4}\delta_G{}^{3/2}/\pi_{3/2}$. The Irving wave function is given by

$$\phi(r,\rho)=A_I \exp[-\delta_I R/2],$$

where

$$A_I=3^{3/4}\delta_I{}^3/(120)^{1/2}\pi^{3/2}.$$

With the values $\delta_G=0.383$ F^{-1}, and $\delta_I=1.263$ F^{-1}, used by Schiff,[23] the predicted normalizations for (t,d) reactions are $N_G=1.43$, and $N_I=3.03$.

These values are considerably smaller than the normalization found with the Irving-Gunn wave function and, as we discuss below, lead to spectroscopic factors somewhat larger than expected on reasonable nuclear models.

These results then agree with the analysis of Griffy and Oakes,[7] who find that both the Gaussian and Irving wave functions underestimate the electron-proton coincidence cross section in the inelastic scattering of electrons off ^{3}He ions.

V. SUMMARY

The results of our application indicate that the distorted-waves theory is capable of predicting both the shape and the magnitude of the (^{3}He,d) cross section. The theoretical angular distribution is most reminiscent of data when the finite-range modifications are included. The magnitude of the cross section is better reproduced in the local form of the theory.

The enhanced cross section, and consequent smaller spectroscopic factor, found in the nonlocal form of the theory is almost wholly due to the fact that the nonlocal bound proton wave function is some 14% larger than the local wave function at large radii. Our preliminary results then suggest that the nonlocality range of the shell-model potential is somewhat less than 0.85 F, at least for the case we are considering here.

An alternative, and perhaps superior, method of examining the consistency of the distorted waves theory is to study the (^{3}He,d) reaction over a range of target nuclei. With the aid of sum rules, a reliable estimate of the normalization could be found. Such a survey has recently been completed by Armstrong and Blair,[4] who have studied this reaction on targets with 28 neutrons. Using the local theory in zero-range approximation, these authors find a normalization factor, $N=3.8\pm0.7$, which is to be compared with our predicted $N=4.4$. Their result, along with our findings, may suggest that the use of the Irving-Gunn wave function leads to an overestimate of order 20%. On the other hand, uncertainties in optical parameters and the bound-state orbital could easily lead to errors of this magnitude.

Studies of the (d,t) reaction provide another test of the normalization. Thus, Fulmer and Daehnick,[24] using our predicted factor of 3.33, have found reasonable agreement with spectroscopic factors from pairing calculations for nickel isotopes, and Bjerregaard *et al.*[25] find an empirical normalization of 3.25 in their studies of the ^{43}Ca(d,t)^{42}Ca reactions.

The weight of evidence seems to indicate that the absolute cross section for these reactions is reasonably predicted by using an Irving-Gunn wave function for the triton or ^{3}He ion. The normalization predicted by the Gaussian wave function is much too small, and the Irving wave function is admissible if the nonlocality of the shell-model potential is of the same order as the nonlocality of the optical potential.

We hope to address these questions in more detail in forthcoming publications. For the nonce we conclude that the distorted-wave theory in its present form gives fairly reliable spectroscopic information.

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[23] L. I. Schiff, Phys. Rev. **133**, B802 (1964).

[24] R. H. Fulmer and W. W. Daehnick, Phys. Rev. **139**, B579 (1965).

[25] J. H. Bjerregaard, H. R. Blieden, O. Hansen, and G. R. Satchler, Phys. Rev. **136**, B1348 (1964).

APPENDIX

We assume that the wave function of the fully space-symmetric state of the three-nucleon system satisfies a Schrödinger equation,

$$V_{dx}\phi(r,\rho)=[B_T-T_r-H_d(\rho)]\phi(r,\rho)\,, \qquad \text{(A1)}$$

with B_T the total binding energy of the system.

Substitution of (A1) into the overlap integral of (5) yields, with the aid of Green's theorem,

$$D(r)=\int d\rho\,\phi_d(\rho)[B_{\text{sep}}-T_r]\phi(r,\rho)\,. \qquad \text{(A2)}$$

Equation (A2) is the result of using the full interaction between the transferred nucleon and the deuteron. For the ^{3}He ion this can be regarded as the sum of nuclear and Coulomb contributions,

$$D(r)=D_{\text{nuclear}}(r)+D_c(r) \qquad \text{(A3)}$$

with

$$D_c(r)=\int d\rho\,\phi d(\rho)V_{\text{Coulomb}}(r,\rho)\phi(r,\rho)\,. \qquad \text{(A4)}$$

Integration over r then yields Eq. (13) for (t,d) reactions and Eqs. (14) and (15) for $(^3\text{He},d)$ reactions.

To evaluate these expressions, it is convenient to work with their Fourier transforms,

$$G(k^2)=\frac{1}{(2\pi)^3}\int d\mathbf{r}\,e^{i\mathbf{k}\cdot\mathbf{r}}D(r)$$

$$=-\frac{\hbar^2}{(2\mu)(2\pi)^3}(\gamma^2+k^2)\int d\mathbf{r}\,e^{i\mathbf{k}\cdot\mathbf{r}}\int d\rho\,\phi_d(\rho)\phi(r,\rho)\,, \qquad \text{(A5)}$$

where we have written $-(\hbar^2/2\mu)\gamma^2=B_{\text{sep}}$, and

$$G_C(k^2)=\frac{1}{(2\pi)^3}\int d\mathbf{r}\,e^{i\mathbf{k}\cdot\mathbf{r}}D_C(r)\,. \qquad \text{(A6)}$$

Except for factors, Eq. (A5) is exactly the integral considered by Griffy and Oakes.[7] Taking over their result, we find

$$G(k^2)=-\frac{256\pi^{3/2}AB(\beta^2-\alpha^2)}{3^{1/2}\delta^5}\frac{\hbar^2}{2\mu}(\gamma^2+k^2)$$

$$\times\int_0^\infty\frac{q^2dq}{(\alpha^2+q^2)(\beta^2+q^2)[1+8q^2/3\delta^2+2k^2/\delta^2]^{5/2}}$$

$$=(\hbar^2/2\mu)(\gamma^2+k^2)I(k^2)\,. \qquad \text{(A7)}$$

To cast (A6) into an expression suitable for numerical evaluation, it is convenient to take the Fourier transform of the Coulomb interaction and then to proceed in the manner described by Griffy and Oakes.[7] After lengthy algebra we find

$$G_C(k^2)=\frac{32\pi^{1/2}ABe^2(\beta^2-\alpha^2)}{3^{1/2}\delta}\int_0^\infty\frac{ds}{s^2}$$

$$\times\int_0^\infty\frac{qdq}{(\alpha^2+q^2)(\beta^2+q^2)}\{[1+Q^-+S^-]^{-1/2}$$

$$-[1+Q^-+S^+]^{-1/2}-[1+Q^++S^-]^{-1/2}$$

$$+[1+Q^++S^+]^{-1/2}\}\,, \qquad \text{(A8)}$$

with

$$Q^\pm=[1+8(q^2+s^2\pm qs)/3\delta^2]$$

and

$$S^\pm=2(k^2\pm 2ks)/\delta^2\,.$$

Expanding $I(k^2)$ and $G_C(k^2)$ in Taylor's series about $k^2=0$, we get

$$G(k^2)=\frac{\hbar^2}{2\mu}(\gamma^2+k^2)\left[I(0)+k^2\frac{d}{dk^2}I(k^2)\Big|_{k^2=0}+\cdots\right], \qquad \text{(A9)}$$

$$G_C(k^2)=G_C(0)+k^2\frac{d}{dk^2}G_C(k^2)\Big|_{k^2=0}+\cdots\,. \qquad \text{(A10)}$$

Then

$$D_0(t,d)=6.26I(0)\,,\quad D_0^a(^3\text{He},d)=5.49I(0)\,,$$

and

$$D_0^b(^3\text{He},d)=D_0^a-G_C(0)\,. \qquad \text{(A11)}$$

Equating the moments of the transform of the Gaussian to the appropriate combination of the moments of G and G_C gives

$$\frac{R^2}{4}(t,d)=-\left[\frac{1}{\gamma_t^2}+\frac{d}{dk^2}\ln I(k^2)\Big|_{k^2=0}\right], \qquad \text{(A12)}$$

$$\frac{R_a^2}{4}(^3\text{He},d)=-\left[\frac{1}{\gamma_3^2}+\frac{d}{dk^2}\ln I(k^2)\Big|_{k^2=0}\right], \qquad \text{(A13)}$$

and

$$\frac{R_b^2}{4}=-\left[\frac{\hbar^2}{2\mu}I(0)\left\{1+\gamma_3^2\frac{d}{dk^2}\ln I(k^2)\Big|_{k^2=0}\right\}\right.$$

$$\left.-\frac{dG_C}{dk^2}(k^2)\Big|_{k^2=0}\right]\left[\frac{\hbar^2}{2\mu}\gamma_3^2I(0)-G_C(0)\right]^{-1}. \qquad \text{(A14)}$$