PHYSICAL REVIEW VOLUME 149, NUMBER 2 16 SEPTEMBER 1966

Elastic Dielectric

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(Received 2 December 1965; revised manuscript received 21 April 1966)

A variational principle is used to derive field equations and boundary conditions for the description of the nonlinear behavior of an elastic dielectric in static equilibrium in an electric field. A discussion of the equilibrium conditions for a rigid dielectric is given to provide a check on the more general theory.

I. INTRODUCTION

IN this paper we derive, using a variational principle, a set of field equations and boundary conditions for the description of the nonlinear behavior of an elastic dielectric in static equilibrium in an electric field. This problem has been discussed in detail by Toupin[1] and Eringen[2]; their papers contain extensive lists of references to previous work. The treatment given below follows Toupin's work fairly closely both in spirit and in notation.

In summary, the method we have used is as follows—introducing a material energy density which is a point function of the displacement gradient and a vector field **D**, later to be identified with the electric displacement field, we demand that the total energy of an isolated system, consisting of a dielectric and a charged conductor, be an extremum with respect to arbitrary variations in these two fields. As constraints on the variations we ask that (a) **D** and its variations satisfy the field equation, div$\mathbf{D}=0$, at all points in space outside the conductor, (b) the surface integral of the normal component of **D** over the conductor is prescribed, and (c) the space displacements of the surface elements of the dielectric are prescribed. Constraint (a) is suggested by the properties of the electric displacement field in the case of a rigid dielectric while (b) and (c) reflect the physical limitations on the constraints one can apply in practice to such a system.

At the present time there is no suitable body of data[3] with which we might test the validity of these nonlinear theories. However, there is one minor test that we can apply. Since the standard theory of (a) the rigid dielectric and of (b) the electrically neutral elastic solid are experimentally well established, any nonlinear theory of the elastic dielectric should contain these theories. In applying our version of the nonlinear theory to the special case (a), we found that the expressions we required for the electrostatic force and torque on a rigid dielectric were not available in the literature. As the derivation of these expressions contains a number of points of interest for the general case, we describe it below.

This paper is divided into two parts. In Sec. II we discuss the rigid dielectric and in Sec. III the compressible or elastic dielectric. Section II consists of two subsections in which we describe a calculation of the electrostatic force and torque on a dielectric. The variational principle and the ensuing calculations of the field equations and boundary conditions are contained in Subsecs. 1 and 2 of Sec. III. This is followed by a description of the effect of the rotational invariance condition on the energy density function, Subsec. 3 and the application of the general theory to the cases (a) and (b) mentioned above, Subsec. 4.

II. THE RIGID DIELECTRIC

1. The Electrostatic Force on a Rigid Dielectric

A dielectric in an electric field will, in general, require the application of mechanical forces to keep it in static equilibrium. In the case of the rigid dielectric the only conditions that these mechanical forces must satisfy is that they balance the net electrostatic force and net electrostatic torque acting on the dielectric. We shall denote the electrostatic force and torque (about the origin of the coordinate axes) on the dielectric by **F** and **Γ**, respectively.

Let us consider, for simplicity, an isolated system consisting only of a rigid homogeneous dielectric and a charged conductor, both of finite extent. Suppose the dielectric is kept in static equilibrium by surface tractions **t**, then the equilibrium conditions are

$$\oint_V t_i dS + F_i = 0\,, \quad i=1,2,3 \tag{1}$$

$$e_{ijk}\oint_V t_k x_j dS + \Gamma_i = 0\,, \quad i=1,2,3 \tag{2}$$

where, for example, t_i is the ith component of the vector **t** and e_{ijk} is the Levi-Civita density; $e_{ijk}=+1(-1)$ if i, j, k is an even (odd) permutation of 1, 2, 3 and zero otherwise. The Einstein summation convention is used in these and all later expressions.

To derive expressions for **F** and **Γ** in terms of the electric field **E** and displacement field **D**, we note that if the dielectric suffers a displacement and at the same time the *net* charge on the conductor remains constant,

[1] R. A. Toupin, J. Rat. Mech. and Anal. **5**, 849 (1956); Arch. Rat. Mech. and Anal. **5**, 440 (1960).
[2] A. C. Eringen, Int. J. Engr. Sci. **1**, 127 (1965).
[3] The ferroelectric crystals are well-known examples of crystals which exhibit strong nonlinear properties. However, the customary measurements and analyses of these effects are based on an *ad hoc* extension of the linear theory. This procedure has not, as yet, been given a rational basis.

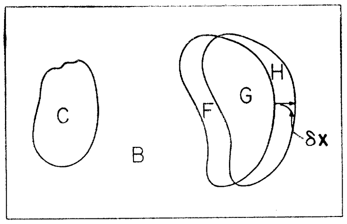

FIG. 1. A schematic diagram of the regions used in the text. C—the region occupied by the conductor; B—the region outside the conductor and dielectric before and after the rigid-body translation; $G+F$—the region occupied by the dielectric before the translation; $G+H$—the region occupied by the dielectric after the translation.

the work done by the surface tractions is equal to the change in the total energy of the system. Thus, by calculating an explicit expression for this energy change and substituting for the electrostatic force and torque, Eqs. (1) and (2), in the energy balance we obtain expressions for F_i and Γ_i.

The total electrostatic energy of the system can be written in the form

$$U=\int_{V+C+A}\Psi dv,$$

where the volume integral is extended throughout all space. The letters V, C, A refer to the dielectric, the conductor and free space, respectively. The energy density Ψ is defined

$$\Psi=\psi(D_i),\quad \text{in } V,$$
$$\Psi\equiv 0,\quad \text{in } C,$$
$$\Psi=D_iD_i/2\epsilon_0,\quad \text{in } A.$$

$\psi(D_i)$ is the material energy density of the dielectric, with the property that $E_i=(\partial\psi/\partial D_i)$, and ϵ_0 the permittivity of free space.[4]

Suppose the dielectric is subjected to an infinitesimal translation described by the vector $\delta\mathbf{x}$; if the net charge on the conductor remains constant during this process, the energy balance is

$$\delta U=\oint_V t_i\delta x_i dS. \tag{3}$$

It is convenient to introduce the following notation for different regions in the system (see Fig. 1): B is the region free of dielectric material (and conductor) before and after translation; G is the region containing dielectric material before and after the translation; F is the region from which dielectric material is removed; H is the region into which dielectric material is moved. Thus, for example, $V=G+F$ and $A=B+H$.

[4] L. D. Landau and E. M. Lifshitz, *Electrodynamics of Continuous Media* (Pergamon Press, Ltd., Oxford, England, 1960).

Let the displacement field after the translation be $\mathbf{D}'$; then the change in energy is

$$\delta U=\int_B[(D_i'D_i'-D_iD_i)/2\epsilon_0]dv+\int_G[\psi(D_i')-\psi(D_i)]dv$$
$$+\int_F\left[\frac{D_i'D_i'}{2\epsilon_0}-\psi(D_i)\right]dv+\int_H\left[\psi(D_i')-\frac{D_iD_i}{2\epsilon_0}\right]dv.$$

Since the translation is infinitesimal the change $\delta\mathbf{D}=\mathbf{D}'-\mathbf{D}$ is infinitesimal in the regions B and G. In the regions F and H, $\mathbf{D}'-\mathbf{D}$ is in general not infinitesimal; thus, to first order in infinitesimals,[5]

$$\delta U=\int_A\frac{D_i\delta D_i}{\epsilon_0}dv+\int_V\frac{\partial\psi}{\partial D_i}\delta D_idv$$
$$+\oint_V\left[\psi(D_i)-\frac{D_i^AD_i^A}{2\epsilon_0}\right]\delta x_jn_jdS.$$

Since $E_i=(\partial\psi/\partial D_i)$ in the dielectric and $E_i=D_i/\epsilon_0$ in free space, the right side can be written as

$$\delta U=\int_{A+V}E_i\delta D_idv+\oint_V\left[\psi(D_i)-\frac{D_i^AD_i^A}{2\epsilon_0}\right]\delta x_jn_jdS. \tag{4}$$

To eliminate the term containing the components δD_i, consider the following identities:

$$\phi_0q=\oint_C\phi_0D_in_idS$$

and

$$\phi_0q=\oint_C\phi_0D_i'n_idS,$$

where q is the net charge and ϕ_0 the potential of the conductor. Since the electrostatic potential and the normal components of $\mathbf{D}$ and $\mathbf{D}'$ are continuous across any dielectric–free-space interface, we may apply Gauss' theorem to the right sides of these identities to get,

$$\phi_0q=\int_{B+F+G+H}E_iD_idv \tag{5}$$

and

$$\phi_0q=\int_{B+F+G+H}E_iD_i'dv. \tag{6}$$

We emphasize that in Eq. (6) $\mathbf{E}$ is the electric field before the translation and $\mathbf{D}'$ the displacement field after the translation.

Subtract (5) from (6),

$$0=\int_{B+F+G+H}E_i(D_i'-D_i)dv.$$

[5] We use, when necessary, the superscripts A and V to distinguish field quantities in free space from those in the dielectric.

To terms linear in infinitesimals, the right side takes the form

$$0=\int_{A+V} E_i\delta D_i dv-\oint_{F\cap V} E_i{}^V(D_i{}^A-D_i{}^V)\delta x_j n_j dS +\oint_{H\cap V} E_i{}^A(D_i{}^V-D_i{}^A)\delta x_j n_j dS, \quad (7)$$

where, for example, the subscripts $F\cap V$ denotes the surface common to the regions F and V. As before, the superscripts distinguish the free space and dielectric components of the vectors **E** and **D**.

Physically, the form of the expression on the right of (7) cannot depend on the direction of the translation, i.e., the sign of $\delta x_j n_j$; thus, we must also have

$$0=\int_{A+V} E_i\delta D_i dv+\oint_{F\cap V} E_i{}^A(D_i{}^V-D_i{}^A)\delta x_j n_j dS -\oint_{H\cap V} E_i{}^V(D_i{}^A-D_i{}^V)\delta x_j n_j dS. \quad (8)$$

Equations (7) and (8) are equivalent, for the following identity holds

$$0=\oint_{F\cap V}(E_i{}^V-E_i{}^A)(D_i{}^A-D_i{}^V)\delta x_j n_j dS +\oint_{H\cap V}(E_i{}^A-E_i{}^V)(D_i{}^V-D_i{}^A)\delta x_j n_j dS. \quad (9)$$

To prove this last statement, we note that the tangential components of **E** and the normal component of **D** are continuous across a dielectric–free-space interface; hence, the integrands appearing in (9) are identically zero.

From (7) and (8),

$$\int_{A+V} E_i\delta D_i dv=-\frac{1}{2}\oint_V (E_i{}^A+E_i{}^V) \times(D_i{}^V-D_i{}^A)\delta x_j n_j dS. \quad (10)$$

Combining (4) and (10), we get

$$\delta U=-\oint_V\left[\frac{1}{2}(E_i{}^V+E_i{}^A)(D_i{}^V-D_i{}^A) +\frac{D_i{}^A D_i{}^A}{2\epsilon_0}-\psi(D_i)\right]\delta x_j n_j dS. \quad (11)$$

Hence, using Eqs. (1), (3), and (11), and the fact that $\delta\mathbf{x}$ is arbitrary we have that the electrostatic force on the dielectric is[6]

$$F_i=\oint_V\left[\frac{1}{2}(E_j{}^V+E_j{}^A)(D_j{}^V-D_j{}^A) +\frac{D_j{}^A D_j{}^A}{2\epsilon_0}-\psi(D_j)\right]n_i dS, \quad i=1,2,3. \quad (12)$$

The results of the following discussion are required in the next section. Consider the expression for δU given by Eq. (4). Apply Gauss' theorem to the surface integral which contains $\psi(D_i)$ in the integrand; thus,

$$\delta U=\int_V E_i\delta D_i dv+\int_V\frac{\partial\psi}{\partial x_j}\delta x_j dv +\int_A E_i\delta D_i dv-\oint_V\frac{D_i{}^A D_i{}^A}{2\epsilon_0}\delta x_j n_j dS.$$

Since (a) $\partial\psi/\partial D_i=E_i$ and (b) the dielectric is homogeneous, i.e., $\partial\psi/\partial x_j=(\partial\psi/\partial D_i)(\partial D_i/\partial x_j)$, an alternative expression for δU, correct to first order in the infinitesimals, is

$$\delta U=\int_V\frac{\partial\psi}{\partial D_i}\left(\delta D_i+\frac{\partial D_i}{\partial x_j}\delta x_j\right)dv +\int_A E_i\delta D_i dv-\int_V\frac{D_i{}^A D_i{}^A}{2\epsilon_0}\delta x_j n_j dS.$$

It is easy to see that the integrand of the first integral on the right can be interpreted as the change in energy density associated with a *volume element of dielectric dv* originally at a point **x** where the displacement field is $\mathbf{D}(\mathbf{x})$ and moved to a point $\mathbf{x}+\delta\mathbf{x}$ where the displacement field is $\mathbf{D}+\delta\mathbf{D}+\delta\mathbf{x}\cdot\mathrm{grad}(\mathbf{D}+\delta\mathbf{D})\cong\mathbf{D}+\delta\mathbf{D}+\delta\mathbf{x}\cdot\mathrm{grad}\mathbf{D}$.[7]

2. The Electrostatic Torque

In the case of an anisotropic dielectric the energy density function ψ depends on the choice of coordinate frame in which we measure the components of **D**; it is thus convenient to introduce a set of coordinate axes, the material axes, fixed in the dielectric and to use only the function ψ referring to these axes. If the dielectric is

[6] It is of interest to note that when this method is applied to the determination of the electrostatic force on one conductor due to another, the term corresponding to the first integral on the right of Eq. (16) vanishes. In addition $\psi\equiv 0$ for a conductor; hence Eq. (16) leads to the correct expression for the force on the conductor, viz.,

$$\oint_{\text{Conductor}}\left(\frac{D_i{}^A D_i{}^A}{2\epsilon_0}\right)n_j dS=\oint_{\text{Conductor}}\left(\frac{\omega E_j{}^A}{2}\right)dS, \quad (j=1,2,3),$$

where ω is the surface charge density on the conductor.

[7] If the point **x** lies outside the dielectric after the translation is carried out, the quantity $\mathbf{D}+\delta\mathbf{D}$ is *not* the displacement field $\mathbf{D}'$ at **x** but rather the *functions* which give the physical displacement field at points within the dielectric; thus $\delta\mathbf{D}$ is an infinitesimal.

both rigid and homogeneous, a single set of material axes is sufficient to describe the total energy U.

Let the material axes be X_1, X_2, X_3. If the origin of this coordinate system and that of the laboratory coordinate system x_1, x_2, x_3 coincide, then the coordinates of any point measured in these two frames satisfy an equation of the form $x_i=\omega_{ij}X_j$, where ω_{ij} is the cosine of the angle between the x_i and X_j axes. If we denote the components of $\mathbf{D}$ measured in the material frame by π_i, then $D_i=\omega_{ij}\pi_j$ and $\pi_j=\omega_{ij}D_i$.

The energy density ψ referred to the material axes is a function of the arguments π_i and $\pi_i\pi_i$. We put $\psi=\psi(\pi_i,a)$ where $a=(\pi_i\pi_i)$. In terms of the D_i, $\psi=\psi(\omega_{ij}D_i,a)$ with $a=(D_iD_i)$. Furthermore, since $E_i=(\partial\psi/\partial D_i)_\omega$ and $(\partial\pi_j/\partial D_i)_\omega=\omega_{ij}$, $(\partial a/\partial D_i)_\omega=2D_i$, then

$$E_i=\left(\frac{\partial\psi}{\partial\pi_j}\right)_a\omega_{ij}+2\left(\frac{\partial\psi}{\partial a}\right)_\pi D_i. \tag{13}$$

Suppose that the dielectric is subjected to an infinitesimal rotation about a line through the common origin. Let the new transformation tensor be $\omega_{ij}+\delta\omega_{ij}$. An element originally at a point with coordinates x_i is displaced to a point with coordinates $x_i+\delta x_i$, where $\delta x_i=\delta\omega_{ij}X_j$; eliminating X_j from this last expression we get

$$\delta x_i=x_k\delta\Omega_{ik}, \tag{14}$$

where $\delta\Omega_{ik}=\omega_{kj}\delta\omega_{ij}$.

It is easy to show that

$$\delta\Omega_{ik}=-\delta\Omega_{ki} \tag{15}$$

and

$$\delta\omega_{ij}=\omega_{kj}\delta\Omega_{ik}. \tag{16}$$

From (14) and (15) we deduce that $\delta_{ik}+\delta\Omega_{ik}$ is the rotation matrix, with respect to the laboratory frame, describing the infinitesimal rotation. δ_{ij} is the Kronecker delta.

Consider now the change in energy density of an element of dielectric moved, as a result of the rotation, from the point $\mathbf{x}$ to the point $\mathbf{x}+\delta\mathbf{x}$; using the argument presented at the end of Subsec. 1, we write for this change

$$\delta\psi=\psi[(\omega_{ij}+\delta\omega_{ij})(D_i+\delta D_i+\partial D_i/\partial x_k\delta x_k),\ a+\delta a]-\psi(\omega_{ij}D_i,a),$$

where

$$\delta a=2D_i(\delta D_i+\partial D_i/\partial x_j\delta x_j) \tag{17}$$

and $\delta\mathbf{D}$ is the infinitesimal change in the field at the point $\mathbf{x}$ resulting from the rotation.[7]

To a first approximation

$$\delta\psi=\left(\frac{\partial\psi}{\partial\pi_j}\right)_a(\omega_{ij}\delta D_i+\omega_{ij}\partial D_i/\partial x_k\delta x_k+D_i\delta\omega_{ij})+\left(\frac{\partial\psi}{\partial a}\right)_\pi\delta a.$$

From (13) and (17),

$$\delta\psi=E_i(\delta D_i+\partial D_i/\partial x_k\delta x_k)+(\partial\psi/\partial\pi_j)_aD_i\delta\omega_{ij}. \tag{18}$$

The total energy change δU is

$$\delta U=\int_V\delta\psi dv-\oint_V\frac{D_j{}^AD_j{}^A}{2\epsilon_0}\delta x_in_idS+\int_A\frac{D_i\delta D_i}{\epsilon_0}dv$$

$$=\int_{V+A}E_i\delta D_idv+\int_V\left[E_i\frac{\partial D_i}{\partial x_k}\delta x_k+\left(\frac{\partial\psi}{\partial\pi_j}\right)_aD_i\delta\omega_{ij}\right]dv-\oint_V\frac{D_j{}^AD_j{}^A}{2\epsilon_0}\delta x_in_idS.$$

Hence from Eq. (10),

$$\delta U=-\frac{1}{2}\oint_V\left[(E_i{}^V+E_i{}^A)(D_i{}^V-D_i{}^A)+\frac{D_i{}^AD_i{}^A}{\epsilon_0}\right]\delta x_jn_jdS+\int_V\left[E_i\frac{\partial D_i}{\partial x_k}\delta x_k+\left(\frac{\partial\psi}{\partial\pi_j}\right)_aD_i\delta\omega_{ij}\right]dv.$$

Eliminating the infinitesimals δx_i and $\delta\omega_{ij}$ by means of (14) and (16), we get

$$\delta U=-\frac{1}{2}\oint_V\left[(E_t{}^V+E_t{}^A)(D_t{}^V-D_t{}^A)+\frac{D_t{}^AD_t{}^A}{\epsilon_0}\right]\times x_kn_j\delta\Omega_{jk}dS+\int_V\left[E_t\frac{\partial D_t}{\partial x_j}x_k+\left(\frac{\partial\psi}{\partial\pi_t}\right)_aD_j\omega_{kt}\right]\delta\Omega_{jk}dv.$$

The energy balance, (3), for this infinitesimal rotation is

$$\delta U=\oint_V t_jx_k\delta\Omega_{jk}dS.$$

Hence, using (a) the equilibrium condition (2), (b) the fact that $\delta\Omega_{ij}$ is an arbitrary antisymmetric tensor, and (c) the identity $e_{ijk}D_jD_k(\partial\psi/\partial a)=0$, we obtain the following expression for the electrostatic torque,

$$\Gamma_i=e_{ijk}\left\{\frac{1}{2}\oint_V\left[(E_t{}^V+E_t{}^A)(D_t{}^V-D_t{}^A)+\frac{D_t{}^AD_t{}^A}{\epsilon_0}\right]\times x_jn_kdS-\int_V\left(E_t\frac{\partial D_t}{\partial x_k}x_j+E_jD_k\right)dv\right\}. \tag{19}$$

Equations (12) and (19) provide explicit expressions for the components of the electrostatic force and torque on a rigid dielectric in the presence of a charged conductor.

III. THE COMPRESSIBLE DIELECTRIC

1. Variational Principle

Consider[8] an isolated system consisting of a conductor and a homogeneous dielectric, both of finite extent. Let us suppose that initially there is no charge distribution on the conductor or dipole moment distribution in the

[8] This is an extension of the variational principle introduced in Ref. 7, Chap. 2; see also H. F. Tiersten, J. Math. Phys. 6, 779 (1965).

dielectric. We introduce a Cartesian coordinate system X_1, X_2, X_3, fixed in the laboratory, to describe the position of each element of the dielectric in this initial state. Let charges be brought from infinity and placed on the conductor and let mechanical surface tractions be applied to both the conductor and dielectric. If this process is carried out quasistatically and the conductor and dielectric kept in static equilibrium throughout, then the total mechanical and electric work done on the system is stored as potential energy, U say. It is convenient to describe the positions of all points in space when the system is in this final state by means of a second Cartesian coordinate system x_1, x_2, x_3 fixed in the laboratory. The deformation suffered by the dielectric, as a result of the charges introduced and the surface tractions applied, is described by a mapping $x_i = x_i(X_K)(i, K=1, 2, 3)$, i.e., the volume element of dielectric originally at the point X_K is moved to the point x_i. The mapping constitutes a field defined within the dielectric; the functions $x_i(X_K)$ are assumed to be continuous and many times differentiable.

The total energy U of the system may be expressed as an integral of an energy density, Ψ, over all space, thus,

$$U=\int_{A+V+C}\Psi dv.$$

The subscripts refer to the regions of the system in the final state, free space—A, conductor—C, and dielectric—V.

Our choice of energy density is

$$\begin{aligned}\Psi&=\psi(x_{i;K},D_i) &&\text{in the dielectric}\\ \Psi&=D_iD_i/2\epsilon_0 &&\text{in free space}\\ \Psi&\equiv 0 &&\text{in the conductor}.\end{aligned}$$

ψ is a function characteristic of the dielectric; the deformation gradients $x_{i;K}=(\partial x_i/\partial X_K)$; $\mathbf{D}$ is some vector spanning all space outside the conductor and the D_i are the components of $\mathbf{D}$ measured in the x_1, x_2, x_3 coordinate system.

We shall demand that this vector $\mathbf{D}$ (i) satisfies the field equation $D_{i;i}=0$, (ii) is related to the net charge q on the conductor by the expression $\oint_C D_in_idS=q$, where the n_i are the direction cosines of the outward normals to the conductor surface. The field equation in (i) leads, in the usual way, to the jump condition, viz., the normal component of $\mathbf{D}$ is continuous across any dielectric–free-space interface.

Let the position vector of each volume element undergo a small variation $\mathbf{x}\rightarrow\mathbf{x}+\delta\mathbf{x}$. Similarly we allow the field $\mathbf{D}$ to undergo a small variation $\mathbf{D}\rightarrow\mathbf{D}'$; the variation in $\mathbf{D}$ is assumed to be small in the sense that the vector $\mathbf{D}'-\mathbf{D}$ is an infinitesimal at all points which are either outside or inside the dielectric throughout the deformation; thus, for these points we put $\mathbf{D}'-\mathbf{D}=\delta\mathbf{D}$.

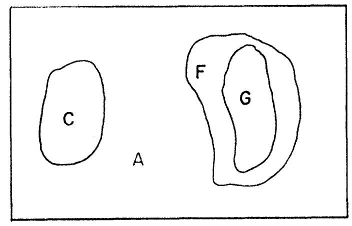

FIG. 2. A schematic diagram of the regions used in the text. C—the region occupied by the conductor; A—the region outside the conductor and dielectric before and after the virtual displacement of the dielectric; $G+F$—the region occupied by the dielectric before the virtual displacement; G—the region occupied by the dielectric after the virtual displacement.

We take as our variational principle that U be an extremum with respect to the small variations in $\mathbf{D}$ and $\mathbf{x}$ which are, however, subject to the following constraints

(a) $\delta x_i=$ constant $(i=1, 2, 3)$ for the surface elements of the dielectric, and

(b) $\delta q=$ constant, i.e., $\oint_C \delta D_in_idS=$ constant.

This choice, (a) and (b), reflects the physical limitations on the constraints one can apply in practice to such a system. In other words, we are free to constrain the net charge on the conductor to any value and the surface elements of the dielectric to any positions we please, but once this choice of constraint is made the corresponding charge distribution on the conductor and the mass distribution in the dielectric, for example, are determined by the physical laws describing the system. It is these physical laws, in the form of field equations and boundary conditions, that we seek from the variational principle.

To simplify the following analysis, we shall assume that for all the surface elements the variation $\delta\mathbf{x}$ is such that $\delta x_in_i<0$, where the n_i are the direction cosines of the outward normals to the dielectric surface. Let the new region occupied by the dielectric after the variation be denoted by G (see Fig. 2) and the region now free of dielectric by F. Thus, $V=F+G$. As a result of this choice of surface displacements, there are no regions free of dielectric before and occupied by dielectric after the variation. This restriction, while simplifying our discussion, in no way impairs the generality of the results.

Introducing Lagrangian multipliers t_i, ϕ_0 for the constraints (a) and (b) above and ϕ for the constraint imposed by the field equation div$\mathbf{D}=0$, we write the condition for an extremum in the form

$$\delta U-\int_{A+G}\phi(\delta D_i)_{;i}dv-\phi_0\oint_C\delta D_in_idS-\oint_V t_i\delta x_idS=0. \qquad (20)$$

The variations δx_i, δD_i may now be treated as completely arbitrary fields.

2. Field Equations and Boundary Conditions

The variation in U is (see II)

$$\delta U=\int_V \psi[x_{i;K}+(\delta x_i)_{;K},\, D_i+\delta D_i+D_{i;j}\delta x_j][1+(\delta x_k)_{;k}]dv-\int_V \psi(x_{i;K},D_i)dv-\oint_V \frac{D_j{}^A D_j{}^A}{2\epsilon_0}\delta x_i n_i dS+\int_A \frac{D_j\delta D_j dv}{\epsilon_0}.$$

The superscripts A are used to emphasize that the $D_i{}^A$ are free-space components of $\mathbf{D}$.

Expand to terms linear in δx_i and δD_i,

$$\delta U=\int_V\left[\frac{\partial\psi}{\partial x_{i;K}}(\delta x_i)_{;K}+\frac{\partial\psi}{\partial D_i}(\delta D_i+D_{i;j}\delta x_j)+\psi(\delta x_k)_{;k}\right]dv-\oint_V\frac{D_j{}^A D_j{}^A}{2\epsilon_0}\delta x_i n_i dS+\int_A\frac{D_j\delta D_j}{\epsilon_0}dv.$$

Using the identity $(\delta x_i)_{;K}=(\delta x_i)_{;j}(x_j)_{;K}$ and Gauss' theorem we obtain the expression

$$\delta U=\int_V\left\{\left[\frac{\partial\psi}{\partial D_i}D_{i;j}-\left(\frac{\partial\psi}{\partial x_{j;K}}x_{i;K}\right)_{;i}-\psi_{;j}\right]\delta x_j+\frac{\partial\psi}{\partial D_i}\delta D_i\right\}dv$$
$$+\oint_V\left[\frac{\partial\psi}{\partial x_{i;K}}x_{j;K}n_j+\psi n_i-\frac{D_j{}^A D_j{}^A}{2\epsilon_0}n_i\right]\delta x_i dS+\int_A\frac{D_j\delta D_j}{\epsilon_0}dv. \quad (21)$$

Consider now the second term in Eq. (20). This can be written in the form

$$\int_{A+G}\phi(\delta D_i)_{;i}dv=\int_{A+G}(\phi\delta D_i)_{;i}dv-\int_{A+G}\phi_{;i}\delta D_i dv$$

or, from Gauss' theorem,

$$\int_{A+G}\phi(\delta D_i)_{;i}dv=-\oint_C\phi\delta D_i n_i dS-\oint_{A\cap F}\phi^A\delta D_i{}^A n_i dS+\oint_G\phi^V\delta D_i{}^V n_i dS-\int_{A+G}\phi_{;i}\delta D_i dv, \quad (22)$$

where $A\cap F$ refers to the surface common to regions A and F and the n_i are the direction cosines of the outward normals to the surfaces of the conductor and dielectric. We have assumed that $\phi\to 0$ and $\delta D_i\to 0$ at infinity sufficiently quickly that the surface integral, $\oint\phi\delta D_i n_i dS$ at infinity vanishes.

In terms of the primed notation, we have

$$-\oint_{A\cap F}\phi^A\delta D_i{}^A n_i dS+\oint_G\phi^V\delta D_i{}^V n_i dS=-\oint_{A\cap F}\phi^A(D_i{}^{A'}-D_i{}^A)n_i dS+\oint_G\phi^V(D_i{}^{V'}-D_i{}^V)n_i dS.$$

Consider the *functions* ϕ^A and $D_i{}^A$, originally introduced for points within A, at points in F. Adding and subtracting a term involving these functions, we get

$$-\oint_{A\cap F}\phi^A\delta D_i{}^A n_i dS+\oint_G\phi^V\delta D_i{}^V n_i dS=\oint_{A\ \ F}(\phi^V-\phi^A)(D_i{}^{A'}-D_i{}^A)n_i dS-\oint_{A\cap F}\phi^V(D_i{}^{A'}-D_i{}^A)n_i dS$$
$$+\oint_G\phi^V(D_i{}^{V'}-D_i{}^V)n_i dS=\oint_{A\cap F}(\phi^V-\phi^A)(D_i{}^{A'}-D_i{}^A)n_i dS+\int_F\phi_{;i}{}^V(D_i{}^{A'}-D_i{}^V)dv.$$

In this last step, we have used Gauss' theorem and the jump conditions $D_i{}^{A'}n_i=D_i{}^{V'}n_i$ on G and $D_i{}^A n_i=D_i{}^V n_i$ on V. Hence,

$$\int_{A+G}\phi(\delta D_i)_{;i}dv=-\int_{A+G}\phi_{;i}\delta D_i dv-\oint_C\phi\delta D_i n_i dS+\oint_{F\cap A}(\phi^V-\phi^A)\delta D_i{}^A n_i dS+\int_F\phi_{;i}{}^V(D_i{}^{A'}-D_i{}^V)dv.$$

In the linear approximation this equation reads

$$\int_{A+V}\phi(\delta D_i)_{;i}dv=-\int_{A+V}\phi_{;i}\delta D_i dv-\oint_C\phi\delta D_i n_i dS+\oint_A(\phi^V-\phi^A)\delta D_i{}^A n_i dS+\oint_V\phi_{;i}{}^V(D_i{}^A-D_i{}^V)\delta x_j n_j dS.$$

Since the normal component of $\mathbf{D}$ and the tangential component of grad ϕ[9] are continuous across a dielectric–free-space interface, the integrand in the last integral on the right may be replaced by an alternative expression; thus,

$$\oint_{A+G}\phi(\delta D_i)_{;i}dv=-\int_{A+V}\phi_{;i}\delta D_i dv-\oint_C\phi\delta D_i n_i dS+\oint_V(\phi^V-\phi^A)\delta D_i{}^A n_i dS+\frac{1}{2}\oint_V(\phi_{;i}{}^A+\phi_{;i}{}^V)(D_i{}^A-D_i{}^V)\delta x_j n_j dS. \quad (23)$$

Hence, the condition for an extremum is, from (20), (21), and (23),

$$\int_V\left\{\left[\frac{\partial\psi}{\partial D_i}D_{i;j}-\left(\frac{\partial\psi}{\partial x_{j;K}}x_{i;K}\right)_{;i}-\psi_{;j}\right]\delta x_j+\left(\frac{\partial\psi}{\partial D_i}+\phi_{;i}\right)\delta D_i\right\}dv+\oint_V\left\{(\phi^A-\phi^V)\delta D_i{}^A n_i+\left[\left(\frac{\partial\psi}{\partial x_{i;K}}x_{j;K}\right)n_j+\psi n_i\right.\right.$$
$$\left.\left.-\tfrac{1}{2}(\phi_{;j}{}^A+\phi_{;j}{}^V)(D_j{}^A-D_j{}^V)n_i-\frac{D_j{}^A D_j{}^A}{2\epsilon_0}n_i-t_i\right]\delta x_i\right\}dS+\int_A\left(\frac{D_j}{\epsilon_0}+\phi_{;j}\right)\delta D_j dv+\oint_C(\phi-\phi_0)\delta D_i n_i dS=0.$$

Equating the coefficients of the arbitrary changes δx_i and δD_i to zero, we obtain the following field equations and boundary conditions.

In the dielectric,

$$\left(\frac{\partial\psi}{\partial x_{i;K}}x_{j;K}\right)_{;j}+\psi_{;i}-\frac{\partial\psi}{\partial D_j}D_{j;i}=0, \quad (24)$$

$$\frac{\partial\psi}{\partial D_i}+\phi_{;i}=0. \quad (25)$$

In free space,

$$D_i/\epsilon_0+\phi_{;i}=0. \quad (26)$$

At the surface of the dielectric

$$\phi^V-\phi^A=0, \quad (27)$$

$$\frac{\partial\psi}{\partial x_{i;K}}x_{j;K}n_j+\psi n_i-\left[\frac{D_j{}^A D_j{}^A}{2\epsilon_0}-\tfrac{1}{2}(\phi_{;j}{}^V+\phi_{;j}{}^A)\times(D_j{}^V-D_j{}^A)\right]n_i-t_i=0. \quad (28)$$

At the surface of the conductor

$$\phi-\phi_0=0. \quad (29)$$

We may now give the Lagrangian multipliers ϕ, ϕ_0, and t_i and the field $\mathbf{D}$ a physical interpretation. ϕ is the electrostatic potential, continuous across the dielectric–free-space interface (27), and constant over the conductor surface, (29). ϕ_0 is the potential of the conductor. We set $\mathbf{E}=-\text{grad}\phi$; thus, $\mathbf{E}$ is the macroscopic electric field and $\mathbf{D}$ the electric displacement field, (25) and (26). Finally, we show that $\mathbf{t}$ is the stress field due to the applied mechanical surface tractions. Consider two solutions to the above field equations and boundary conditions D_i, x_i and $D_i+\delta D_i$, $x_i+\delta x_i$, where δD_i and δx_i are infinitesimals. In particular, we have $(D_i+\delta D_i)_{;i}=0$ so that from Eq. (20) we obtain

$$\delta U=\phi_0\oint_C\delta D_i n_i dS+\oint_V t_i\delta x_i dS.$$

The terms on the right are the electric and mechanical work and hence $\mathbf{t}$ is the applied stress field.

To recapitulate, we have derived the following field equations and boundary conditions to describe the behavior of an elastic dielectric in static equilibrium in the presence of a charged conductor.

$$((\partial\psi/\partial x_{i;K})x_{j;K}+\psi\delta_{ij})_{;j}-E_j D_{j;i}=0, \quad \text{in } V, \quad (30)$$

$$\text{div}\mathbf{D}=0, \quad \text{in } V \text{ and } A, \quad (31)$$

$$\text{curl}\mathbf{E}=0, \quad \text{in } V \text{ and } A, \quad (32)$$

$$t_i=\left(\frac{\partial\psi}{\partial x_{i;K}}x_{j;K}+\psi\delta_{ij}\right)n_j-\left[\frac{D_j{}^A D_j{}^A}{2\epsilon_0}+\tfrac{1}{2}(E_j{}^A+E_j{}^V)(D_j{}^V-D_j{}^A)\right]n_i, \quad \text{on } V, \quad (33)$$

$$D_i{}^V n_i=D_i{}^A n_i, \quad \text{on } V, \quad (34)$$

$$e_{ijk}n_j E_k{}^V=e_{ijk}n_j E_k{}^A, \quad \text{on } V, \quad (35)$$

$$\oint_C D_i n_i dS=q. \quad (36)$$

[9] The proof of this latter statement follows immediately from the identity curl grad $\phi\equiv 0$.

The following results are required in the discussion in Subsec. 4.

The net mechanical force acting on the dielectric is $\oint_V t_i dS$. From Eq. (33),

$$\oint_V t_i dS = \oint_V \left\{ \left(\frac{\partial \psi}{\partial x_{i;K}} x_{j;K} + \psi \delta_{ij} \right) n_j - \left[\frac{D_j^A D_j^A}{2\epsilon_0} + \tfrac{1}{2}(E_j^A + E_j^V)(D_j^V - D_j^A) \right] n_i \right\} dS.$$

Using Gauss' theorem and Eq. (30) we may write the right-hand side in the form

$$\oint_V t_i dS = \oint_V E_j D_{j;i} dv - \frac{1}{2} \oint_V \left[\frac{D_j^A D_j^A}{\epsilon_0} + (E_j^A + E_j^V)(D_j^V - D_j^A) \right] n_i dS. \tag{37}$$

Similarly, the net mechanical torque about the origin of the coordinate system is

$$e_{ijk} \oint x_j t_k dS = e_{ijk} \left\{ \int_V \left(\frac{\partial \psi}{\partial x_{k;K}} x_{j;K} + x_j E_s D_{s;k} \right) dv - \frac{1}{2} \oint_V \left[\frac{D_s^A D_s^A}{\epsilon_0} + (E_s^A + E_s^V)(D_s^V - D_s^A) \right] n_k x_j dS. \tag{38}$$

3. Rotational Invariance

Since we are dealing with an isolated system, the energy density Ψ must be invariant with respect to an arbitrary rotation of the coordinate axes x_1, x_2, x_3. It is sufficient to consider only an infinitesimal rotation. Suppose such a rotation is described by the antisymmetric tensor $\delta_{ij} + \Omega_{ij}$, where $\Omega_{ij} = -\Omega_{ji}$ and $\Omega_{ij} \ll 1$; the transformed position coordinates and displacement components are then

$$x_i' = (\delta_{ij} + \Omega_{ij}) x_j, \tag{39}$$

$$D_i' = (\delta_{ij} + \Omega_{ij}) D_j. \tag{40}$$

The rotational invariance[1] condition is

$$\Psi(x_{i;K}', D_i') = \Psi(x_{i;K}, D_i). \tag{41}$$

This identity is satisfied trivially by the free-space energy density $\Psi = D_i D_i / 2\epsilon_0$ but not by the material energy density $\Psi = \psi$. To determine what restrictions are imposed on ψ by (41), we substitute for x_i' and D_i' from (39) and (40) into (41) and expand to terms linear in Ω_{ij}; thus,

$$(\partial \psi / \partial x_{i;K}) \Omega_{ij} x_{j;K} + (\partial \psi / \partial D_i) \Omega_{ij} D_j = 0.$$

Since Ω_{ij} is an arbitrary anti symmetric tensor,

$$e_{ijk} \left[\frac{\partial \psi}{\partial x_{j;K}} x_{k;K} + \frac{\partial \psi}{\partial D_j} D_k \right] = 0. \tag{42}$$

Equation (42) provides three partial differential equations to be satisfied by ψ. From a theorem due to Cauchy[1] we conclude that ψ is a function of the 12 variables $x_{i;K}$, D_i through the 10 variables $x_{i;K} x_{i;L}$, $x_{i;K} D_i$, $D_i D_i$. We set $\eta_{KL} = \frac{1}{2}(x_{i;K} x_{i;L} - \delta_{KL})$, $\pi_K = x_{i;K} D_i$, and $a = D_i D_i$ and put $\psi = \psi(\eta_{KL}, \pi_K, a)$. The quantity η_{KL} is the Green strain tensor.

The following identities are easily proved.

$$(\partial \eta_{AB} / \partial x_{i;C}) x_{j;C} = \tfrac{1}{2}(x_{i;B} x_{j;A} + x_{i;A} x_{j;B}),$$

$$(\partial \pi_A / \partial x_{i;B}) x_{j;B} = x_{j;A} D_i,$$

$$\partial \pi_A / \partial D_i = x_{i;A}.$$

Hence,

$$\frac{\partial \psi}{\partial x_{i;K}} x_{j;K} = \frac{1}{2} \left(\frac{\partial \psi}{\partial \eta_{AB}} \right)_{\pi, a} (x_{i;B} x_{j;A} + x_{i;A} x_{j;B}) + \left(\frac{\partial \psi}{\partial \pi_C} \right)_{\eta, a} x_{j;C} D_i \tag{43}$$

and

$$E_i = \left(\frac{\partial \psi}{\partial D_i} \right) = \left(\frac{\partial \psi}{\partial \pi_C} \right) x_{i;C} + 2 \left(\frac{\partial \psi}{\partial a} \right) D_i. \tag{44}$$

4. Two Limiting Cases

We now show that the field equations and boundary conditions derived in Secs. 1 and 2 exhibit, as special cases, the standard treatment of (a) the rigid dielectric and of (b) the electrically inert elastic solid. This exercise serves as a minor check on our results.

(a) For a rigid dielectric, Eqs. (31) and (32) and the corresponding boundary conditions (34), (35), and (36) are unchanged; these agree with the standard results, (see part II). Equation (30) is satisfied identically; this can be seen most easily if we regard the rigid dielectric as the limiting case of an elastic dielectric for which the elastic constants tend to infinity. Under these circumstances, Eq. (30) yields as a solution a zero strain tensor independently of the electrical conditions.

The boundary condition (33) provides an expression for the net mechanical force and net mechanical torque, (37), (38). To bring the right side of Eq. (37) into the form given in Sec. II, we note that for the rigid dielectric $(x_{i;K})_{;j} = 0$ and so $\psi_{;i} \equiv (\partial \psi / \partial D_j) D_{j;i}$. Hence, by applying Gauss' theorem and this last identity to Eq. (37), we get

$$\oint_V t_i dS = \oint_V \left\{ \psi - \left[\frac{D_j^A D_j^A}{2\epsilon_0} + \tfrac{1}{2}(E_j^A + E_j^V) \times (D_j^A - D_j^V) \right] \right\} n_i dS. \tag{45}$$

From Eqs. (38) and (43) we have, since $\partial\psi/\partial\eta_{AB}\equiv 0$,

$$e_{ijk}\oint_V x_j t_k dS = e_{ijk}\left\{\int_V\left[\left(\frac{\partial\psi}{\partial\pi_C}\right) x_{j;C}D_i + x_j E_s D_{s;k}\right]dv - \frac{1}{2}\oint_V\left[\frac{D_s^A D_s^A}{\epsilon_0} + (E_s^A + E_s^V)(D_s^V - D_s^A)\right]n_k x_j dS\right.$$

$$= e_{ijk}\left\{\int_V (E_j D_i + x_j E_s D_{s;k})dv - \frac{1}{2}\oint_V\left[\frac{D_s^A D_s^A}{\epsilon_0} + (E_s^A + E_s^V)(D_s^V - D_s^A)\right]n_k x_j dS.\right. \quad (46)$$

In this last step we have used the identity

$$e_{ijk}D_j D_k(\partial\psi/\partial a) = 0$$

and the expression for E_i given by Eq. (43). The expressions in Eqs. (45) and (46) agree with the results obtained in Sec. II, Eqs. (12), (19).

Finally, we note that for a rigid body the deformation gradients $x_{i;K}$ are constants which may be identified with the direction cosines defining the rotational portion of the coordinate transformation relating the two coordinate axes x_1, x_2, x_3 and X_1, X_2, X_3. In the notation of I, $x_{i;K} = \omega_{iK}$.

(b) For the electrically neutral elastic solid, $\mathbf{D}\equiv 0$, $\mathbf{E}\equiv 0$, and $\phi\equiv 0$; hence, the field equations and boundary conditions reduce to the following:

$$((\partial\psi/\partial x_{i;K})x_{j;K} + \psi\delta_{ij})_{;j} = 0 \quad (47)$$

and

$$t_i = ((\partial\psi/\partial x_{i;K})x_{j;K} + \psi\delta_{ij})n_j. \quad (48)$$

Set $\psi = \rho\chi$, where ρ is the mass density. Then,

$$\frac{\partial\psi}{\partial x_{i;K}} = \rho\frac{\partial\chi}{\partial x_{i;K}} + \chi\frac{\partial\rho}{\partial x_{i;K}},$$

but $\rho = \rho_0/J$, where ρ_0 is the mass density in the initial state and $J = \det[x_{i;K}]$; hence,

$$\partial\rho/\partial x_{i;K} = -\rho_0/J^2(\text{cofactor of } x_{i;K}).$$

Thus,

$$\frac{\partial\psi}{\partial x_{i;K}}x_{j;K} = \rho\frac{\partial\chi}{\partial x_{i;K}}x_{j;K} - \rho_0\frac{\chi}{J^2}[\text{cofactor of } x_{i;K}]x_{j;K}$$

or

$$\frac{\partial\psi}{\partial x_{i;K}}x_{j;K} + \psi\delta_{ij} = \rho\frac{\partial\chi}{\partial x_{i;K}}x_{j;K}. \quad (49)$$

The expression on the right of Eq. (49) is the stress tensor of elasticity theory; Thus, Eq. (47) and (48) are the standard forms for the elastic field equation and boundary condition, respectively.[10]

5. Comment

The form of the boundary condition (33) is convenient to use as a basis for the analysis of the thermodynamics of nonlinear elastic dielectrics. We hope to report on such a study in a later paper.

ACKNOWLEDGMENT

Part of this work was carried out at the Division of Pure Chemistry, National Research Council, Ottawa, Canada.

[10] F. D. Murnaghan, Am. J. Math. 59, 235 (1937).