Theory of Toeplitz Determinants and the Spin Correlations of the Two-Dimensional Ising Model. I

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We study in detail the asymptotic behavior, for large separations, of the correlation between two spins in the case of the two-dimensional Ising model without magnetic field. In the limit of infinite separation, this correlation is equal to the square of the spontaneous magnetization per spin. This paper is devoted to answering the question how, for fixed temperature, the correlation approaches this limiting value. The investigation is restricted to the situation where the two spins under consideration lie on the same row of the two-dimensional lattice. There are three distinct cases where the fixed temperature is above, below, or equal to the critical temperature or the Curie temperature. In the first two cases, the limiting value is approached exponentially as a function of the separation, while at the critical temperature the correlation behaves asymptotically as the inverse fourth root of the separation. In this paper, we evaluate exactly the first four terms of the asymptotic expansion in the first case, the first three terms in the second case, and the coefficients of the terms proportional to the $-1/4$ and the $-9/4$ powers of the separation in the third case. All these results are obtained by first expressing the correlation as a Toeplitz determinant, and the method used is capable of giving in principle the entire asymptotic expansion for large classes of such determinants, when the size of the determinant approaches infinity. The explicit results for the two-dimensional Ising model also serve as an example where the prescription of summing the leading terms, or the most divergent terms, fails.

1. INTRODUCTION

It is the purpose of this paper to study the long-range order in the two-dimensional Ising lattice without magnetic field. More precisely, we investigate the question how the two-spin correlation function approaches, for large separations, the limiting value, which is the square of the spontaneous magnetization.

Over two decades ago, Onsager[1] gave a most remarkable treatment of the Ising model. His approach was later greatly simplified by Kaufman.[2] Shortly thereafter, Yang[3] calculated exactly the spontaneous magnetization. This calculation was in turn simplified by Montroll, Potts, and Ward,[4] who expressed the correlation function

$$S_N=\langle\sigma_{0,0}\sigma_{0,N}\rangle \tag{1.1}$$

between a spin at the site (0,0) and one at $(0,N)$ in the form of a Toeplitz determinant, and then used a theorem of Szego[5] to find the spontaneous magnetization

$$M=\sqrt{S_\infty}. \tag{1.2}$$

The theorem of Szego gives only the limiting value of S_N as $N\to\infty$. Here we pose the problem of finding the asymptotic behavior for S_N, the leading term being M^2. The basic idea is very roughly as follows. Consider an $N\times N$ Toeplitz determinant D_N, the elements of which are d_{i-j}. Let x_j satisfy the linear equations

$$\sum_{j=0}^{N-1} d_{i-j}x_j=\delta_{i0}. \tag{1.3}$$

Then, on the one hand,

$$D_{N-1}/D_N=x_0; \tag{1.4}$$

while on the other hand, when N is large, (1.3) may be solved approximately by iterating a Weiner-Hopf sum equation.[6,7] Therefore, we can find the asymptotic behavior of D_{N-1}/D_N, and hence, by combining with Szego's theorem,[5] that of D_N itself.

It must be emphasized that the main idea here is to make use of the similarity between a Toeplitz determinant and the corresponding Wiener-Hopf sum equation. This similarity may be utilized to gain detailed information about the asymptotic behaviors of Toeplitz determinants under a variety of circumstances. In this paper, we shall restrict our attention to the application to S_N of the two-dimensional Ising model when N is large but T is *fixed*. In order of increasing complexity, we shall treat three cases: $T>T_c$, $T<T_c$, and $T=T_c$. We shall see in these cases that the procedure, although simple in principle, can be technically very complicated.

The results are summarized in Sec. 8, subsections A and B.

We follow the notation employed by Montroll, Potts, and Ward,[4] who used $\pm E_1=\pm kTK_1$ and $\pm E_2=\pm kTK_2$ as the energy of interaction between horizontal and vertical pairs of neighboring spins, respectively. Moreover,

$$z_1=\tanh K_1,\quad z_2=\tanh K_2, \tag{1.5}$$

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[6] N. Wiener and E. Hopf, Sitzber. Deut. Akad. Wiss. Berlin, 1931.

[7] A very good discussion of the equation of Wiener and Hopf is to be found in M. G. Krein, Am. Math. Soc. Transl. **22**, 163 (1962).

and

$$z_2^* = (1-z_2)/(1+z_2). \quad (1.6)$$

Then the critical temperature T_c is given by

$$z_1 = z_2^*, \quad (1.7)$$

and the correlation function S_N of (1.1) can be expressed as a Toeplitz determinant[4,8]

$$S_N = \begin{vmatrix} a_0 & a_{-1} & a_{-2} & \cdots & a_{-N+1} \\ a_1 & a_0 & a_{-1} & \cdots & a_{-N+2} \\ a_2 & a & a_0 & \cdots & a_{-N+3} \\ \vdots & \vdots & \vdots & \vdots\vdots\vdots & \vdots \\ a_{N-1} & a_{N-2} & a_{N-3} & \cdots & a_0 \end{vmatrix}, \quad (1.8)$$

where

$$a_n = (2\pi)^{-1} \int_0^{2\pi} \varphi(\theta) e^{-in\theta} d\theta, \quad (1.9)$$

with

$$\varphi(\theta) = \left[\frac{(1-\alpha_1 e^{i\theta})(1-\alpha_2 e^{-i\theta})}{(1-\alpha_1 e^{-i\theta})(1-\alpha_2 e^{i\theta})} \right]^{1/2}, \quad (1.10)$$

$$\alpha_1 = z_1 z_2^*, \quad \alpha_2 = z_2^*/z_1. \quad (1.11)$$

In (1.10), the square root is taken such that $\varphi(\pi) > 0$. We note the following qualitative differences in the three cases:

(a) when $T > T_c$, we have $\alpha_1 < 1 < \alpha_2$ and

$$\ln\varphi(2\pi) - \ln\varphi(0) = -2\pi i; \quad (1.12)$$

(b) when $T < T_c$, we have $\alpha_1 < \alpha_2 < 1$ and

$$\ln\varphi(2\pi) - \ln\varphi(0) = 0; \quad (1.13)$$

and

(c) when $T = T_c$, we have $\alpha_1 < \alpha_2 = 1$ and

$$\varphi(0) = -\varphi(2\pi) = -i. \quad (1.14)$$

Note that the quantity $\ln\varphi(2\pi) - \ln\varphi(0)$ on the left-hand sides of (1.12) and (1.13) is just the index of the corresponding Wiener-Hopf equation[7] and plays an important role in its theory.

2. SPIN CORRELATIONS ABOVE THE CRITICAL TEMPERATURE

We treat first the case $T > T_c$. From the theory of Wiener-Hopf sum equation, it is convenient to work with a kernel whose index is zero, that is, a kernel with the property that its logarithm is continuous and periodic. Because of (1.12), we introduce

$$\varphi_1(\theta) = \varphi(\theta) e^{i\theta}, \quad (2.1)$$

and

$$b_n = (2\pi)^{-1} \int_0^{2\pi} \varphi_1(\theta) e^{-in\theta} d\theta. \quad (2.2)$$

Therefore

$$b_n = a_{n-1}. \quad (2.3)$$

[8] R. B. Potts and J. C. Ward, Progr. Theoret. Phys. (Kyoto) **13**, 38 (1955).

Let R_N be the $N \times N$ Toeplitz determinant formed from b_n, i.e., R_N is given by the right-hand side of (1.8) with all the a's replaced by b's. Szego's theorem can be immediately applied to R_N to give

$$\lim_{N\to\infty} (-1)^N R_N = [(1-\alpha_1^2)(1-\alpha_2^{-2})(1-\alpha_1/\alpha_2)^2]^{1/4}. \quad (2.4)$$

For large N, the difference between R_N and this limiting value is exponentially small in N.

Consider the linear equations

$$\sum_{m=0}^{N} b_{n-m} x_m = \delta_{n0} \quad (2.5)$$

for $0 \le n \le N$. Because of (2.3), S_N is given by

$$S_N = (-1)^N R_{N+1} x_N. \quad (2.6)$$

To determine S_N asymptotically, it is therefore sufficient to find x_N for large N. For this purpose, we develop first the Wiener-Hopf procedure in a form suitable for iterations.

As a generalization of (2.5), consider the equation

$$\sum_{m=0}^{N} c_{n-m} x_m = y_n \quad (2.7)$$

for $0 \le n \le N$. We assume that $\sum_{n=-\infty}^{\infty} |c_n|$ converges so that

$$C(\xi) = \sum_{n=-\infty}^{\infty} c_n \xi^n \quad (2.8)$$

is continuous on the unit circle. We further assume that $\ln C(\xi)$ is continuous and periodic on the unit circle.[7] We define

$$x_n = y_n = 0 \quad (2.9)$$

for $n < 0$ and for $n > N$;

$$u_n = \sum_{m=0}^{N} c_{N+n-m} x_m, \quad \text{for} \quad n > 0$$
$$= 0, \qquad \text{for} \quad n \le 0, \quad (2.10)$$

and also

$$v_n = \sum_{m=0}^{N} c_{-n-m} x_m, \quad \text{for} \quad n > 0$$
$$= 0, \qquad \text{for} \quad n \le 0. \quad (2.11)$$

We further define

$$X(\xi) = \sum_{n=0}^{N} x_n \xi^n, \quad (2.12)$$

$$Y(\xi) = \sum_{n=0}^{N} y_n \xi^n, \quad (2.13)$$

$$U(\xi) = \sum_{n=1}^{\infty} u_n \xi^n, \quad (2.14)$$

and

$$V(\xi)=\sum_{n=1}^{\infty} v_n\xi^n. \tag{2.15}$$

It then follows from (2.7) that for $|\xi|=1$,

$$C(\xi)X(\xi)=Y(\xi)+U(\xi)\xi^N+V(\xi^{-1}). \tag{2.16}$$

Under the present assumptions, $C(\xi)$ has a unique factorization,[7] up to a multiplicative constant, in the form

$$[C(\xi)]^{-1}=P(\xi)Q(\xi^{-1}) \tag{2.17}$$

for $|\xi|=1$, such that $P(\xi)$ and $Q(\xi)$ are both analytic for $|\xi|<1$, and continuous and nonzero for $|\xi|\leq 1$. Equation (2.16) can thus be rewritten in the form

$$\begin{aligned}[P(\xi)]^{-1}X(\xi)-[Q(\xi^{-1})Y(\xi)]_+-[Q(\xi^{-1})U(\xi)\xi^N]_+ \\ =Q(\xi^{-1})V(\xi^{-1})+[Q(\xi^{-1})Y(\xi)]_- \\ +[Q(\xi^{-1})U(\xi)\xi^N]_-,\end{aligned} \tag{2.18}$$

again for $|\xi|=1$, where the subscript $+$ $(-)$ means that we should expand the quantity in the brackets into a Laurent series and keep only those terms where ξ is raised to a non-negative (negative) power. We then apply the standard Wiener-Hopf argument by noticing that the left-hand side of (2.18) is analytic inside the unit circle, while the right-hand side is analytic outside the unit circle and furthermore approaches zero at infinity. Therefore

$$X(\xi)=P(\xi)\{[Q(\xi^{-1})Y(\xi)]_+ +[Q(\xi^{-1})U(\xi)\xi^N]_+\}, \tag{2.19a}$$

and

$$V(\xi^{-1})=-[Q(\xi^{-1})]^{-1}\{[Q(\xi^{-1})Y(\xi)]_- +[Q(\xi^{-1})U(\xi)\xi^N]_-\}. \tag{2.20a}$$

Similarly,

$$X(\xi^{-1})\xi^N=Q(\xi)\{[P(\xi^{-1})Y(\xi^{-1})\xi^N]_+ +[P(\xi^{-1})V(\xi)\xi^N]_+\}, \tag{2.19b}$$

and

$$U(\xi^{-1})=-[P(\xi^{-1})]^{-1}\{[P(\xi^{-1})Y(\xi^{-1})\xi^N]_- +[P(\xi^{-1})V(\xi)\xi^N]_-\}. \tag{2.20b}$$

These are the equations that we shall use.

We now specialize to the problem of the Ising model with $T>T_c$. In this case,

$$C(\xi)=\left[\frac{(1-\alpha_1\xi)(1-\alpha_2^{-1}\xi)}{(1-\alpha_1\xi^{-1})(1-\alpha_2^{-1}\xi^{-1})}\right]^{1/2}, \tag{2.21}$$

$$P(\xi)=[(1-\alpha_1\xi)(1-\alpha_2^{-1}\xi)]^{-1/2}, \tag{2.22}$$

and

$$Q(\xi)=[(1-\alpha_1\xi)(1-\alpha_2^{-1}\xi)]^{1/2}. \tag{2.23}$$

Moreover, a comparison of (2.5) with (2.7) shows that

$$Y(\xi)=1. \tag{2.24}$$

We find $V(\xi)$ approximately by using (2.20a) with the $U(\xi)$ term neglected:

$$V(\xi^{-1})\sim-[Q(\xi^{-1})]^{-1}[Q(\xi^{-1})]_-=[Q(\xi^{-1})]^{-1}-1,$$

or

$$V(\xi)\sim[Q(\xi)]^{-1}-1. \tag{2.25}$$

Equation (2.25) is then substituted into (2.19b) to give

$$X(\xi^{-1})\xi^N\sim Q(\xi)[P(\xi^{-1})Q(\xi)^{-1}\xi^N]_+. \tag{2.26}$$

The desired x_N is found by setting $\xi=0$ in (2.26):

$$x_N\doteqdot(2\pi i)^{-1}\oint d\xi\,\xi^{N-1}P(\xi^{-1})Q(\xi)^{-1}, \tag{2.27}$$

where the integration is around the unit circle. In (2.27), the symbol $\doteqdot$ means that, for $N\rightarrow\infty$ but T fixed, the right-hand side and the left-hand side have the same asymptotic expansion. More explicitly, by (2.22) and (2.23),

$$x_N\doteqdot(2\pi i)^{-1}\oint d\xi\,\xi^{N-1}[(1-\alpha_1\xi)(1-\alpha_1\xi^{-1})(1-\alpha_2^{-1}\xi)\times(1-\alpha_2^{-1}\xi^{-1})]^{-1/2}, \tag{2.28a}$$

or

$$x_N\doteqdot-(2\pi)^{-1}\int_0^{2\pi}d\theta\,e^{iN\theta}|(1-\alpha_1e^{i\theta})(1-\alpha_2^{-1}e^{i\theta})|^{-1}. \tag{2.28b}$$

The error involved in (2.28) is exponentially small in N even when compared with x_N. In (2.28b), the minus sign on the right hand side is due to the fact that, from (2.1), the square root under the integral of (2.28a) is taken to be negative at the point $\xi=-1$. Finally, the substitution of (2.4) and (2.28) into (2.6) gives

$$S_N\doteqdot-(2\pi i)^{-1}[(1-\alpha_1^2)(1-\alpha_2^{-2})(1-\alpha_1/\alpha_2)^2]^{1/4}\times\oint d\xi\,\xi^{N-1}[(1-\alpha_1\xi)(1-\alpha_1\xi^{-1})(1-\alpha_2^{-1}\xi)\times(1-\alpha_2^{-1}\xi^{-1})]^{-1/2}. \tag{2.29}$$

Equation (2.29) is the desired answer; it only remains to evaluate its right-hand side asymptotically for large N. This is straightforward but tedious. We deform the contour of integration around the branch cut from α_1 to α_2^{-1} to get the result

$$S_N\doteqdot\pi^{-1}\alpha_2^{-N}[(1-\alpha_1^2)(1-\alpha_2^{-2})(1-\alpha_1/\alpha_2)^2]^{1/4}\times\int_{\alpha_1\alpha_2}^{1}d\xi_1\xi_1^{N-1}[(1-\alpha_1\alpha_2^{-1}\xi_1)(1-\alpha_1\alpha_2\xi_1^{-1})\times(1-\alpha_2^{-2}\xi_1)(\xi_1^{-1}-1)]^{-1/2}. \tag{2.30}$$

From this point on, we can proceed in many slightly different ways. For example, the simplest thing to do is to expand the integrand of (2.30) about the point $\xi_1=1$ and then integrate term by term. We shall follow a procedure which is only slightly different. Let

$$x_1=(1-\alpha_1/\alpha_2)^{-1}(1+\alpha_1/\alpha_2)=\cosh 2K_1, \quad (2.31)$$

$$x_2=(1-\alpha_1\alpha_2)^{-1}(1+\alpha_2\alpha_2)=\coth 2K_2, \quad (2.32)$$

and

$$x_3=(\alpha_2^2-1)^{-1}(\alpha_2^2+1). \quad (2.33)$$

In (2.31) and (2.32), we have used (1.11), (1.5), and (1.6). These three x's are related by

$$x_1x_2+x_1x_3-x_2x_3=1. \quad (2.34)$$

We use these three x's in (2.30); for example,

$$(1-\alpha_1\alpha_2^{-1}\xi)^{-1/2}=(1+x_1)^{1/2}(1+\xi_1)^{-1/2} \times[1+x_1(1-\xi_1)/(1+\xi_1)]^{-1/2}. \quad (2.35)$$

The result is

$$S^N \doteqdot \pi^{-1}\alpha_2^{-N}(1-\alpha_1^2)^{1/4}(1-\alpha_2^{-2})^{-1/4}(1-\alpha_1\alpha_2)^{-1/2} \times 2^{3/2}\int_{\alpha_1\alpha_2}^{1} d\xi_1\xi_1^N(1+\xi_1)^{-3/2}(1-\xi_1)^{-1/2} \times A_>[(1-\xi_1)/(1+\xi\ )], \quad (2.36)$$

where

$$A_>(z)=[(1+x_1z)(1-x_2z)(1+x_3z)]^{-1/2}. \quad (2.37)$$

If we expand $A_>(z)$ into a power series

$$A_>(z)=\sum_{n=0}^{\infty}A_{n>}z^n, \quad (2.38)$$

then the first few coefficients are

$$A_{0>}=1,$$
$$A_{1>}=-\tfrac{1}{2}(x_1-x_2+x_3),$$
$$A_{2>}=\tfrac{3}{8}(x_1^2+x_2^2+x_3^2)-\tfrac{1}{4}(x_2x_3-x_3x_1+x_1x_2),$$

and

$$A_{3>}=-\tfrac{5}{16}(x_1^3-x_2^3+x_3^3) +\tfrac{3}{16}(x_1^2x_2-x_1x_2^2-x_2^2x_3+x_2x_3^2-x_3^2x_1-x_3x_1^2) +\tfrac{1}{8}x_1x_2x_3. \quad (2.39)$$

If we substitute (2.38) into (2.36) in order to integrate term by term, we can replace the lower limit of integration $\alpha_1\alpha_2$ by 0 without changing the asymptotic series:

$$S_N \doteqdot \pi^{-1}\alpha_2^{-N}(1-\alpha_1^2)^{1/4}(1-\alpha_2^{-2})^{-1/4}(1-\alpha_1\alpha_2)^{-1/2}N! \times\sum_{n=0}^{\infty}A_{n>}2^{-n}\Gamma(n+\tfrac{1}{2})[\Gamma(N+n+\tfrac{3}{2})]^{-1} \times F(n+\tfrac{3}{2},n+\tfrac{1}{2};N+n+\tfrac{3}{2};\tfrac{1}{2}). \quad (2.40)$$

In (2.40), the sum over n is to be interpreted in the sense of asymptotic series, and we have used there Euler's integral representation of the hypergeometric function F.[9] It is then trivial to rearrange the series to get

$$S_N \doteqdot \pi^{-1/2}\alpha_2^{-N}(1-\alpha_1^2)^{1/4}(1-\alpha_2^{-2})^{-1/4}(1-\alpha_1\alpha_2)^{-1/2}N! \times\sum_{m=0}^{\infty}[\Gamma(N+m+\tfrac{3}{2})]^{-1}\pi^{-1/2}\Gamma(m+\tfrac{3}{2})\Gamma(m+\tfrac{1}{2}) \times 2^{-m}\sum_{n=0}^{m}A_{n>}[(m-n)!\Gamma(n+\tfrac{3}{2})]^{-1}. \quad (2.41)$$

This is the desired answer.

For completeness we write down the first few terms:

$$S_N\sim\pi^{-1/2}\alpha_2^{-N}(1-\alpha_1^2)^{1/4}(1-\alpha_2^{-2})^{-1/4}(1-\alpha_1\alpha_2)^{-1/2} \times N![\Gamma(N+\tfrac{3}{2})]^{-1}\{1+\tfrac{3}{8}(N+\tfrac{3}{2})^{-1}(1+\tfrac{2}{3}A_{1>}) +(45/64)[(N+\tfrac{3}{2})(N+\tfrac{5}{2})]^{-1}(\tfrac{1}{2}+\tfrac{2}{3}A_{1>}+(4/15)A_{2>}) +(1575/512)[(N+\tfrac{3}{2})(N+\tfrac{5}{2})(N+\tfrac{7}{2})]^{-1} \times(\tfrac{1}{6}+\tfrac{1}{3}A_{1>}+(4/15)A_{2>}+(8/105)A_{3>})+\cdots\}, \quad (2.42)$$

or, more explicitly, as $N\to\infty$,

$$S_N\sim(\pi N)^{-1/2}\alpha_2^{-N}(1-\alpha_1^2)^{1/4}(1-\alpha_2^{-2})^{-1/4}(1-\alpha_1\alpha_2)^{-1/2} \times[1+\tfrac{1}{4}N^{-1}A_{1>}+\tfrac{3}{16}N^{-2}(A_{2>}-\tfrac{5}{6}) +(15/64)N^{-3}(A_{3>}-(7/6)A_{1>})+\cdots], \quad (2.43)$$

where $A_{1>}$, $A_{2>}$, and $A_{3>}$ are given by (2.39). These are the first four terms of the long-range correlation along the lattice sites above the critical temperature.

3. SPIN CORRELATIONS BELOW THE CRITICAL TEMPERATURE

In this section, we consider the somewhat more complicated case $T<T_c$. In view of (1.13), the logarithm of $\varphi(\theta)$ is continuous and periodic in the present case. Accordingly, instead of (2.5), we study more directly here the equations

$$\sum_{m=0}^{N}a_{n-m}x_m=\delta_{n0} \quad (3.1)$$

for $0\leq n\leq N$. The solution of (3.1) is related to the correlation function S_N by

$$x_0=x_{0N}=S_N/S_{N+1}. \quad (3.2)$$

In order to determine S_N from x_{0N}, we need the known result of spontaneous magnetization[3]

$$S_\infty=(1-\alpha_1^2)^{1/4}(1-\alpha_2^2)^{1/4}(1-\alpha_1\alpha_2)^{-1/2}, \quad (3.3)$$

so that

$$S_N=(1-\alpha_1^2)^{1/4}(1-\alpha_2^2)^{1/4}(1-\alpha_1\alpha_2)^{-1/2}\prod_{n=N}^{\infty}x_{0n}. \quad (3.4)$$

To calculate x_0 approximately for large N, we use the formalism developed in the last section, or more specifically Eqs. (2.19) and (2.20). By (1.10),

$$C(\xi)=\left[\frac{(1-\alpha_1\xi)(1-\alpha_2\xi^{-1})}{(1-\alpha_1\xi^{-1})(1-\alpha_2\xi)}\right]^{1/2}, \quad (3.5)$$

[9] Bateman Manuscript Project, *Higher Transcendental Functions*, edited by A. Erdelyi (McGraw-Hill Book Company, Inc., New York, 1953), Vol. I, Chap. 2.

for $T<T_c$, so that

$$P(\xi)=[(1-\alpha_2\xi)/(1-\alpha_1\xi)]^{1/2}, \tag{3.6}$$

and

$$Q(\xi)=[(1-\alpha_1\xi)/(1-\alpha_2\xi)]^{1/2}. \tag{3.7}$$

Note that, both in this case and in the case treated in the last section,

$$P(\xi)Q(\xi)=1. \tag{3.8}$$

The procedure to be followed is (1) calculate $V(\xi)$ approximately from (2.20a) with the $U(\xi)$ term neglected; (2) get $U(\xi)$ from (2.20b) with the $V(\xi)$ of step 1; and finally (3) compute $X(0)$ from (2.19a) with the $U(\xi)$ of step 2. We therefore see that the case $T<T_c$ is more complicated than $T>T_c$ in two respects: first, one more step is needed here to get x_0; and, secondly, from (3.4) an infinite product of the x_0's is required to obtain S_N.

Equations (2.24) and (2.25) still hold here. By (2.20b)

$$U(\xi^{-1})\sim-[P(\xi^{-1})]^{-1}[P(\xi^{-1})Q(\xi)^{-1}\xi^N]_-. \tag{3.9}$$

For a function $F(\xi)$ given on the unit circle, let

$$[F(\xi)]_+'=[F(\xi)]_+-(2\pi i)^{-1}\oint d\xi\, F(\xi)/\xi, \tag{3.10}$$

where the path of integration is around the unit circle. Then

$$[F(\xi)]_+'=(2\pi i)^{-1}\xi\oint d\xi' F(\xi')\xi'^{-1}(\xi'-\xi)^{-1}, \tag{3.11}$$

provided that the path of integration is indented outward near $\xi'=\xi$. With this notation, (3.9) can be rewritten as

$$U(\xi)\sim-[P(\xi)]^{-1}[P(\xi)Q(\xi^{-1})^{-1}\xi^{-N}]_+'. \tag{3.12}$$

The substitution of (3.12) into (2.19a) then gives

$$x_0=X(0)\doteqdot 1-(2\pi i)^{-1}\oint d\xi\, \xi^{N-1}Q(\xi^{-1})P(\xi)^{-1} \times[P(\xi)Q(\xi^{-1})^{-1}\xi^{-N}]_+', \tag{3.13}$$

or more explicitly

$$x_0\doteqdot 1+(2\pi)^{-2}\oint d\xi\, \xi^N(1-\alpha_1\xi)^{1/2}(1-\alpha_1\xi^{-1})^{1/2}(1-\alpha_2\xi)^{-1/2}(1-\alpha_2\xi^{-1})^{-1/2} \times\oint d\xi'(\xi'-\xi)^{-1}\xi'^{-N-1}(1-\alpha_1\xi')^{-1/2}(1-\alpha_1\xi'^{-1})^{-1/2}(1-\alpha_2\xi')^{1/2}(1-\alpha_2\xi'^{-1})^{1/2}. \tag{3.14}$$

Again, the error involved in (3.14) is exponentially small in N compared with the double integral, or roughly of the order of α_2^{4N}. The double integral is of course roughly of the order α_2^{2N}. We finally substitute (3.14) into (3.4) to obtain, for large N,

$$\begin{aligned}(1-\alpha_1^2)^{-1/4}(1-\alpha_2^2)^{-1/4}(1-\alpha_1\alpha_2)^{1/2}S_N &\doteqdot 1+\sum_{n=N}^{\infty}(x_{0n}-1)\\ &\doteqdot 1+(2\pi)^{-2}\oint d\xi\, \xi^N(1-\alpha_1\xi)^{1/2}(1-\alpha_1\xi^{-1})^{1/2}(1-\alpha_2\xi)^{-1/2}(1-\alpha_2\xi^{-1})^{-1/2}\\ &\quad\times\oint d\xi'(\xi'-\xi)^{-2}\xi'^{-N}(1-\alpha_1\xi')^{-1/2}(1-\alpha_1\xi'^{-1})^{-1/2}(1-\alpha_2\xi')^{1/2}(1-\alpha_2\xi'^{-1})^{1/2}.\end{aligned} \tag{3.15}$$

In both (3.14) and (3.15), the path of integration for the variable ξ' is to be indented outward near $\xi'=\xi$. Equation (3.15) is the desired answer, but it remains to evaluate the right-hand side more explicitly for large N.

Since the author is unable to find any elegant way of carrying out this evaluation, we shall proceed by brute force. We deform the contour of integration in the variable ξ around the branch cut from α_1 to α_2, that in the variable ξ' around the cut from α_2^{-1} to α_1^{-1}, and change the variables

$$\xi=\alpha_2\xi_1,\quad \xi'=(\alpha_2\xi_2)^{-1}, \tag{3.16}$$

and

$$\eta_1=(1-\xi_1)/(1+\xi_1),\quad \eta_2=(1-\xi_2)/(1+\xi_2). \tag{3.17}$$

In terms of these variables and the x's of (2.31)–(2.33), we get

$$S_N\doteqdot(1-\alpha_1^2)^{1/4}(1-\alpha_2^2)^{1/4}(1-\alpha_1\alpha_2)^{-1/2}\Big\{1+16\pi^{-2}\alpha_2^{2N+2}(1-\alpha_2^2)^{-2}\int_{\alpha_1/\alpha_2}^{1}d\xi_1\int_{\alpha_1/\alpha_2}^{1}d\xi_2\xi_1^N\xi_2^N \times(1+\xi_1)^{-3/2}(1-\xi_1)^{-1/2}(1+\xi_2)^{-5/2}(1-\xi_2)^{1/2}(1-x_3\eta_1-x_3\eta_2+\eta_1\eta_2)^{-2}A_<(\eta_1)[A_<(\eta_2)]^{-1}\Big\}, \tag{3.18}$$

where

$$A_<(\eta)=(1-\eta x_1)^{1/2}(1+\eta x_2)^{1/2}(1-\eta x_3)^{-1/2}. \tag{3.19}$$

We need the series expansions

$$A_<(\eta)=\sum_{n=0}^{\infty} A_{n<}\eta^n, \tag{3.20}$$

$$[A_<(\eta)]^{-1}=\sum_{n=0}^{\infty} \bar{A}_{n<}\eta^n, \tag{3.21}$$

and

$$(1-x_3\eta_1-x_3\eta_2+\eta_1\eta_2)^{-2}=\sum_{p,q=0}^{\infty} C_{pq}\eta_1{}^p\eta_2{}^q. \tag{3.22}$$

After substituting (3.20)–(3.22) into (3.18), we replace the lower limits of integration α_1/α_2 by zero to obtain

$$S_N \doteqdot (1-\alpha_1{}^2)^{1/4}(1-\alpha_2{}^2)^{1/4}(1-\alpha_1\alpha_2)^{-1/2} \times\{1+2\pi^{-1}N^{-2}\alpha_2{}^{2N}(\alpha_2{}^{-1}-\alpha_2)^{-2}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{p=0}^{\infty}\sum_{q=0}^{\infty} N^{-m-n-p-q}A_{m<}\bar{A}_{n<}C_{pq}\mathcal{G}_{m+p}\mathcal{G}_{n+q+1}\}, \tag{3.23}$$

where[9]

$$\mathcal{G}_n=(8N/\pi)^{1/2}N^n\int_0^1 d\xi\, \xi^N(1+\xi)^{-n-3/2}$$
$$=(N/\pi)^{1/2}(\tfrac{1}{2}N)^n\Gamma(N+1)\Gamma(n+\tfrac{1}{2})[\Gamma(N+n+\tfrac{3}{2})]^{-1}F(n+\tfrac{1}{2}, n+\tfrac{3}{2}; N+n+\tfrac{3}{2}; \tfrac{1}{2}). \tag{3.24}$$

Equation (3.23) gives the desired asymptotic behavior of S_N as $N\to\infty$ for $T<T_c$.

The first few terms are explicitly

$$S_N\sim(1-\alpha_1{}^2)^{1/4}(1-\alpha_2{}^2)^{1/4}(1-\alpha_1\alpha_2)^{-1/2}\{1+(2\pi N^2)^{-1} \times\alpha_2{}^{2N}(\alpha_2{}^{-1}-\alpha_2)^{-2}[1+(2N)^{-1}(-A_{1<}+4x_3)+3(2N)^{-2} \times(-A_{2<}+A_{1<}{}^2-2x_3A_{1<}+6x_3{}^2-13/6)+\cdots]\}, \tag{3.25}$$

where, by (3.20) and (3.19),

$$A_{1<}=\tfrac{1}{2}(-x_1+x_2+x_3), \tag{3.26}$$

and

$$A_{2<}=-\tfrac{1}{8}(x_1{}^2+2x_1x_2+x_2{}^2+2x_1x_3 -2x_2x_3-3x_3{}^2). \tag{3.27}$$

4. SPECIAL CASE $\alpha_1=0,\ \alpha_2=1$

Because of (1.14), the case $T=T_c$ is much more complicated, and the relevance of the Wiener-Hopf procedure more obscure. In this section, we treat the special case $\alpha_1=0$. Physically, this is the limit where, at $T=T_c$,

$$K_1\to 0 \quad\text{and}\quad K_2\to\infty, \tag{4.1}$$

such that

$$K_1=\exp(-2K_2). \tag{4.2}$$

Throughout this and the next three sections, we take $T=T_c$, or, by (1.7) and (1.11), $\alpha_2=1$. We shall denote this special case by the superscript 0; hence, by (1.9) and (1.10),

$$\varphi^{(0)}(\theta)=ie^{-i\theta/2}, \tag{4.3}$$

and

$$a_n{}^{(0)}=2\pi^{-1}(2n+1)^{-1}. \tag{4.4}$$

Let M be the $(N+1)\times(N+1)$ matrix whose elements are, for $m, n=0, \cdots, N$,

$$M_{mn}=\tfrac{1}{2}\pi a_{m-n}{}^{(0)}=(2m-2n+1)^{-1}, \tag{4.5}$$

then

$$S_{N+1}{}^{(0)}=\det(2\pi^{-1}M)=(2\pi^{-1})^{N+1}\det M. \tag{4.6}$$

To evaluate $\det M$, we use the following theorem, which can be easily proved algebraically[10]: if

$$M_{mn}=(\mu_m+\nu_n)^{-1}, \tag{4.7}$$

then

$$\det M=[\prod_{0\le m<n\le N}(\mu_m-\mu_n)(\nu_m-\nu_n)] \times[\prod_{m=0}^{N}\prod_{n=0}^{N}(\mu_m+\nu_n)]^{-1}. \tag{4.8}$$

Since

$$\mu_m=2m+1 \quad\text{and}\quad \nu_n=-2n, \tag{4.9}$$

we get from (4.8)

$$\det M=2^{2N(N+1)}[G(N+1)]^4[G(2N+2)]^{-1}, \tag{4.10}$$

where

$$G(N)=1^{N-1}2^{N-2}3^{N-3}\cdots(N-1). \tag{4.11}$$

Barnes[11] has given the asymptotic expansion of $G(N)$

[10] N. I. Achieser, *Theory of Approximation* (Frederick Ungar Publishing Company, New York, 1956), p. 19.

[11] E. W. Barnes, Quart. J. Math. 31, 264 (1900). See in particular p. 285. Note that our $G(N)$ is the $G(N+1)$ of Barnes.

for large N,

$$\ln G(N) \doteq \tfrac{1}{12} - \ln A + \tfrac{1}{2}N \ln 2\pi + (\tfrac{1}{2}N^2 - \tfrac{1}{12}) \ln N - \tfrac{3}{4}N^2 + \sum_{s=1}^{\infty} (-1)^s [2s(2s+2)]^{-1} B_{s+1} N^{-2s}, \quad (4.12)$$

where the B's are the Bernoulli's numbers, and A is Glaisher's constant,[12]

$$A \sim 1.282427130. \quad (4.13)$$

The substitution of (4.12) and (4.10) into (4.6) then gives, approximately for large N,

$$S_N^{(0)} \sim e^{1/4} 2^{1/12} A^{-3} N^{-1/4} (1 - \tfrac{1}{64} N^{-2} + \cdots). \quad (4.14)$$

Equation (4.14) is the desired result for this special case.

In the next section, we obtain the leading term of S_N in the more general case where α_1 is not necessarily zero. This is carried out by comparing with the special case $\alpha_1 = 0$. In order to get further terms of S_N, more detailed information about this special case is needed, and this is derived in Sec. 6.

5. SPIN CORRELATIONS AT THE CRITICAL TEMPERATURE

The case $T > T_c$ has been treated in Sec. 2. Consider the leading term on the right-hand side of (2.43), which is

$$S_N \sim (\pi N)^{-1/2} \alpha_2^{-N} (1-\alpha_1^2)^{1/4} (1-\alpha_2^{-2})^{-1/4} \times (1-\alpha_1\alpha_2)^{-1/2}, \quad (5.1)$$

as $N \to \infty$. In particular, if $\alpha_1 = 0$, (5.1) reduces to

$$S_N^{(0)} \sim (\pi N)^{-1/2} \alpha_2^{-N} (1-\alpha_2^{-2})^{-1/4}. \quad (5.2)$$

Therefore, as $N \to \infty$ for fixed $T > T_c$,

$$S_N \sim (1-\alpha_1^2)^{1/4} (1-\alpha_1\alpha_2)^{-1/2} S_N^{(0)}. \quad (5.3)$$

On the other hand, we may start with the leading terms of (3.25) for $T < T_c$,

$$S_N \sim (1-\alpha_1^2)^{1/4} (1-\alpha_2^2)^{1/4} (1-\alpha_1\alpha_2)^{-1/2} \times \{1 + (2\pi N^2)^{-1} \alpha_2^{2N} (\alpha_2^{-1} - \alpha_2)^{-2}\}, \quad (5.4)$$

as $N \to \infty$. For $\alpha_1 = 0$, (5.4) is more simply

$$S_N^{(0)} \sim (1-\alpha_2^2)^{1/4} \{1 + (2\pi N^2)^{-1} \alpha_2^{2N} (\alpha_2^{-1} - \alpha_2)^{-2}\}. \quad (5.5)$$

Therefore, (5.3) also holds as $N \to \infty$ for fixed $T < T_c$. Note that $S_N^{(0)}$ is obtained from S_N by setting $\alpha_1 = 0$ without changing α_2.

It is therefore not unreasonable to expect (5.3) to hold also as $N \to \infty$ for $T = T_c$. More explicitly, at $T = T_c$,

$$S_N \sim (1+\alpha_1)^{1/4} (1-\alpha_1)^{-1/4} S_N^{(0)}, \quad (5.6)$$

as $N \to \infty$. With (4.14), we get

$$S_N \sim e^{1/4} 2^{1/12} A^{-3} (1+\alpha_1)^{1/4} (1-\alpha_1)^{-1/4} N^{-1/4}, \quad (5.7)$$

approximately for large N. It is the purpose of this section to obtain (5.6) by a direct calculation.

Consider two $N \times N$ Toeplitz determinants

$$D_N = \begin{vmatrix} c_0 & c_{-1} & c_{-2} & \cdots & c_{-N+1} \\ c_1 & c_0 & c_{-1} & \cdots & c_{-N+2} \\ c_2 & c_1 & c_0 & \cdots & c_{-N+3} \\ \vdots & \vdots & \vdots & \vdots\vdots\vdots & \vdots \\ c_{N-1} & c_{N-2} & c_{N-3} & \cdots & c_0 \end{vmatrix}, \quad (5.8)$$

and

$$\bar{D}_N = \begin{vmatrix} \bar{c}_0 & \bar{c}_{-1} & \bar{c}_{-2} & \cdots & \bar{c}_{-N+1} \\ \bar{c}_1 & \bar{c}_0 & \bar{c}_{-1} & \cdots & \bar{c}_{-N+2} \\ \bar{c}_2 & \bar{c}_1 & \bar{c}_0 & \cdots & \bar{c}_{-N+3} \\ \vdots & \vdots & \vdots & \vdots\vdots\vdots & \vdots \\ \bar{c}_{N-1} & \bar{c}_{N-2} & \bar{c}_{N-3} & \cdots & \bar{c}_0 \end{vmatrix}, \quad (5.9)$$

where

$$c_n = (2\pi)^{-1} \int_0^{2\pi} \psi(\theta) e^{-in\theta} d\theta, \quad (5.10)$$

and

$$\bar{c}_n = (2\pi)^{-1} \int_0^{2\pi} \bar{\psi}(\theta) e^{-in\theta} d\theta. \quad (5.11)$$

We ask the question how D_N and $\bar{D}_N$ are related for large N if

$$\bar{\psi}(\theta) = \psi(\theta)(1 - \alpha e^{-i\theta}), \quad (5.12)$$

where $|\alpha| < 1$.

By (5.9)–(5.12)

$$\bar{c}_n = c_n - \alpha c_{n+1}, \quad (5.13)$$

and

$$\bar{D}_N = \begin{vmatrix} c_0 - \alpha c_1 & c_{-1} - \alpha c_0 & c_{-2} - \alpha c_{-1} & \cdots & c_{-N+1} - \alpha c_{-N+2} \\ c_1 - \alpha c_2 & c_0 - \alpha c_1 & c_{-1} - \alpha c_0 & \cdots & c_{-N+2} - \alpha c_{-N+3} \\ c_2 - \alpha c_3 & c_1 - \alpha c_2 & c_0 - \alpha c_1 & \cdots & c_{-N+3} - \alpha c_{-N+4} \\ \vdots & \vdots & \vdots & \vdots\vdots\vdots & \vdots \\ c_{N-1} - \alpha c_N & c_{N-2} - \alpha c_{N-1} & c_{N-3} - \alpha c_{N-2} & \cdots & c_0 - \alpha c_1 \end{vmatrix}. \quad (5.14)$$

Accordingly, $\bar{D}_N$ can be expressed as an $(N+1) \times (N+1)$ determinant in the form

$$\bar{D}_N = \begin{vmatrix} 1 & \alpha & \alpha^2 & \alpha^3 & \cdots & \alpha^N \\ c_1 & c_0 & c_{-1} & c_{-2} & \cdots & c_{-N+1} \\ c_2 & c_1 & c_0 & c_{-1} & \cdots & c_{-N+2} \\ c_3 & c_2 & c_1 & c_0 & \cdots & c_{-N+3} \\ \vdots & \vdots & \vdots & \vdots & \vdots\vdots\vdots & \vdots \\ c_N & c_{N-1} & c_{N-2} & c_{N-3} & \cdots & c_0 \end{vmatrix}. \quad (5.15)$$

Suppose we define $x_0, x_1, x_2, \cdots, x_N$ by the linear equations

$$\sum_{m=0}^{N} \alpha^m x_m = 1, \quad (5.16)$$

and

$$\sum_{m=0}^{N} c_{n-m} x_m = 0 \quad (5.17)$$

[12] J. W. L. Glaisher, Messenger of Math. 24, 1 (1894).

for $1\leq n\leq N$. Then, by (5.8) and (5.15), we have

$$x_0=D_N/\bar{D}_N. \tag{5.18}$$

For very large N, the system of equations (5.16) and (5.17) are approximately

$$\sum_{m=0}^{\infty}\alpha^m x_m=1, \tag{5.19}$$

and

$$\sum_{m=0}^{\infty}c_{n-m}x_m=0 \tag{5.20}$$

for $n\leq 1$. Define y_0 by

$$\sum_{m=0}^{\infty}c_{-m}x_m=y_0, \tag{5.21}$$

$C(\xi)$ by (2.8), $X(\xi)$ by (2.12), and $P(\xi)$ and $Q(\xi)$ by (2.17), then by (2.19a)

$$X(\xi)=P(\xi)[Q(\xi^{-1})y_0]_+ = y_0Q(0)P(\xi), \tag{5.22}$$

provided that the Wiener-Hopf sum equation under consideration has a unique solution. In order to determine y_0, we note that (5.19) is just

$$X(\alpha)=1. \tag{5.23}$$

Accordingly

$$y_0Q(0)P(\alpha)=1, \tag{5.24}$$

and thus

$$X(\xi)=P(\xi)/P(\alpha). \tag{5.25}$$

It then follows from (5.18) and (5.25) that

$$\lim_{N\to\infty}\bar{D}_N/D_N=P(\alpha)/P(0). \tag{5.26}$$

It remains to write (5.26) in a more familiar form. Let

$$g_n=(2\pi)^{-1}\int_0^{2\pi}d\theta\, e^{-in\theta}\ln\psi(\theta), \tag{5.27}$$

and

$$\bar{g}_n=(2\pi)^{-1}\int_0^{2\pi}d\theta\, e^{-in\theta}\ln\bar{\psi}(\theta). \tag{5.28}$$

Then, by (5.12),

$$\bar{g}_n=g_n \quad \text{for } n\geq 0, \quad\text{and}\quad \bar{g}_n=g_n+\alpha^{-n}/n \quad \text{for } n<0. \tag{5.29}$$

On the other hand, it follows from (2.8) and (2.17) that

$$P(\xi)=\text{const}\times\exp[-\sum_{n=1}^{\infty}g_n\xi^n]. \tag{5.30}$$

The substitution into (5.26) gives

$$\lim_{N\to\infty}\bar{D}_N/D_N=\exp[-\sum_{n=1}^{\infty}g_n\alpha^n] =\exp\sum_{n=1}^{\infty}n(\bar{g}_n\bar{g}_{-n}-g_ng_{-n}). \tag{5.31}$$

This is the desired answer.

Next we consider the same problem except that the condition (5.12) is replaced by

$$\bar{\psi}(\theta)=\psi(\theta)(1-\alpha e^{i\theta}). \tag{5.32}$$

In this case, we can apply (5.31) to the complex conjugate functions to obtain

$$\lim_{N\to\infty}\bar{D}_N^*/D_N^*=\exp\sum_{n=1}^{\infty}n(\bar{g}_{-n}^*\bar{g}_n^*-g_{-n}^*g_n^*). \tag{5.33}$$

Therefore (5.31) also holds in this case.

If (5.31) is used repeatedly a finite number of times, then we get the following result. Let D_N and $\bar{D}_N$ be two $N\times N$ Toeplitz determinants with their elements the Fourier coefficients of $\psi(\theta)$ and $\bar{\psi}(\theta)$, as shown in (5.8)–(5.11), let the coefficients c_n have the property that the corresponding Wiener-Hopf sum equation has a unique solution, let $\bar{\psi}(\theta)/\psi(\theta)$ be a trigonometric polynomial normalized so that

$$\bar{\psi}(\theta)=\psi(\theta)[\prod_{n=1}^{n_1}(1-\alpha^{(n)}e^{-i\theta})][\prod_{n=1}^{n_2}(1-\bar{\alpha}^{(n)}e^{i\theta})], \tag{5.34}$$

with all the α's and $\bar{\alpha}$'s less than 1 in magnitude; then (5.31) holds.

Let $S_N^{(0)}$ be D_N, and S_N be $\bar{D}_N$, then

$$g_n=\tfrac{1}{2}n^{-1},\quad n\neq 0, \qquad =0,\quad n=0, \tag{5.35}$$

and

$$\begin{aligned}\bar{g}_n&=\tfrac{1}{2}n^{-1}(1-\alpha_1^n), & n&>0,\\ &0, & n&=0,\\ &=\tfrac{1}{2}n^{-1}(1-\alpha_1^{-n}), & n&<0.\end{aligned} \tag{5.36}$$

Therefore, by (5.31),

$$\lim_{N\to\infty}S_N/S_N^{(0)}=\exp\sum_{n=1}n\{-[\tfrac{1}{2}n^{-1}(1-\alpha_1^n)]^2+[\tfrac{1}{2}n^{-1}]^2\} =(1+\alpha_1)^{1/4}(1-\alpha_1)^{-1/4}. \tag{5.37}$$

Thus we have obtained the leading term of the asymptotic expansion of S_N for large N at the critical temperature. In Sec. 7, we shall show how to obtain the asymptotic series in principle; because of algebraic complications, we shall carry out the computation only for the next term. For this purpose, some further properties of the special case $\alpha_1=0$ are needed.

6. SPECIAL CASE $\alpha_1=0$, $\alpha_2=1$

We consider the inverse of the matrix M. More precisely, let

$$L=\tfrac{1}{2}\pi M^{-1}, \quad (6.1)$$

then, by (4.8),

$$L_{pq}=\tfrac{1}{2}\pi(-1)^{p+q}(\mu_q+\nu_p)^{-1}\Big[\prod_{m=0}^{N}(\mu_m+\nu_p)\Big]\Big[\prod_{n=0}^{N}(\mu_q+\nu_n)\Big]$$
$$\times\Big[\prod_{m=0}^{q-1}(\mu_q-\mu_m)\Big]^{-1}\Big[\prod_{m=q+1}^{N}(\mu_m-\mu_q)\Big]^{-1}$$
$$\times\Big[\prod_{n=0}^{p-1}(\nu_p-\nu_n)\Big]^{-1}\Big[\prod_{n=p+1}^{N}(\nu_n-\nu_p)\Big]^{-1}. \quad (6.2)$$

With (4.9), (6.2) can be simplified to

$$\begin{aligned}L_{pq}&=\tfrac{1}{2}\pi(2q-2p+1)^{-1}2^{-4N}(2N-2p+1)!(2p)!\\&\quad\times(2N-2q)!(2q+1)![(N-p)!p!(N-q)!q!]^{-2}\\&=\tfrac{1}{2}\pi(2q-2p+1)^{-1}2^{-2N}(2N-2p+1)!!(2p-1)!!\\&\quad\times(2N-2q-1)!!(2q+1)!![(N-p)!p!(N-q)!q!]^{-1}\\&=2\pi^{-1}(2q-2p+1)^{-1}\Gamma(N-p+\tfrac{3}{2})\Gamma(p+\tfrac{1}{2})\\&\quad\times\Gamma(N-q+\tfrac{1}{2})\Gamma(q+\tfrac{3}{2})[(N-p)!p!(N-q)!q!]^{-1},\end{aligned} \quad (6.3)$$

where

$$(2p-1)!!=1\cdot3\cdot5\cdot7\cdots(2p-1), \quad (6.4)$$

for example. As given by (6.1), L_{pq} is defined only for $0\leq p\leq N$ and $0\leq q\leq N$. Thus (6.3) may be used to extend the domain of definition for L_{pq} to all integers p and q. It is, however, immediately seen from (6.3) that with this extension,

$$L_{pq}=0, \quad (6.5)$$

unless $0\leq p\leq N$ and $0\leq q\leq N$. It is also interesting to note that

$$L_{N-q,N-p}=L_{pq}, \quad (6.6)$$

and that, for large N and fixed p and q,

$$L_{pq}\sim\tfrac{1}{2}(2q-2p+1)^{-1}2^{-p-q+1}(2p-1)!!(2q+1)!!\times[p!q!]^{-1}, \quad (6.7)$$

with an error of the order of N^{-1}. In view of (6.6), it is convenient to define the matrix operation ★ such that, for an $(N+1)\times(N+1)$ matrix $\mathfrak{M}$,

$$(\mathfrak{M}^\star)_{pq}=\mathfrak{M}_{N-p,N-q}. \quad (6.8)$$

Thus,

$$M^\star=M \quad\text{and}\quad L^\star=L. \quad (6.9)$$

Finally, we proceed to compute the quantities

$$K_{pn}=2\pi^{-1}\sum_m L_{pm}M_{mn}. \quad (6.10)$$

By (6.5), $K_{pn}=0$ unless $0\leq p\leq N$; by (6.1),

$$K_{pn}=\delta_{pn}, \quad (6.11)$$

if $0\leq p\leq N$ and $0\leq n\leq N$. We are therefore interested in the case where $0\leq p\leq N$ while either $n>N$ or $n<0$. This in particular means $p\neq n$. Accordingly, for these ranges of values for p and n, by partial fraction

$$K_{pn}=\pi^{-2}\frac{\Gamma(N-p+\frac{3}{2})\Gamma(p+\frac{1}{2})}{(N-p)!p!(p-n)}\times[pK^{(0)}(p)-nK^{(0)}(n)], \quad (6.12)$$

where

$$K^{(0)}(z)=\sum_{m=0}^{\infty}(m-z+\tfrac{1}{2})^{-1}\Gamma(m+\tfrac{1}{2})\times\Gamma(N-m+\tfrac{1}{2})[(N-m)!m!]^{-1}. \quad (6.13)$$

This sum for $K^{(0)}(z)$ can be carried out to yield[13]

$$K^{(0)}(z)=\pi(-1)^N\Gamma(z)\Gamma(\tfrac{1}{2}-z)\times[\Gamma(-N+z)\Gamma(N-z+\tfrac{3}{2})]^{-1}. \quad (6.14)$$

Therefore, for $0\leq p\leq N$ and $n>N$,

$$K_{pn}=\pi^{-1}(p-n)^{-1}\frac{\Gamma(N-p+\frac{3}{2})\Gamma(p+\frac{1}{2})n!\Gamma(n-N-\frac{1}{2})}{(N-p)!p!(n-N-1)!\Gamma(n+\frac{1}{2})}; \quad (6.15)$$

while, for $0\leq p\leq N$ and $n<0$,

$$K_{pn}=\pi^{-1}(p-n)^{-1}\frac{\Gamma(N-p+\frac{3}{2})\Gamma(p+\frac{1}{2})(N-n)!\Gamma(\frac{1}{2}-n)}{(N-p)!p!(-n-1)!\Gamma(N-n+\frac{3}{2})}. \quad (6.16)$$

Equations (6.11), (6.15), and (6.16) give K_{pn} in all cases.

We now turn our attention once more to the general case where α_1 is not necessarily zero.

7. SPIN CORRELATIONS AT THE CRITICAL TEMPERATURE

At the end of Sec. 4, we have obtained the first two terms in the asymptotic expansion of S_N at the critical temperature when $\alpha_1=0$. In this section we shall derive a result more accurate than (5.6), namely,

$$S_N/S_N^{(0)}=(1+\alpha_1)^{1/4}(1-\alpha_1)^{-1/4}\times[1+\tfrac{1}{8}N^{-2}\alpha_1(1-\alpha_1)^{-2}+O(N^{-3})]. \quad (7.1)$$

Thus the generalization of (4.14) is, as $N\to\infty$,

$$S_N=e^{1/4}2^{1/12}A^{-3}N^{-1/4}(1+\alpha_1)^{1/4}(1-\alpha_1)^{-1/4}\times\{1+\tfrac{1}{8}N^{-2}[\alpha_1(1-\alpha_1)^{-2}-\tfrac{1}{8}]+O(N^{-3})\}. \quad (7.2)$$

Even though these results are rather simple, the actual calculation, to be presented below, is quite long.

The following two observations form the basis of the calculational procedure.

[13] See Eq. 2.4 (5) on p. 79 of Ref. 9.

(1) Define

$$d_n=(2\pi)^{-1}\int_0^{2\pi} d\theta\, e^{-in\theta}\varphi(\theta)/\varphi^{(0)}(\theta)$$

$$=(2\pi)^{-1}\int_0^{2\pi} d\theta\, e^{-in\theta}(1-\alpha_1e^{i\theta})^{1/2}(1-\alpha_1e^{-i\theta})^{-1/2}, \quad (7.3)$$

so that

$$a_{m-n}=\sum_{p=-\infty}^{\infty} a_{m-p}{}^{(0)}d_{p-n}, \quad (7.4)$$

where a_n and $a_n{}^{(0)}$ are given by (1.9) and (4.4), respectively. For large $|n|$, d_n is *exponentially small* in $|n|$.

(2) It is seen from (6.3) that, for large N and fixed p and q between 0 and N,

$$L_{pq}=O(1),$$
$$L_{p,N-q}=O(1), \quad (7.5)$$
$$L_{N-p,q}=O(N^{-2}),$$

and

$$L_{N-p,N-q}=O(1).$$

A. Formulation of the Problem

Let $\mathfrak{A}$ be the $(N+1)\times(N+1)$ matrix whose elements are, for $0\leq m\leq N$ and $0\leq n\leq N$,

$$\mathfrak{A}_{mn}=a_{m-n}, \quad (7.6)$$

so that

$$S_{N+1}=\det\mathfrak{A}. \quad (7.7)$$

By (4.5) and (6.1),

$$\mathfrak{A}^{(0)}=2\pi^{-1}M=L^{-1}. \quad (7.8)$$

Furthermore, it is clear from (1.8) that

$$S_N/S_{N+1}=(\mathfrak{A}^{-1})_{00}, \quad (7.9)$$

where the right-hand side denotes the 00th element of the $(N+1)\times(N+1)$ matrix $\mathfrak{A}^{-1}$. Similarly, by (7.8),

$$S_N{}^{(0)}/S_{N+1}{}^{(0)}=L_{00}. \quad (7.10)$$

We propose to obtain (7.1) by calculating the quantity

$$\mathfrak{R}=\frac{S_N}{S_{N+1}}\bigg/\frac{S_N{}^{(0)}}{S_{N+1}{}^{(0)}}=(\mathfrak{A}^{-1})_{00}/L_{00}. \quad (7.11)$$

Let B be the $(N+1)\times(N+1)$ matrix

$$B=L\mathfrak{A}, \quad (7.12)$$

so that

$$\mathfrak{A}^{-1}=B^{-1}L. \quad (7.13)$$

If γ is the column matrix whose elements are

$$\gamma_n=L_{n0}/L_{00}$$
$$=-\tfrac{1}{2}\pi^{-1/2}\frac{\Gamma(n-\frac{1}{2})\Gamma(N-n+\frac{3}{2})N!}{n!(N-n)!\Gamma(N+\frac{3}{2})}, \quad (7.14)$$

then by (7.13) $\mathfrak{R}$ can be expressed by

$$\mathfrak{R}=(B^{-1}\gamma)_0, \quad (7.15)$$

where the right-hand side denotes the zeroth element of the column matrix $B^{-1}\gamma$.

We partition the matrix B as follows:

$$B=\begin{bmatrix} B^{(11)} & B^{(12)} \\ B^{(21)} & B^{(22)}\end{bmatrix}, \quad (7.16)$$

where the sizes of the four matrices $B^{(11)}$, $B^{(12)}$, $B^{(21)}$, and $B^{(22)}$ are, respectively, $N_1\times N_1$, $N_1\times N_2$, $N_2\times N_1$, and $N_2\times N_2$, with $N_1+N_2=N+1$. We chose N_1 and N_2 to be roughly $\frac{1}{2}N$, the precise value being unimportant. A possible choice is, for example,

$$N_1=N_2=\tfrac{1}{2}(N+1)$$

for N odd, and

$$N_1=N_2-1=\tfrac{1}{2}N$$

for even. The inverse of B can be expressed in terms of these $B^{(ij)}$ by

$$B^{-1}=\begin{bmatrix} (B^{(11)}-B^{(12)}B^{(22)-1}B^{(21)})^{-1} & -B^{(11)-1}B^{(12)}(B^{(22)}-B^{(21)}B^{(11)-1}B^{(12)})^{-1} \\ -B^{(22)-1}B^{(21)}(B^{(11)}-B^{(12)}B^{(22)-1}B^{(21)})^{-1} & (B^{(22)}-B^{(21)}B^{(11)-1}B^{(12)})^{-1}\end{bmatrix}. \quad (7.17)$$

It is convenient to renumber the indices by introducing four matrices $\dot{B}^{(ij)}$:

$$\dot{B}_{mn}{}^{(11)}=B_{mn}, \quad \text{for } 0\leq m<N_1,\ 0\leq n<N_1;$$
$$\dot{B}_{mn}{}^{(12)}=B_{m,N-n}, \quad \text{for } 0\leq m<N_1,\ 0\leq n<N_2;$$
$$\dot{B}_{mn}{}^{(21)}=B_{N-m,n}, \quad \text{for } 0\leq m<N_2,\ 0\leq n<N_1; \quad (7.18)$$

and

$$\dot{B}_{mn}{}^{(22)}=B_{N-m,N-n}, \quad \text{for } 0\leq m<N_2,\ 0\leq n<N_2.$$

Similarly, we use two column matrices $\gamma^{(1)}$ and $\gamma^{(2)}$ defined by

$$\gamma_n{}^{(1)}=\gamma_n, \quad \text{for } 0\leq n<N_1,$$

and

$$\gamma_n{}^{(2)}=\gamma_{N-n}, \quad \text{for } 0\leq n<N_2. \quad (7.19)$$

Then, by (7.15) and (7.17), $\mathfrak{R}$ can be expressed by

$$\mathfrak{R}=\{(\dot{B}^{(11)}-\dot{B}^{(12)}\dot{B}^{(22)-1}\dot{B}^{(21)})^{-1}\gamma^{(1)} -\dot{B}^{(11)-1}\dot{B}^{(12)}(\dot{B}^{(22)}-\dot{B}^{(21)}\dot{B}^{(11)-1}\dot{B}^{(12)})^{-1}\gamma^{(2)}\}_0. \quad (7.20)$$

As seen from (6.3) for example, our expressions contain a rather large number of gamma functions. It is convenient to remove some of these by changing the $\dot{B}$'s slightly. Let

$$\bar{B}_{mn}{}^{(11)}=\frac{(N-m)!\Gamma(N-n+\frac{3}{2})}{\Gamma(N-m+\frac{3}{2})(N-n)!}\dot{B}_{mn}{}^{(11)},$$

$$\bar{B}_{mn}{}^{(12)}=\frac{N(N-m)!\Gamma(N-n+\frac{1}{2})}{\Gamma(N-m+\frac{3}{2})(N-n)!}\dot{B}_{mn}{}^{(12)},$$

$$\bar{B}_{mn}{}^{(21)}=\frac{N(N-m)!\Gamma(N-n+\frac{3}{2})}{\Gamma(N-m+\frac{1}{2})(N-n)!}\dot{B}_{mn}{}^{(21)},$$

and

$$\bar{B}_{mn}{}^{(22)}=\frac{(N-m)!\Gamma(N-n+\frac{1}{2})}{\Gamma(N-m+\frac{1}{2})(N-n)!}\dot{B}_{mn}{}^{(22)}. \tag{7.21}$$

By (7.5) and (7.18), for large N but fixed m and n, all four $\bar{B}_{mn}{}^{(ij)}$ are of the order of magnitude 1. The substitution of (7.21) into (7.20) yields

$$\mathcal{R}=\{(\bar{B}^{(11)}-N^{-2}\bar{B}^{(12)}\bar{B}^{(22)-1}\bar{B}^{(21)})^{-1}\bar{\gamma}^{(1)}-N^{-2}\bar{B}^{(11)-1}\bar{B}^{(12)} \times(\bar{B}^{(22)}-N^{-2}\bar{B}^{(21)}\bar{B}^{(11)-1}\bar{B}^{(12)})^{-1}\bar{\gamma}^{(2)}\}_0, \tag{7.22}$$

where

$$\bar{\gamma}_n{}^{(1)}=\frac{(N-n)!\Gamma(N+\frac{3}{2})}{\Gamma(N-n+\frac{3}{2})N!}\gamma_n{}^{(1)} \tag{7.23}$$
$$=-\tfrac{1}{2}\pi^{-1/2}\Gamma(n-\tfrac{1}{2})/n!,$$

and

$$\bar{\gamma}_n{}^{(2)}=\frac{(N-n)!\Gamma(N+\frac{3}{2})}{\Gamma(N-n+\frac{1}{2})N!}\gamma_n{}^{(2)}$$
$$=-\tfrac{1}{2}\pi^{-1/2}N(N-n-\tfrac{1}{2})^{-1}\Gamma(n+\tfrac{3}{2})/n!. \tag{7.24}$$

Equation (7.22) is to be used for further development in this section. It only remains to write down the numerous $\bar{B}_{mn}{}^{(ij)}$ explicitly. By (7.12) and (7.4),

$$B_{mn}=\sum_{q=0}^{N}\sum_{p=-\infty}^{\infty}L_{mq}a_{q-p}{}^{(0)}d_{p-n}$$
$$=\sum_{p=-\infty}^{\infty}K_{mp}d_{p-n}, \tag{7.25}$$

where K_{mp} is defined by (6.10). Using (6.11), (6.15), and (6.16), we find

$$\bar{B}_{mn}{}^{(11)}=d_{mn}'+\pi^{-1}\sum_{p=-\infty}^{-1}(m-p)^{-1}\frac{\Gamma(m+\frac{1}{2})\Gamma(\frac{1}{2}-p)}{m!(-p-1)!}d_{pn}' +\pi^{-1}\sum_{p=N+1}^{\infty}(m-p)^{-1}\frac{\Gamma(m+\frac{1}{2})p!\Gamma(p-N-\frac{1}{2})\Gamma(N-n+\frac{3}{2})}{m!(p-N-1)!\Gamma(p+\frac{1}{2})(N-n)!}d_{p-n}, \tag{7.26}$$

$$\bar{B}_{mn}{}^{(12)}=-\pi^{-1}\sum_{p=-\infty}^{-1}N(N-m-p)^{-1}\frac{\Gamma(m+\frac{1}{2})\Gamma(-p-\frac{1}{2})}{m!(-p-1)!}d_{pn}''+\frac{N(N-m)!\Gamma(N-n+\frac{1}{2})}{\Gamma(N-m+\frac{3}{2})(N-n)!}d_{m+n-N} -\pi^{-1}N\sum_{p=N+1}^{\infty}(N-m-p)^{-1}\frac{\Gamma(m+\frac{1}{2})p!\Gamma(p-N+\frac{1}{2})\Gamma(N-n+\frac{1}{2})}{m!(p-N-1)!\Gamma(p+\frac{3}{2})(N-n)!}d_{-p+n}, \tag{7.27}$$

$$\bar{B}_{mn}{}^{(21)}=\pi^{-1}\sum_{p=-\infty}^{-1}N(N-m-p)^{-1}\frac{\Gamma(m+\frac{3}{2})\Gamma(\frac{1}{2}-p)}{m!(-p-1)!}d_{pn}'+\frac{N(N-m)!\Gamma(N-n+\frac{3}{2})}{\Gamma(N-m+\frac{1}{2})(N-n)!}d_{N-m-n} +\pi^{-1}N\sum_{p=N+1}^{\infty}(N-m-p)^{-1}\frac{\Gamma(m+\frac{1}{2})p!\Gamma(p-N-\frac{1}{2})\Gamma(N-n+\frac{3}{2})}{m!(p-N-1)!\Gamma(p+\frac{1}{2})(N-n)!}d_{p-n}, \tag{7.28}$$

and

$$\bar{B}_{mn}{}^{(22)}=d_{mn}''-\pi^{-1}\sum_{p=-\infty}^{-1}(m-p)^{-1}\frac{\Gamma(m+\frac{3}{2})\Gamma(-p-\frac{1}{2})}{m!(-p-1)!}d_{pn}'' -\pi^{-1}\sum_{p=N+1}^{\infty}(m-p)^{-1}\frac{\Gamma(m+\frac{3}{2})p!\Gamma(p-N+\frac{1}{2})\Gamma(N-n+\frac{1}{2})}{m!(p-N-1)!\Gamma(p+\frac{3}{2})(N-n)!}d_{-p+n}. \tag{7.29}$$

In (7.26)–(7.29), we have used the notation

$$d_{pn}'=\frac{(N-p)!\Gamma(N-n+\frac{3}{2})}{\Gamma(N-p+\frac{3}{2})(N-n)!}d_{p-n},$$

and

$$d_{pn}''=\frac{(N-p)!\Gamma(N-n+\frac{1}{2})}{\Gamma(N-p+\frac{1}{2})(N-n)!}d_{-p+n}. \tag{7.30}$$

B. Asymptotic Expansion of $\mathcal{R}$

So far, no approximation has been made and (7.22) is exact. Since (7.26)–(7.29) are rather complicated, we restrict our attention to asymptotic expansions for large N. As d_n is exponentially small for large $|n|$, (7.26)–(7.29) are much simpler asymptotically. We need to keep only the first two terms in the cases of

(7.26) and (7.29), and only the first term in (7.27) and (7.28):

$$\bar{B}_{mn}{}^{(11)} \doteqdot d_{mn}{}' + \pi^{-1} \sum_{p=-\infty}^{-1} (m-p)^{-1} [m!(-p-1)!]^{-1} \times \Gamma(m+\tfrac{1}{2})\Gamma(\tfrac{1}{2}-p) d_{pn}{}', \quad (7.31)$$

$$\bar{B}_{mn}{}^{(12)} \doteqdot -\pi^{-1} \sum_{p=-\infty}^{-1} N(N-m-p)^{-1} [m!(-p-1)!]^{-1} \times \Gamma(m+\tfrac{1}{2})\Gamma(-\tfrac{1}{2}-p) d_{pn}{}'', \quad (7.32)$$

$$\bar{B}_{mn}{}^{(21)} \doteqdot \pi^{-1} \sum_{p=-\infty}^{-1} N(N-m-p)^{-1} [m!(-p-1)!]^{-1} \times \Gamma(m+\tfrac{3}{2})\Gamma(\tfrac{1}{2}-p) d_{pn}{}', \quad (7.33)$$

and

$$B_{mn}{}^{(22)} \doteqdot d_{mn}{}'' - \pi^{-1} \sum_{p=-\infty}^{-1} (m-p)^{-1} [m!(-p-1)!]^{-1} \times \Gamma(m+\tfrac{3}{2})\Gamma(-\tfrac{1}{2}-p) d_{pn}{}''. \quad (7.34)$$

[Strictly speaking, (7.32) is not valid if m is close to N_1 and n close to N_2, while (7.33) is not valid if m is close to N_2 and n close to N_1. However, these elements of the matrices do not contribute to the asymptotic expansion of $\mathcal{R}$.]

Let $\tilde{d}_{pn}{}'$ and $\tilde{d}_{pn}{}''$ be respectively the asymptotic series for $d_{pn}{}'$ and $d_{pn}{}''$ for large N and fixed p and n. Thus $\tilde{d}_{pn}{}'$ and $\tilde{d}_{pn}{}''$ are defined, term by term, for all p and all n. Similarly, let $\tilde{B}_{mn}{}^{(ij)}$, $i=1$, 2 and $j=1$, 2, be the asymptotic series for $\bar{B}_{mn}{}^{(ij)}$, again for large N and fixed m and n. Thus $\tilde{B}_{mn}{}^{(ij)}$ is defined, term by term for all $m \geq 0$ and all $n \geq 0$. We can accordingly form the infinite matrices $\tilde{B}^{(ij)}$, term by term in powers of N^{-1}. The row and column indices for $\tilde{B}^{(ij)}$ each run from zero to infinity. In the same fashion, we define infinite column matrices $\tilde{\gamma}^{(i)}$ from the $\bar{\gamma}^{(i)}$ of (7.23) and (7.24), again as asymptotic series in N^{-1}. With the help of these infinite matrices and (7.22), we get asymptotically

$$\mathcal{R} \doteqdot \{(\tilde{B}^{(11)} - N^{-2}\tilde{B}^{(12)}\tilde{B}^{(22)-1}\tilde{B}^{(21)})^{-1}\tilde{\gamma}^{(1)} - N^{-2}\tilde{B}^{(11)-1}\tilde{B}^{(12)} \times (\tilde{B}^{(22)} - N^{-2}\tilde{B}^{(21)}\tilde{B}^{(11)-1}\tilde{B}^{(12)})^{-1}\tilde{\gamma}^{(2)}\}. \quad (7.35)$$

[It is worthwhile to keep in mind that infinite matrices may not be associative.]

Let $\tilde{A}$ be the infinite matrix whose elements are

$$\tilde{A}_{mn} = a_{m-n}{}^{(0)} = 2\pi^{-1}(2m-2n+1)^{-1} \quad (7.36)$$

for $m \geq 0$ and $n \geq 0$, and let $\tilde{A}^T$ be the transpose of $\tilde{A}$. We also define

$$\mathcal{B}^{(1)} = \tilde{A}\tilde{B}^{(11)}, \quad (7.37)$$

and

$$\mathcal{B}^{(2)} = \tilde{A}^T\tilde{B}^{(22)}. \quad (7.38)$$

Therefore the required matrices $\tilde{B}^{(11)-1}$ and $\tilde{B}^{(22)-1}$ are

$$\tilde{B}^{(11)-1} = \mathcal{B}^{(1)-1}\tilde{A}, \quad (7.39)$$

and

$$\tilde{B}^{(22)-1} = \mathcal{B}^{(2)-1}\tilde{A}^T. \quad (7.40)$$

We proceed to calculate the elements of $\mathcal{B}^{(1)}$ and $\mathcal{B}^{(2)}$. Since, for $p<0$,

$$\sum_{q=0}^{\infty} (m-q+\tfrac{1}{2})^{-1}(q-p)^{-1}\Gamma(q+\tfrac{1}{2})/q! = (m-p+\tfrac{1}{2})^{-1} \sum_{q=0}^{\infty} [(m-q+\tfrac{1}{2})^{-1} + (q-p)^{-1}] \times \Gamma(q+\tfrac{1}{2})/q! = \pi(m-p+\tfrac{1}{2})^{-1}\Gamma(-p)/\Gamma(\tfrac{1}{2}-p), \quad (7.41)$$

we must have

$$\mathcal{B}_{mn}{}^{(1)} = 2\pi^{-1} \sum_{p=0}^{\infty} (2m-2p+1)^{-1}\tilde{d}_{pn}{}' + 2\pi^{-1} \sum_{p=-\infty}^{-1} (2m-2p+1)^{-1}\tilde{d}_{pn}{}' = 2\pi^{-1} \sum_{p=-\infty}^{\infty} (2m-2p+1)^{-1}\tilde{d}_{pn}{}'. \quad (7.42)$$

Similarly

$$\mathcal{B}_{mn}{}^{(2)} = -2\pi^{-1} \sum_{p=-\infty}^{\infty} (2m-2p-1)^{-1}\tilde{d}_{pn}{}'. \quad (7.43)$$

Both (7.42) and (7.43) are to be understood in the sense of term-by-term equality for each power of N^{-1}.

A very similar calculation yields

$$(\tilde{A}\tilde{\gamma}^{(1)})_n = \delta_{n0}. \quad (7.44)$$

Unlike (7.42) and (7.43), (7.44) does not involve N and hence is exact. With this result (7.35) can be written alternatively in the form

$$\mathcal{R} \doteqdot \{(1 - N^{-2}\tilde{B}^{(11)-1}\tilde{B}^{(12)}\tilde{B}^{(22)-1}\tilde{B}^{(21)})^{-1}\mathcal{B}^{(1)-1}\}_{00} - N^{-2}\{\tilde{B}^{(11)-1}\tilde{B}^{(12)}(1 - N^{-2}\tilde{B}^{(22)-1}\tilde{B}^{(21)}\tilde{B}^{(11)-1}\tilde{B}^{(12)})^{-1} \times \tilde{B}^{(22)-1}\tilde{\gamma}^{(2)}\}_0. \quad (7.45)$$

We have kept the entire asymptotic series so far. From here on, we shall keep only enough terms to get (7.1). For this more limited purpose, we write

$$\mathcal{R} \sim \mathcal{R}_1 + \mathcal{R}_2 + \mathcal{R}_3, \quad (7.46)$$

where

$$\mathcal{R}_1 = (\mathcal{B}^{(1)-1})_{00}, \quad (7.47)$$

$$\mathcal{R}_2 = N^{-2}\{\tilde{B}^{(11)-1}\tilde{B}^{(12)}\tilde{B}^{(22)-1}\tilde{B}^{(21)}\mathcal{B}^{(1)-1}\}_{00}, \quad (7.48)$$

and

$$\mathcal{R}_3 = -N^{-2}\{\tilde{B}^{(11)-1}\tilde{B}^{(12)}\tilde{B}^{(22)-1}\tilde{\gamma}^{(2)}\}_0. \quad (7.49)$$

We need to calculate each of these three $\mathcal{R}$'s to the accuracy N^{-3} for large N.

It is useful to note, in connection with (7.42) and (7.43), that it follows from (7.30)

$$\tilde{d}_{pn}{}' = d_{p-n}[1 + \tfrac{1}{2}\bar{N}^{-1}(p-n) + O(N^{-2})],$$

and

$$\tilde{d}_{pn}{}'' = d_{-p+n}[1 - \tfrac{1}{2}\bar{N}^{-1}(p-n) + O(N^{-2})], \quad (7.50)$$

where

$$\bar{N}=N+O(1). \quad (7.51)$$

We therefore define an infinite matrix $\mathcal{B}$ by, for $m\geq 0$ and $n\geq 0$,

$$\mathcal{B}_{mn}=2\pi^{-1}\sum_{p=-\infty}^{\infty}(2m-2p+1)^{-1}\times d_{p-n}[1+\tfrac{1}{2}\bar{N}^{-1}(p-n)]. \quad (7.52)$$

Thus, to order N^{-1},

$$\mathcal{B}^{(1)}\sim\mathcal{B} \quad\text{and}\quad \mathcal{B}^{(2)}\sim\mathcal{B}^{T}. \quad (7.53)$$

This approximation (7.53) may be used in (7.48) and (7.49) for the purposes of obtaining $\mathcal{R}_2$ and $\mathcal{R}_3$, but is not accurate enough for $\mathcal{R}_1$ as given by (7.47).

C. Approximate Calculation of $\mathcal{B}^{-1}$

Let

$$\mathcal{S}=\mathcal{B}^{-1}. \quad (7.54)$$

Since, by (7.52), $\mathcal{B}_{mn}$ depends only on $m-n$, we may compute $\mathcal{S}$ again by the method of Wiener and Hopf.[6] Define $C(\xi)$ for $|\xi|=1$ such that

$$\mathcal{B}_{mn}=(2\pi)^{-1}\int_0^{2\pi}d\theta\, e^{-i(m-n)\theta}C(e^{i\theta}), \quad (7.55)$$

then

$$C(\xi)=i\xi^{-1/2}\left(\frac{1-\alpha_1\xi}{1-\alpha_1/\xi}\right)^{1/2}\times\left[1-\frac{\alpha_1}{4\bar{N}}\frac{\xi-2\alpha_1+1/\xi}{(1-\alpha_1\xi)(1-\alpha_1/\xi)}\right]. \quad (7.56)$$

Let

$$\alpha_1'=\tfrac{1}{2}\alpha_1^{-1}(1+\tfrac{1}{4}\bar{N}^{-1})^{-1}(1+\alpha_1^2(1+\tfrac{1}{2}\bar{N}^{-1})-\{[1+\alpha_1^2(1+\tfrac{1}{2}\bar{N}^{-1})]^2-4\alpha_1^2(1+\tfrac{1}{4}\bar{N}^{-1})^2\}^{1/2}), \quad (7.57)$$

then

$$C(\xi)=i\xi^{-1/2}(1-\alpha_1\xi)^{1/2}(1-\alpha_1/\xi)^{-1/2}\times\mathcal{C}[(1-\alpha_1'\xi)/(1-\alpha_1'/\xi)][(\xi-\alpha_1')/(\xi-\alpha)], \quad (7.58)$$

where

$$\mathcal{C}=\alpha_1(1+\tfrac{1}{4}\bar{N}^{-1})/\alpha_1'. \quad (7.59)$$

Suppose that we define P and Q on the basis of (2.17) using this $C(\xi)$ of (7.58), then one possible choice is

$$P(\xi)=(1-\xi)^{1/2}(1-\alpha_1\xi)^{1/2}(1-\alpha_1'\xi)^{-1}, \quad (7.60)$$

and

$$Q(\xi)=\mathcal{C}^{-1}(1-\xi)^{-1/2}(1-\alpha_1\xi)^{3/2}(1-\alpha_1'\xi)^{-1}. \quad (7.61)$$

When $\bar{N}\to\infty$, the constant $\mathcal{C}$ is easily found to be

$$\mathcal{C}=1-\tfrac{1}{16}\alpha_1^2(1-\alpha_1^2)^{-1}\bar{N}^{-2}-\tfrac{1}{32}\alpha_1^4(1-\alpha_1^2)^{-2}\bar{N}^{-3}+O(N^{-4}). \quad (7.62)$$

By (7.54), the elements of the matrix $\mathcal{S}$ satisfy

$$\sum_{m=0}^{\infty}\mathcal{B}_{pm}\mathcal{S}_{mn}=\delta_{pn}, \quad (7.63)$$

and

$$\sum_{n=0}^{\infty}\mathcal{S}_{mn}\mathcal{B}_{np}=\delta_{mp}. \quad (7.64)$$

Both of these equations can be solved by the method of Wiener and Hopf and the results are, using the notation of Sec. 2 with (7.60) and (7.61),

$$\sum_{m=0}\mathcal{S}_{mn}\xi^m=\mathcal{C}^{-1}(1-\xi)^{1/2}(1-\alpha_1\xi)^{1/2}(1-\alpha_1'\xi)^{-1}\times[\xi^n(\xi-1)^{-1/2}(\xi-\alpha_1)^{3/2}(\xi-\alpha_1')^{-1}]_+ \quad (7.65)$$

and

$$\sum_{n=0}^{\infty}\mathcal{S}_{mn}\xi^n=\mathcal{C}^{-1}(1-\xi)^{-1/2}(1-\alpha_1\xi)^{3/2}(1-\alpha_1'\xi)^{-1}\times[\xi^m(\xi-1)^{1/2}(\xi-\alpha_1)^{1/2}(\xi-\alpha_1')^{-1}]_+. \quad (7.66)$$

Equations (7.65) and (7.66) are of course merely different versions of the same formula. A more symmetrical way to write this result is as follows: Let

$$P(\xi)=\sum_{n=0}^{\infty}p_n\xi^n,$$

and

$$Q(\xi)=\sum_{n=0}^{\infty}q_n\xi^n, \quad (7.67)$$

for $|\xi|<1$, then

$$\mathcal{S}_{mn}=\sum_{j=0}^{\min(m,n)}p_{m-j}q_{n-j}. \quad (7.68)$$

In particular, it follows from (7.62) that

$$\mathcal{S}_{00}=\mathcal{C}^{-1}=1+\tfrac{1}{16}\alpha_1^2(1-\alpha_1^2)^{-1}\bar{N}^{-2}+\tfrac{1}{32}\alpha_1^4(1-\alpha_1^2)^{-2}\bar{N}^{-3}+O(N^{-4}). \quad (7.69)$$

D. Approximate Calculation of $\mathcal{R}_1$

In order to obtain $\mathcal{R}_1$ of (7.47) to the accuracy N^{-3}, we need a formula for $\tilde{d}_{pn}'$ more accurate than (7.50). Since, for $z\to\infty$,

$$\Gamma(z+\tfrac{3}{2})/\Gamma(z+1)=(z+\tfrac{3}{4})^{1/2}[1+\tfrac{1}{64}(z+\tfrac{3}{4})^{-2}+O(z^{-4})], \quad (7.70)$$

we have from (7.30), for $N\to\infty$,

$$d_{pn}'=d_{p-n}(N+\tfrac{3}{4}-n)^{1/2}(N+\tfrac{3}{4}-p)^{-1/2}\times[1+\tfrac{1}{32}N^{-3}(n-p)+O(N^{-4})]=d_{p-n}(\bar{N}-n)^{1/2}(\bar{N}-p)^{-1/2}[1+O(N^{-4})], \quad (7.71)$$

provided that

$$\bar{N}=N+\tfrac{3}{4}+\tfrac{1}{16}N^{-1}. \quad (7.72)$$

On the other hand, note that the term $\bar{N}^{-1}$ is missing on the right-hand side of (7.69). Thus it is sufficient, for the purpose of obtaining (7.1), to use

$$\bar{N}=N+\tfrac{3}{4}. \quad (7.73)$$

By (7.71), (7.42), and (7.52), we have, to the required accuracy,

$$\tilde{B}^{(1)}\sim\mathcal{B}+\mathcal{B}', \tag{7.74}$$

where

$$\mathcal{B}_{mn}'=2\pi^{-1}\sum_{p=-\infty}^{\infty}(2m-2p+1)^{-1}d_{p-n}(p-n)\times\tfrac{1}{8}\bar{N}^{-2}[(3p+n)+\tfrac{1}{2}\bar{N}^{-1}(5p^2+2pn+n^2)]. \tag{7.75}$$

Therefore, the required $\mathcal{R}_1$ is

$$\mathcal{R}_1\sim\mathcal{S}_{00}-\{(\mathcal{S}\mathcal{B}')\mathcal{S}\}_{00}. \tag{7.76}$$

Since $\mathcal{S}_{00}$ is given by (7.69), we concentrate on the second term here.

Let

$$\mathcal{T}_p=2\pi^{-1}\sum_{n=0}^{\infty}\mathcal{S}_{0n}(2n-2p+1)^{-1} \tag{7.77}$$

for all integers p, then

$$\mathcal{R}_1\sim\mathcal{S}_{00}-\tfrac{1}{8}\bar{N}^{-2}\sum_{p=-\infty}^{\infty}\sum_{m=0}^{\infty}\mathcal{T}_p d_{p-m}(p-m)\times[(3p+m)+\tfrac{1}{2}\bar{N}^{-1}(5p^2+2pm+m^2)]\mathcal{S}_{m0}. \tag{7.78}$$

Since, by (7.66), the generating function for $\mathcal{S}_{0n}$ is

$$\sum_{n=0}^{\infty}\mathcal{S}_{0n}\xi^n=\mathcal{C}^{-1}(1-\xi)^{-1/2}(1-\alpha_1\xi)^{3/2}(1-\alpha_1'\xi)^{-1}, \tag{7.79}$$

$\mathcal{T}_p$ is generated by

$$\sum_{p=-\infty}^{\infty}\mathcal{T}_p\xi^p=\mathcal{C}^{-1}\xi^{1/2}(\xi-1)^{-1/2}\times(1-\alpha_1\xi)^{3/2}(1-\alpha_1'\xi)^{-1}. \tag{7.80}$$

The point of greatest importance here is that, while the right-hand side (7.79) has a branch cut from 1 to infinity, the right-hand side of (7.80) is *analytic in the region*

$$1<|\xi|<1/\alpha_1'. \tag{7.81}$$

Accordingly

$$\mathcal{T}_p\rightarrow 0 \tag{7.82}$$

exponentially as $p\rightarrow+\infty$, and

$$\mathcal{T}_{-p}=(2\pi i\mathcal{C})^{-1}\oint_{-}d\xi\,\xi^{-p-1}\xi^{-1/2}(1-\xi)^{-1/2}\times(\xi-\alpha_1)^{3/2}(\xi-\alpha_1')^{-1}, \tag{7.83}$$

where $\oint_-$ denotes a contour integral in the counterclockwise direction along a circular path of radius between α_1' and 1. Note that the integrand in (7.83) is positive for $\alpha_1'<\xi<1$.

With (7.65) and (7.83), the evaluation of $\mathcal{R}_1$ by (7.78) is straightforward but tedious. By (7.50), it is convenient to rewrite (7.78) in the form

$$\mathcal{R}_1\sim\mathcal{S}_{00}-\tfrac{1}{8}\bar{N}^{-2}\sum_{p=-\infty}^{\infty}\sum_{m=0}^{\infty}\mathcal{T}_p\{d_{p-m}[1+\tfrac{1}{2}\bar{N}^{-1}(p-m)]\}\times\{(p-m)[(3p+m)+\bar{N}^{-1}(p+m)^2]\}\mathcal{S}_{m0}. \tag{7.84}$$

Let

$$\mathcal{G}_1(\xi)=\xi^{-1/2}(1-\xi)^{-1/2}(\xi-\alpha_1)^{3/2}(\xi-\alpha_1')^{-1}=\sum_{p=-\infty}^{\infty}\mathcal{C}\mathcal{T}_{-p}\xi^p, \tag{7.85}$$

$$\mathcal{G}_2(\xi)=\xi^{1/2}(1-\alpha_1\xi)^{-1/2}(1-\alpha_1'\xi)(\xi-\alpha_1)^{-3/2}(\xi-\alpha_1')=\sum_{p=-\infty}^{\infty}d_p(1+\tfrac{1}{2}\bar{N}^{-1}p)\xi^p, \tag{7.86}$$

and

$$\mathcal{G}_3(\xi)=(1-\xi)^{1/2}(1-\alpha_1\xi)^{1/2}(1-\alpha_1'\xi)^{-1}=\sum_{m=0}^{\infty}\mathcal{C}\mathcal{S}_{m0}\xi^m. \tag{7.87}$$

Then (7.84) is equivalent to

$$\mathcal{R}_1\sim\mathcal{S}_{00}-\tfrac{1}{8}\bar{N}^{-2}(2\pi i)^{-1}\oint_{-}d\xi\times\xi^{-1}[(2\mathcal{G}_1''\mathcal{G}_2\mathcal{G}_3+\mathcal{G}_1\mathcal{G}_2''\mathcal{G}_3)-\tfrac{1}{3}\bar{N}^{-1}\times(4\mathcal{G}_1'''\mathcal{G}_2\mathcal{G}_3+\mathcal{G}_1\mathcal{G}_2'''\mathcal{G}_3)], \tag{7.88}$$

where each prime means $\xi d/d\xi$. In obtaining (7.88), we have made use of the fact that $\mathcal{G}_3(\xi)$ is analytic inside the unit circle. The various functions have been chosen in such a way that

$$\mathcal{G}_1(\xi)\mathcal{G}_2(\xi)\mathcal{G}_3(\xi)=1. \tag{7.89}$$

Because of (7.89), the evaluation of the right-hand side of (7.88) is not difficult with the help of the formulas

$$\oint_{-}d\xi\,\xi^{-1}\mathcal{G}_i^{-1}\mathcal{G}_i''=\oint_{-}\xi\,d\xi(d\ln\mathcal{G}_i/d\xi)^2,$$

and

$$\oint_{-}d\xi\,\xi^{-1}\mathcal{G}_i^{-1}\mathcal{G}_i'''=\oint_{-}\xi^2d\xi(d\ln\mathcal{G}_i/d\xi)^3, \tag{7.90}$$

for $i=1,2$. By (7.62) and (7.73), the result for $\mathcal{R}_1$ is

$$\begin{aligned}\mathcal{R}_1&=1-\tfrac{1}{8}\bar{N}^{-2}\alpha_1(1-\alpha_1)^{-1}+\tfrac{1}{16}\bar{N}^{-3}\alpha_1(1+2\alpha_1)(1-\alpha_1)^{-2}+O(N^{-4})\\&=1-\tfrac{1}{8}N^{-2}\alpha_1(1-\alpha_1)^{-1}+\tfrac{1}{16}N^{-3}\alpha_1(4-\alpha_1)(1-\alpha_1)^{-2}+O(N^{-4}).\end{aligned} \tag{7.91}$$

E. Approximate Calculation of $\mathcal{R}_3$

Let

$$\rho_n=-\tfrac{1}{2}\pi^{-1/2}N(N-n+\tfrac{1}{2})^{-1}\Gamma(n+\tfrac{1}{2})/n!, \tag{7.92}$$

and

$$\tau_n=2\pi^{-1/2}\sum_{p=-\infty}^{-1}[(-p-1)!]^{-1}\Gamma(-p-\tfrac{1}{2})\times[1+N^{-1}(p+\tfrac{1}{2})]\tilde{d}_{pn}'', \quad (7.93)$$

then by (7.32), to the required accuracy, for fixed $m\geq 0$ and $n\geq 0$,

$$\bar{B}_{mn}^{(12)}\sim\rho_m\tau_n. \quad (7.94)$$

Let ρ be the infinite column matrix with elements ρ_n, and τ be the infinite row matrix with elements τ_n, then by (7.49)

$$\mathcal{R}_3\sim -N^{-2}\mathcal{R}_3'\mathcal{R}_3'', \quad (7.95)$$

where

$$\mathcal{R}_3'=(\tilde{B}^{(11)-1}\rho)_0, \quad (7.96)$$

and

$$\mathcal{R}_3''=\tau\tilde{B}^{(22)-1}\tilde{\gamma}^{(2)}. \quad (7.97)$$

Because of (7.96), for the purpose of computing $\mathcal{R}_3$, we need by (7.39), (7.53), and (7.54)

$$(\tilde{B}^{(11)-1})_{0n}=\sum_{m=0}^{\infty}(\mathcal{B}^{(1)-1})_{0m}a_{m-n}^{(0)}\sim\sum_{m=0}^{\infty}\mathcal{S}_{0m}a_{m-n}^{(0)}. \quad (7.98)$$

Accordingly, by (7.66), $(\tilde{B}^{(11)-1})_{0n}$ are generated approximately by

$$\sum_{n=-\infty}^{\infty}(\tilde{B}^{(11)-1})_{0n}\xi^n\sim\xi^{1/2}(\xi-1)^{-1/2}\times(1-\alpha_1\xi)^{3/2}(1-\alpha_1'\xi)^{-1}. \quad (7.99)$$

We note that the right-hand side of (7.99) is analytic in the region (7.81). On the other hand, ρ_n are generated by, also approximately,

$$\sum_{n=0}^{\infty}\rho_n\xi^n\sim -\tfrac{1}{2}(1-N^{-1})(1-\xi)^{-1/2}\times[1+\tfrac{1}{2}N^{-1}(1-\xi)^{-1}], \quad (7.100)$$

which is analytic in the unit circle. A straightforward calculation from (7.96), (7.99), and (7.100) then gives

$$\mathcal{R}_3'\sim -\tfrac{1}{2}(1-N^{-1})(2\pi i)^{-1}\oint_+ d\xi(\xi-1)^{-1}(1-\alpha_1\xi)^{3/2}\times(1-\alpha_1'\xi)^{-1}[1+\tfrac{1}{2}N^{-1}\xi(\xi-1)^{-1}], \quad (7.101)$$

where $\oint_+$ denotes a contour integration in the counterclockwise direction along a circular path lying in the region (7.81), and hence

$$\mathcal{R}_3'\sim -\tfrac{1}{2}(1-\alpha_1)^{1/2}(1-\tfrac{1}{2}N^{-1}) \quad (7.102)$$

as $N\to\infty$ with an error of the order N^{-2}.

We proceed to calculate $\mathcal{R}_3''$. By (7.97), (7.40), and (7.53),

$$\mathcal{R}_3''\sim(\tau\mathcal{S}^T\tilde{A}^T)\tilde{\gamma}^{(2)}. \quad (7.103)$$

It follows from (7.50) and (7.93) that the τ_n are generated approximately by

$$\sum_{n=-\infty}^{\infty}\tau_n\xi^{-n}\sim 2\xi(1-\xi)^{-1/2}(1-\alpha_1\xi)^{-3/2}(1-\alpha_1'\xi)(1-\alpha_1/\xi)^{-1/2}\times(1-\alpha_1'/\xi)[1-\tfrac{1}{2}N^{-1}(1-\xi)^{-1}]. \quad (7.104)$$

Hence, by (7.66), (7.60), and (7.67), for $|\zeta|\leq 1$,

$$\begin{aligned}\sum_{n=0}^{\infty}(\tau\mathcal{S}^T)_n\zeta^n &\sim(\pi i)^{-1}\oint_- d\xi(1-\xi)^{-1}(1-\alpha_1/\xi)^{-1/2}(1-\alpha_1'/\xi)[1-\tfrac{1}{2}N^{-1}(1-\xi)^{-1}]\sum_{n=0}^{\infty}\zeta^n\sum_{m=0}^{n}p_m\xi^{n-m}\\ &=(\pi i)^{-1}\oint_- d\xi(1-\xi)^{-1}(1-\alpha_1/\xi)^{-1/2}(1-\alpha_1'/\xi)[1-\tfrac{1}{2}N^{-1}(1-\xi)^{-1}](1-\xi\zeta)^{-1}(1-\zeta)^{-1}(1-\alpha_1\zeta)^{1/2}(1-\alpha_1'\zeta)^{-1}\\ &=2(1-\zeta)^{-1/2}[(1-\alpha_1)^{1/2}(1-\alpha_1\zeta)^{1/2}(1-\alpha_1'\zeta)^{-1}-1]+\tfrac{1}{2}N^{-1}\zeta(1-\zeta)^{-1}]. \end{aligned}\quad (7.105)$$

On the other hand, by (7.24), $\tilde{\gamma}_n^{(2)}$ are generated approximately by

$$\sum_{n=0}^{\infty}\tilde{\gamma}_n^{(2)}\xi^{-n}\sim -\tfrac{1}{4}(1-N^{-1})\xi^{3/2}(\xi-1)^{-3/2}\times[1+\tfrac{3}{2}N^{-1}\xi(\xi-1)^{-1}], \quad (7.106)$$

for $|\xi|>1$. The substitution of (7.105) and (7.106) gives, after some algebraic simplification very similar to that encountered previously,

$$\mathcal{R}_3''\sim\tfrac{1}{4}\alpha_1(1-\alpha_1)^{-1}\{1+\tfrac{1}{2}N^{-1}(1+2\alpha_1)(1-\alpha_1)^{-1}\}. \quad (7.107)$$

And the substitution of (7.102) and (7.107) into (7.95) yields

$$\mathcal{R}_3\sim\tfrac{1}{8}N^{-2}\alpha_1(1-\alpha_1)^{-1/2}\{1+\tfrac{3}{2}N^{-1}\alpha_1(1-\alpha_1)^{-1}\}, \quad (7.108)$$

again for $N\to\infty$ with an error of the order N^{-4}.

F. Approximate Calculation of $\mathcal{R}_2$

Entirely similar to (7.93), let

$$\bar{\tau}_n=-2\pi^{-1/2}\sum_{p=-\infty}^{-1}[(-p-1)!]^{-1}\Gamma(\tfrac{1}{2}-p)\times[1+N^{-1}(p-\tfrac{1}{2})]\tilde{d}_{pn}', \quad (7.109)$$

then, by (7.33) and (7.24), we get to the required accuracy, for fixed $m\geq 0$ and $n\geq 0$,

$$\bar{B}_{mn}^{(21)}\sim\tilde{\gamma}_m^{(2)}\bar{\tau}_n. \quad (7.110)$$

The substitution of (7.110) into (7.48) gives, because of (7.49),

$$\mathcal{R}_2 \sim -\mathcal{R}_3(\bar{\tau}\mathcal{B}^{(1)-1})_0, \qquad (7.111)$$

where $\bar{\tau}$, like τ, is an infinite row matrix whose elements are the $\bar{\tau}_n$ of (7.109) for $n \geq 0$.

The computation of $(\bar{\tau}\mathcal{B}^{(1)-1})_0$ is again based on the method of generating functions and contains no new feature. The result is

$$(\tau\mathcal{B}^{(1)-1})_0 \sim 1-(1-\alpha_1)^{-1/2}+N^{-1}\alpha_1(1-\alpha_1)^{-3/2}. \qquad (7.112)$$

When (7.112) is used together with (7.108) in (7.111), we get

$$\mathcal{R}_2+\mathcal{R}_3 \sim \mathcal{R}_3\{(1-\alpha_1)^{-1/2}-N^{-1}\alpha_1(1-\alpha_1)^{-3/2}\}$$
$$\sim \tfrac{1}{8}N^{-2}\alpha_1(1-\alpha_1)^{-1}\{1+\tfrac{1}{2}N^{-1}\alpha_1(1-\alpha_1)^{-1}\}. \qquad (7.113)$$

All square roots have disappeared.

G. Final Result

It only remains to substitute (7.91) and (7.113) into (7.46) to find that, as $N \to \infty$,

$$\mathcal{R} = 1+\tfrac{1}{4}N^{-3}\alpha_1(1-\alpha_1)^{-2}+O(N^{-4}). \qquad (7.114)$$

It is difficult not to be impressed by the remarkable amount of cancellation. We now recall the definition of $\mathcal{R}$ as given by (7.11) and we find that

$$S_N/S_N^{(0)} = \text{const}[1+\tfrac{1}{8}N^{-2}\alpha_1(1-\alpha_1)^{-2}+O(N^{-3})] \qquad (7.115)$$

for $N \to \infty$. This multiplicative constant is then found to be

$$(1+\alpha_1)^{1/4}(1-\alpha_1)^{-1/4} \qquad (7.116)$$

on the basis of (5.37). The required result (7.1) is just (7.115) with (7.116).

8. CONCLUSION AND DISCUSSIONS

(A). In this paper, we have studied in great detail the asymptotic behavior of the two-spin correlation $\langle\sigma_{00}\sigma_{0N}\rangle$ for the two-dimensional Ising model at fixed temperatures. The explicit results are given by (2.43) for $T>T_c$, (3.25) for $T<T_c$, and (7.2) for $T=T_c$. In principle, we can obtain as many terms of the asymptotic expansion as we desire, but the computational labor may be prohibitive.

The method of computation is much more general than that indicated by the explicit results. The extension of the results in the following two directions are clearly desirable.

(a) We may ask what is the asymptotic behavior of the more general correlation $\langle\sigma_{00}\sigma_{MN}\rangle$ at fixed temperatures, when either M or N is large. Although there is no way to express this correlation in terms of a Toeplitz determinant, the procedure of Sec. 2 and Sec. 3 are directly applicable. More precisely, the asymptotic behavior of $\langle\sigma_{00}\sigma_{MN}\rangle$ can also be obtained by studying a pair of Wiener-Hopf sum equations, at least for fixed $T \neq T_c$.

(b) The asymptotic expansions (2.43) and (3.25) hold $N \to \infty$ at fixed temperatures $T \neq T_c$. Because of the explicit form, we see that they hold if

$$N \gg |x_3|, \qquad (8.1)$$

or

$$N \gg |1-T/T_c|^{-1}. \qquad (8.2)$$

Therefore the present consideration gives little information about the correlation function for T near T_c. More precisely, we may ask what is the asymptotic behavior of S_N as $N \to \infty$ for fixed $(1-T/T_c)N$. This question can also be answered by the present method.

(B). We discuss briefly the special case $E_1=E_2$, or $K=K_1=K_2$. In this case, by (1.11),

$$\alpha_1 = zz^*, \quad \alpha_2 = z^*/z, \qquad (8.3)$$

where

$$z = \tanh K, \quad \text{and} \quad z^* = e^{-2K}. \qquad (8.4)$$

Consider first $T=T_c$. In this case

$$z=\sqrt{2}-1, \quad \text{and} \quad \alpha_1 = 3-2\sqrt{2}. \qquad (8.5)$$

Therefore, by (7.1) and (7.2),

$$S_N/S_N^{(0)} = 2^{1/8}[1+\tfrac{1}{32}N^{-2}+O(N^{-3})], \qquad (8.6)$$

or

$$S_N = e^{1/4}2^{5/24}A^{-3}N^{-1/4}[1+\tfrac{1}{64}N^{-2}+O(N^{-3})]. \qquad (8.7)$$

An approximate result for $S_N/S_N^{(0)}$ has been given previously by Kaufman and Onsager.[2] Their result is simply

$$\tfrac{1}{2}(1+\sqrt{2}). \qquad (8.8)$$

Our asymptotic result (8.6) for large N is about 10% smaller than this approximate result of Kaufman and Onsager.

Consider next $T>T_c$. It is convenient to use the variables

$$\delta = \tfrac{1}{2}(z^{-1}-z) = \text{csch}2K > 0, \qquad (8.9)$$

and

$$\omega = \delta^{-1}+\delta-1 \geq 1. \qquad (8.10)$$

Then, by (2.31)-(2.33),

$$x_2-x_1 = (\omega^2-1)^{1/2},$$
$$x_1x_2 = \omega+1,$$

and

$$x_3 = \omega(\omega^2-1)^{-1/2}. \qquad (8.11)$$

Accordingly, by (2.39), we have

$$A_{1>} = \tfrac{1}{2}(\omega^2-1)^{1/2}[1-\omega(\omega^2-1)^{-1}],$$
$$A_{2>} = \tfrac{1}{8}[3\omega^2+2\omega+4+3(\omega^2-1)^{-1}],$$

and

$$A_{3>} = \tfrac{1}{16}(\omega^2-1)^{1/2}[5\omega^2+9\omega+6-10\omega(\omega^2-1)^{-1} -5\omega(\omega^2-1)^{-2}]. \qquad (8.12)$$

The substitution of (8.12) into (2.43) gives finally

$$S_N \sim (\pi N)^{-1/2}(z/z^*)^N \times(1-z^2z^{*2})^{1/4}(1-z^2z^{*-2})^{-1/4}(1-z^{*2})^{-1/2} \times\{1+\tfrac{1}{8}N^{-1}(\omega^2-1)^{1/2}[1+\omega(\omega^2-1)^{-1}] +128^{-1}N^{-2}[9\omega^2+6\omega-8+9(\omega^2-1)^{-1}] +(5/1024)N^{-3}(\omega^2-1)^{1/2}[15\omega^2+27\omega-10 -2\omega(\omega^2-1)^{-1}-15\omega(\omega^2-1)^{-2}]\}. \quad (8.13)$$

A similar result can be written down for $T<T_c$.

(C). We discuss further the case $T=T_c$. One interesting question is: What is the next term on the right hand side of (7.1)? For this purpose, we have carried out, in Appendix A, a series expansion of S_N for small α_1 but fixed N. We note that the N^{-2} terms on the right hand sides of (A9) and (7.1) agree up to α_1^{-2}. On the basis of (A9) we make the following conjectures.

Conjecture (*a*). The asymptotic expansion of $S_N/S_N^{(0)}$ at $T=T_c$ is of the following form:

$$S_N/S_N^{(0)}=(1+\alpha_1)^{1/4}(1-\alpha_1)^{-1/4}\sum_{n=0}^{\infty} f_nN^{-2n}, \quad (8.14)$$

where

$$f_0=1, \quad f_1=\tfrac{1}{8}\alpha_1(1-\alpha_1)^{-2}. \quad (8.15)$$

and

In other words, all odd powers of $1/N$ do not appear.

Conjecture (*b*): In the above notation,

$$f_2=\tfrac{1}{32}\alpha_1(1-\alpha_1)^{-2}+(81/128)\alpha_1^2(1-\alpha_1)^{-4}. \quad (8.16)$$

In other words, for large N at $T=T_c$,

$$S_N/S_N^{(0)}=(1+\alpha_1)^{1/4}(1-\alpha_1)^{-1/4}\{1+\tfrac{1}{8}N^{-2}\alpha_1(1-\alpha_1)^{-2} +128^{-1}N^{-4}\alpha_1(1-\alpha_1)^{-2}[4+81\alpha_1(1-\alpha_1)^{-2}] +O(N^{-5})\}. \quad (8.17)$$

In particular, if $E_1=E_2$, then

$$S_N/S_N^{(0)}=2^{1/8}[1+\tfrac{1}{32}N^{-2} +97N^{-4}/2048+O(N^{-5})]. \quad (8.18)$$

We note that, if conjecture (a) is correct, then the $O(N^{-3})$ in (7.1) and (8.6) may be replaced by $O(N^{-4})$, and the $O(N^{-5})$ in (8.17) and (8.18) replaced by $O(N^{-6})$.

(D). There are numerous reasons why it is not trivial to render the calculation in the present paper rigorous. For example, we may ask what is the relation between (5.17) and (5.20). The author is unable to answer this question in general.

The considerations in Sec. 5 have other difficulties besides the one just mentioned. We list two others.

(a) The procedure of Wiener and Hopf, as given by Krein,[7] is mathematically rigorous under the assumption

$$\sum_{-\infty}^{\infty}|c_n|<\infty. \quad (8.19)$$

This condition is not satisfied in Sec. 5. The necessary generalization is rather straightforward, and is considered in Appendix B.

(b) We have obtained in Sec. 5 some relation between two Toeplitz determinants if they are generated by two functions of which the quotient is a trigonometric polynomial, as shown in (5.34). Since $\varphi(\theta)/\varphi^{(0)}(\theta)$ is not a trigonometric polynomial, a limiting process is involved. It should be emphasized that we have *not* studied this limiting process, which is certainly not trivial. As an example, we mention that, if

$$F_N^{(n)}=N/(N+n), \quad (8.20)$$

then

$$F_N^{(n+1)}/F_n^{(n)}\to 1 \quad (8.21)$$

uniformly as $N\to\infty$, but, for all N,

$$F_N^{(\infty)}=0\neq 1. \quad (8.22)$$

Since the d_n of (7.3) goes to zero exponentially as $n\to\pm\infty$, it is believed that such difficulties do not actually occur.

(E). If the limiting process just mentioned can indeed be carried out, then we get the following modified form of Szego's theorem.[5]

Conjecture (*s*): Let

$$\varphi^{(0)}(\theta)=ie^{-i\theta/2}, \quad \text{for} \quad 0\le\theta<2\pi,$$

and

$$\varphi(\theta)=\varphi^{(0)}(\theta)\varphi^{(1)}(\theta), \quad (8.23)$$

where $\varphi^{(1)\prime}(\theta)$ satisfies a Lipschitz condition and $\ln\varphi^{(1)}(\theta)$ is continuous, periodic, and

$$\int_0^{2\pi}\ln\varphi^{(1)}(\theta)d\theta=0. \quad (8.24)$$

Let S_N be the $N\times N$ Toeplitz determinant formed with the Fourier coefficients of $\varphi(\theta)$, and $S_N^{(0)}$ with those of $\varphi^{(0)}(\theta)$, then

$$\lim_{N\to\infty}S_N/S_N^{(0)} =\exp\sum_{n=1}^{\infty}-\tfrac{1}{4}n^{-1}[(1+2nk_n)(1-2nk_{-n})-1], \quad (8.25)$$

where

$$\ln\varphi^{(1)}(\theta)=\sum_{n=-\infty}^{\infty}k_ne^{in\theta}. \quad (8.26)$$

[The asymptotic behavior of $S_N^{(0)}$ is given by (4.14).]

If this conjecture is true, it can undoubtedly be generalized to a large class of possible $\varphi^{(0)}$'s.

(F). Lee and Yang[14] have studied the "spontaneous magnetization" of the Ising model in an imaginary magnetic field of a particular value. In this case, the two-spin correlation $\langle\sigma_{00}\sigma_{0N}\rangle$ can be expressed in terms of a block Toeplitz determinant, i.e., a determinant of

[14] T. D. Lee and C. N. Yang, Phys. Rev. 87, 410 (1952).

the form (1.8) except that each a_n is itself a $j\times j$ matrix. If we apply our procedure to this case, we must consider coupled Wiener-Hopf sum equation of the form

$$\sum_{m=0}^{\infty} c_{n-m}x_m = a\delta_{n0}, \tag{8.27}$$

where each c_{n-m} is a $j\times j$ matrix, and a and each x_m is a $j\times 1$ column matrix. In general, an equation of the form (8.27) cannot be exactly solved. However, if the matrix

$$C(\xi)=\sum_{n=-\infty}^{\infty} c_n\xi^n \tag{8.28}$$

is of the form

$$C(\xi)=C^{(0)}(\xi)C^{(1)}(\xi), \tag{8.29}$$

where $C^{(0)}(\xi)$ is a single function (not a matrix), and $C^{(1)}(\xi)$ is a matrix whose elements are polynomials in ξ, then (8.27) can be solved by the procedure of Wiener and Hopf.[6,7] In the case considered by Lee and Yang,[14] $j=2$ and (8.29) is indeed satisfied. Thus the methods of this paper are applicable. We should add that, since Szego's theorem cannot be used for block Toeplitz matrices, the considerations of Sec. 5 here are needed to evaluate the asymptotic series for these matrices.

(G). When $T=0$, only two configurations of the two-dimensional Ising lattice are of importance. These are the configurations where $\sigma_{mn}=\pm1$, independent of m and n. When $0<T<T_c$, we picture these spins as predominantly $+1$ (or -1) with "islands" of spins taking on the opposite value. What is the rough "size" of these islands?

Since this is a rather vague question, the answer cannot be precise. Consider the expectation value of σ_{0N} under the conditions that

$$\sigma_{00}=1 \quad\text{and}\quad \sigma_{0\infty}=-1. \tag{8.30}$$

If this expectation value is called $\langle\sigma_{0N}\rangle_c$, then

$$\langle\sigma_{0N}\rangle_c=\langle\tfrac{1}{2}(1+\sigma_{00})\tfrac{1}{2}(1-\sigma_{0\infty})\sigma_{0N}\rangle/\langle\tfrac{1}{2}(1+\sigma_{00})\tfrac{1}{2}(1-\sigma_{0\infty})\rangle. \tag{8.31}$$

Since

$$\langle\sigma_{mn}\sigma_{m'n'}\sigma_{m''n''}\rangle=0, \tag{8.32}$$

we get, by (1.1),

$$\langle\sigma_{0N}\rangle_c=(S_N-S_\infty)/(1-S_\infty). \tag{8.33}$$

If the first term on the right-hand side of (3.25) is used, the result is that, for large N,

$$\langle\sigma_{0N}\rangle_c\sim(2\pi N^2)^{-1}\alpha_2^{2N}(\alpha_2^{-1}-\alpha_2)^{-2} \times[(1-\alpha_1^2)^{-1/4}(1-\alpha_2^2)^{-1/4}(1-\alpha_1\alpha_2)^{1/2}-1]^{-1}.$$

Thus a very rough measure of the "size" is $-(\ln\alpha_2)^{-1}$. This "size" approaches infinity as $T\rightarrow T_c$.

It seems very difficult to make these vague statements any more precise. In particular, it may be remembered that there are islands inside islands *ad infinitum*.

(H). We conclude this paper with an attempt to relate the present results on the Ising model to the so-called procedure of summing the most divergent terms. This procedure has been discussed in detail by Lee, Huang, and Yang[15] in connection with the boson system of hard spheres, and is used to about the same time by Gell-Mann and Brueckner[16] to calculate the correlation energy of a dense electron gas. In both of these problems of statistical mechanics, the procedure is quite successful and yields the correct answers.

More recently, procedures of this variety have been applied to problems of quantum-field theory. By far the two most outstanding examples are the work of Lee[17] and collaborators on the radiative correction to processes involving intermediate bosons and that of Feinberg and Pais[18] on higher-order weak interactions. Both lines of investigation depend very much on the assumption that the sum of the most divergent terms gives approximately the correct answer when the coupling constant, electromagnetic and weak respectively, is small. For this reason, there has been a great deal of effort to check whether this assumption is valid in solvable cases.[19] Unfortunately, in almost all of these solvable cases, the problem reduces to the Born series or the series of ladder diagrams.[20] It therefore seems desirable to look for examples of a different nature, and for this purpose the attention may be turned to statistical mechanics again.

For $T>T_c$, the series expansion of S_N in powers of N^{-1} may be found from (2.36). Except the factor α_2^{-N}, the series is a power series in N^{-1}, with each coefficient a polynomial in x_3, which is large for T near T_c. Suppose that, for each n, we keep in the coefficient of N^{-n} only the terms with the highest power of x_3. Can we get any information about the case $T=T_c$ by summing this series and take the limit $T\rightarrow T_c$ with fixed N? The N^{-1} here plays the role of the coupling constant.

Since we need to keep only terms with the highest power of x_3, we can approximate the $A_>(z)$ of (2.37) by

$$A_>(z)\sim[(1+x_1z_0)(1-x_2z_0)(1+x_3z)]^{-1/2},$$

where

$$z_0=[(1-\xi_1)/(1+\xi_1)]|_{\xi_1=1}=0.$$

That is,

$$A_>(z)\sim(1+x_3z)^{-1/2}. \tag{8.34}$$

[15] T. D. Lee, K. Huang, and C. N. Yang, Phys. Rev. **106**, 1135 (1957).

[16] M. Gell-Mann and K. A. Brueckner, Phys. Rev. **106**, 364 (1957).

[17] T. D. Lee, Phys. Rev. **128**, 899 (1962); J. Bernstein and T. D. Lee, Phys. Rev. Letters **11**, 512 (1963); T. D. Lee and A. Sirlin, Rev. Mod. Phys. **36**, 666 (1964); and T. D. Lee, Phys. Rev. Letters **12**, 569 (1964).

[18] G. Feinberg and A. Pais, Phys. Rev. **131**, 2724 (1963); **133**, B477 (1964).

[19] See, for example, N. N. Khuri and A. Pais, Rev. Mod. Phys. **36**, 590 (1964).

[20] This is almost like trying to learn properties of infinite series by studying only the geometric series.

We substitute (8.34) into (2.36), and use the approximations

$$\alpha_2^{-N}\sim 1\,,$$
$$1-\alpha_1\alpha_2\sim 1-\alpha\,,$$
$$1+\xi_1\sim 2\,,$$
$$x_3\sim 2(1-\alpha_2^{-2})^{-1}\,,$$

and

$$z\sim\tfrac{1}{2}(1-\xi_1)\,. \tag{8.35}$$

The result is then

$$S_N\sim\pi^{-1}(1+\alpha_1)^{1/4}(1-\alpha_1)^{-1/4}(1-\alpha_2^{-2})^{-1/4}\times\int_0^1 d\xi_1\xi_1^N(1-\xi_1)^{-1/2}[1+(1-\alpha_2^{-2})^{-1}(1-\xi_1)]^{-1/2}.$$

We expand the last factor by the binomial theorem and then integrate term by term to get

$$\pi S_N(1+\alpha_1)^{-1/4}(1-\alpha_1)^{1/4}\sim(1-\alpha_2^{-2})^{-1/4}\sum_{n=0}^{\infty}\binom{-\frac{1}{2}}{n}(1-\alpha_2^{-2})^{-n}\int_0^1 d\xi_1\xi_1^N(1-\xi_1)^{n-1/2}$$
$$=(1-\alpha_2^{-2})^{-1/4}\sum_{n=0}^{\infty}\binom{-\frac{1}{2}}{n}(1-\alpha_2^{-2})^{-n}N!\Gamma(n+\tfrac{1}{2})/\Gamma(N+n+\tfrac{3}{2})\,. \tag{8.36}$$

Since, for fixed n and large N,

$$N!/\Gamma(N+n+\tfrac{3}{2})=N^{-n-1/2}[1+O(N^{-1})]\,, \tag{8.37}$$

we *must*, by the prescription of summing the most divergent terms in the limit $\alpha_2\rightarrow 1+$, replace the factor $N!/\Gamma(N+n+\frac{3}{2})$ in (8.36) by $N^{-n-1/2}$. Therefore

$$\pi S_N(1+\alpha_1)^{-1/4}(1-\alpha_1)^{1/4}\sim(1-\alpha_2^{-2})^{-1/4}N^{-1/2}\sum_{n=0}^{\infty}\binom{-\frac{1}{2}}{n}\Gamma(n+\tfrac{1}{2})[N(1-\alpha_2^{-2})]^{-n}. \tag{8.38}$$

The sum on the right-hand side of (8.38) is a formal power series in the variable $[N(1-\alpha_2^{-2})]^{-1}$. In this variable, the radius of convergence R is

$$R=0\,. \tag{8.39}$$

Thus the series is quite similar to that encountered by Feinberg and Pais.[18] There is, strictly speaking, no unique way of assigning a meaning to (8.38).

In view of the way (8.38) is obtained, the following way of summing the divergent series is perhaps reasonable. Replacing the gamma function by its integral representation

$$\Gamma(n+\tfrac{1}{2})=\int_0^\infty dt\, e^{-t}t^{n-1/2}\,,$$

and reversing the order of summing over n and integrating over t, we get from (8.38)

$$\pi S_N(1+\alpha_1)^{-1/4}(1-\alpha_1)^{1/4}\sim(1-\alpha_2^{-2})^{-1/4}N^{-1/2}\int_0^\infty dt\, e^{-t}t^{-1/2}\sum_{n=0}^{\infty}\binom{-\frac{1}{2}}{n}t^n[N(1-\alpha_2^{-2})]^{-n}$$
$$=(1-\alpha_2^{-2})^{-1/4}N^{-1/2}\int_0^\infty dt\, e^{-t}t^{-1/2}[1+N^{-1}(1-\alpha_2^{-2})^{-1}t]^{-1/2} \tag{8.40}$$
$$=(1-\alpha_2^{-2})^{1/4}K_0(\tfrac{1}{2}N(1-\alpha_2^{-2}))\exp\tfrac{1}{2}N(1-\alpha_2^{-2})\,,$$

where K_0 is the modified Bessel function. The limit of the right hand side of (8.40), as $\alpha_2\rightarrow 1+$, is zero for all N. *Therefore, the procedure of summing the most divergent terms gives, in this case, a wrong answer.*

We make the following four remarks.

(a) It is a most interesting question whether it is justified to blame the wrong answer on (8.39). This point is not understood by the author.

(b) As already noted in Sec. 5, we do get from (8.40) the correct dependence of S_N on α_1. In other words, (5.6) follows from either (8.36) or the fact that the right hand side of (8.36), and hence (8.38) and (8.40), is independent of α_1.

(c) Suppose that we know the form of (8.38) but not the actual coefficients, that is, suppose that we know

$$S_N(1+\alpha_1)^{-1/4}(1-\alpha_1)^{1/4}\sim(1-\alpha_2^{-2})^{-1/4}N^{-1/2}\times\text{function of }[N(1-\alpha_2^{-2})]^{-1}. \tag{8.41}$$

Let us also make the assumption that, for N fixed but $\alpha_2\rightarrow 1+$, the right-hand side of (8.41) is finite. Then, we hope that, as $\alpha_2 \to 1+$,

function of $[N(1-\alpha_2^{-2})]^{-1}$

$$\sim \text{const.}\,[N(1-\alpha_2^{-2})]^{1/4}. \quad (8.42)$$

The substitution of (8.42) into (8.41) then gives

$$S_N \sim \text{const.}(1+\alpha_1)^{1/4}(1-\alpha_1)^{-1/4}N^{-1/4}. \quad (8.43)$$

This is fortuitously consistent with (5.7). What is wrong is that the explicit result (8.40) shows that this constant in (8.43) is actually zero.

(d) The correct answer for $T=T_c$ is given by (5.7), while the sum of the most divergent terms is zero. That the sum is zero is perhaps worth noting in connection with the argument of Bernstein and Lee.[17]

We list below the three known major pitfalls when the procedure of summing the most divergent terms is applied to problems of quantum-field theory.

(i) There is no reason to believe that the sum of an arbitrary, small subset of diagrams should give an answer relevant to the physical problem described by a nonrenormalizable field theory. In this connection, considerations applicable to ladder diagrams only must be regarded as devoid of physical content.[21]

(ii) It does not seem possible to justify neglecting a larger coupling constant while keeping higher order terms in a smaller coupling constant. Since the electromagnetic coupling constant is much larger than the weak coupling constant unless the mass of the intermediate boson of weak interactions is enormous, much of the work where electromagnetic effects are neglected is open to criticism.[21]

(iii) The sum of the most divergent terms may not be a good approximation to the sum of all terms, even if the coupling constant is small. This makes it rather difficult to extract useful information without explicit calculation.

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APPENDIX A

We have seen that, at the critical temperature, the spin correlation along the lattice sites is given by the S_N of (1.8) with (1.9) and

$$\varphi(\theta) = ie^{-i\theta/2}(1-\alpha_1 e^{i\theta})^{1/2}(1-\alpha_1 e^{-i\theta})^{-1/2}. \quad (A1)$$

In this Appendix, we try to expand this S_N into a power series in α_1, with N fixed. We note that

$$\int_0^{2\pi} d\theta \ln[\varphi(\theta)/\varphi^{(0)}(\theta)] = 0. \quad (A2)$$

Because of the rather large number of terms involved, we shall keep only terms up to and including α_1^2. Thus

$$\varphi(\theta) \sim ie^{-i\theta/2}[\tfrac{3}{8}\alpha_1^2 e^{-2i\theta} + \tfrac{1}{2}\alpha_1 e^{-i\theta} + (1-\tfrac{1}{4}\alpha_1^2) - \tfrac{1}{2}\alpha_1 e^{i\theta} - \tfrac{1}{8}\alpha_1^2 e^{2i\theta}]. \quad (A3)$$

Accordingly, with the $a_n^{(0)}$ of (4.4), we can write a_n in the form

$$a_n = \tfrac{3}{8}\alpha_1^2 a_{n+2}^{(0)} + \tfrac{1}{2}\alpha_1 a_{n+1}^{(0)} + (1-\tfrac{1}{4}\alpha_1^2)a_n^{(0)} - \tfrac{1}{2}\alpha_1 a_{n-1}^{(0)} - \tfrac{1}{8}\alpha_1^2 a_{n-2}^{(0)} + O(\alpha_1^3). \quad (A4)$$

And to this order S_N consists of the following terms

$$S_N \sim \begin{vmatrix} a_0^{(0)} & a_{-1}^{(0)} & a_{-2}^{(0)} & \cdots & a_{-N+3}^{(0)} & a_{-N+2}^{(0)} & a_{-N+1}^{(0)} \\ a_1^{(0)} & a_0^{(0)} & a_{-1}^{(0)} & \cdots & a_{-N+4}^{(0)} & a_{-N+3}^{(0)} & a_{-N+2}^{(0)} \\ a_2^{(0)} & a_1^{(0)} & a_0^{(0)} & \cdots & a_{-N+5}^{(0)} & a_{-N+4}^{(0)} & a_{-N+3}^{(0)} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{N-1}^{(0)} & a_{N-2}^{(0)} & a_{N-3}^{(0)} & \cdots & a_2^{(0)} & a_1^{(0)} & a_0^{(0)} \end{vmatrix}$$

$$+\tfrac{1}{2}\alpha_1 \begin{vmatrix} a_1^{(0)} & a_{-1}^{(0)} & a_{-2}^{(0)} & \cdots & a_{-N+3}^{(0)} & a_{-N+2}^{(0)} & a_{-N+1}^{(0)} \\ a_2^{(0)} & a_0^{(0)} & a_{-1}^{(0)} & \cdots & a_{-N+4}^{(0)} & a_{-N+3}^{(0)} & a_{-N+2}^{(0)} \\ a_3^{(0)} & a_1^{(0)} & a_0^{(0)} & \cdots & a_{-N+5}^{(0)} & a_{-N+4}^{(0)} & a_{-N+3}^{(0)} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_N^{(0)} & a_{N-2}^{(0)} & a_{N-3}^{(0)} & \cdots & a_2^{(0)} & a_1^{(0)} & a_0^{(0)} \end{vmatrix}$$

$$-\tfrac{1}{2}\alpha_1 \begin{vmatrix} a_0^{(0)} & a_{-1}^{(0)} & a_{-2}^{(0)} & \cdots & a_{-N+3}^{(0)} & a_{-N+2}^{(0)} & a_{-N}^{(0)} \\ a_1^{(0)} & a_0^{(0)} & a_{-1}^{(0)} & \cdots & a_{-N+4}^{(0)} & a_{-N+3}^{(0)} & a_{-N+1}^{(0)} \\ a_2^{(0)} & a_1^{(0)} & a_0^{(0)} & \cdots & a_{-N+5}^{(0)} & a_{-N+4}^{(0)} & a_{-N+2}^{(0)} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{N-1}^{(0)} & a_{N-2}^{(0)} & a_{N-3}^{(0)} & \cdots & a_2^{(0)} & a_1^{(0)} & a_{-1}^{(0)} \end{vmatrix}$$

$$-\tfrac{1}{4}\alpha_1^2 \begin{vmatrix} a_0^{(0)} & a_{-1}^{(0)} & a_{-2}^{(0)} & \cdots & a_{-N+3}^{(0)} & a_{-N+2}^{(0)} & a_{-N+1}^{(0)} \\ a_1^{(0)} & a_0^{(0)} & a_{-1}^{(0)} & \cdots & a_{-N+4}^{(0)} & a_{-N+3}^{(0)} & a_{-N+2}^{(0)} \\ a_2^{(0)} & a_1^{(0)} & a_0^{(0)} & \cdots & a_{-N+5}^{(0)} & a_{-N+4}^{(0)} & a_{-N+3}^{(0)} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{N-1}^{(0)} & a_{N-2}^{(0)} & a_{N-3}^{(0)} & \cdots & a_2^{(0)} & a_1^{(0)} & a_0^{(0)} \end{vmatrix}$$

(Continued on next page)

[21] For example, Y. Pwu and T. T. Wu, Phys. Rev. **133**, B778 (1964). As far as physics is concerned, this work is completely incorrect on account of both (i) and (ii).

$$
-\tfrac{1}{4}\alpha_1^2\begin{vmatrix}
a_1^{(0)} & a_{-1}^{(0)} & a_{-2}^{(0)} & \cdots & a_{-N+3}^{(0)} & a_{-N+2}^{(0)} & a_{-N}^{(0)} \\
a_2^{(0)} & a_0^{(0)} & a_{-1}^{(0)} & \cdots & a_{-N+4}^{(0)} & a_{-N+3}^{(0)} & a_{-N+1}^{(0)} \\
a_3^{(0)} & a_1^{(0)} & a_0^{(0)} & \cdots & a_{-N+5}^{(0)} & a_{-N+4}^{(0)} & a_{-N+2}^{(0)} \\
\vdots & \vdots & \vdots & \vdots\vdots\vdots & \vdots & \vdots & \vdots \\
a_N^{(0)} & a_{N-2}^{(0)} & a_{N-3}^{(0)} & \cdots & a_2^{(0)} & a_1^{(0)} & a_{-1}^{(0)}
\end{vmatrix}
$$

$$
+\tfrac{3}{8}\alpha_1^2\begin{vmatrix}
a_2^{(0)} & a_{-1}^{(0)} & a_{-2}^{(0)} & \cdots & a_{-N+3}^{(0)} & a_{-N+2}^{(0)} & a_{-N+1}^{(0)} \\
a_3^{(0)} & a_0^{(0)} & a_{-1}^{(0)} & \cdots & a_{-N+4}^{(0)} & a_{-N+3}^{(0)} & a_{-N+2}^{(0)} \\
a_4^{(0)} & a_1^{(0)} & a_0^{(0)} & \cdots & a_{-N+5}^{(0)} & a_{-N+4}^{(0)} & a_{-N+3}^{(0)} \\
\vdots & \vdots & \vdots & \vdots\vdots\vdots & \vdots & \vdots & \vdots \\
a_{N+1}^{(0)} & a_{N-2}^{(0)} & a_{N-3}^{(0)} & \cdots & a_2^{(0)} & a_1^{(0)} & a_0^{(0)}
\end{vmatrix}
$$

$$
-\tfrac{1}{8}\alpha_1^2\begin{vmatrix}
a_0^{(0)} & a_{-1}^{(0)} & a_{-2}^{(0)} & \cdots & a_{-N+3}^{(0)} & a_{-N+2}^{(0)} & a_{-N-1}^{(0)} \\
a_1^{(0)} & a_0^{(0)} & a_{-1}^{(0)} & \cdots & a_{-N+4}^{(0)} & a_{-N+3}^{(0)} & a_{-N}^{(0)} \\
a_2^{(0)} & a_1^{(0)} & a_0^{(0)} & \cdots & a_{-N+5}^{(0)} & a_{-N+4}^{(0)} & a_{-N+1}^{(0)} \\
\vdots & \vdots & \vdots & \vdots\vdots\vdots & \vdots & \vdots & \vdots \\
a_{N-1}^{(0)} & a_{N-2}^{(0)} & a_{N-3}^{(0)} & \cdots & a_2^{(0)} & a_1^{(0)} & a_{-2}^{(0)}
\end{vmatrix}
$$

$$
+\tfrac{3}{8}\alpha_1^2\begin{vmatrix}
a_0^{(0)} & a_1^{(0)} & a_{-2}^{(0)} & \cdots & a_{-N+3}^{(0)} & a_{-N+2}^{(0)} & a_{-N+1}^{(0)} \\
a_1^{(0)} & a_2^{(0)} & a_{-1}^{(0)} & \cdots & a_{-N+4}^{(0)} & a_{-N+3}^{(0)} & a_{-N+2}^{(0)} \\
a_2^{(0)} & a_3^{(0)} & a_0^{(0)} & \cdots & a_{-N+5}^{(0)} & a_{-N+4}^{(0)} & a_{-N+3}^{(0)} \\
\vdots & \vdots & \vdots & \vdots\vdots\vdots & \vdots & \vdots & \vdots \\
a_{N-1}^{(0)} & a_N^{(0)} & a_{N-3}^{(0)} & \cdots & a_2^{(0)} & a_1^{(0)} & a_0^{(0)}
\end{vmatrix}
$$

$$
-\tfrac{1}{8}\alpha_1^2\begin{vmatrix}
a_0^{(0)} & a_{-1}^{(0)} & a_{-2}^{(0)} & \cdots & a_{-N+3}^{(0)} & a_{-N}^{(0)} & a_{-N+1}^{(0)} \\
a_1^{(0)} & a_0^{(0)} & a_{-1}^{(0)} & \cdots & a_{-N+4}^{(0)} & a_{-N+1}^{(0)} & a_{-N+2}^{(0)} \\
a_2^{(0)} & a_1^{(0)} & a_0^{(0)} & \cdots & a_{-N+5}^{(0)} & a_{-N+2}^{(0)} & a_{-N+3}^{(0)} \\
\vdots & \vdots & \vdots & \vdots\vdots\vdots & \vdots & \vdots & \vdots \\
a_{N-1}^{(0)} & a_{N-2}^{(0)} & a_{N-3}^{(0)} & \cdots & a_2^{(0)} & a_{-1}^{(0)} & a_0^{(0)}
\end{vmatrix}
$$

$$
+\tfrac{1}{4}\alpha_1^2\begin{vmatrix}
a_1^{(0)} & a_0^{(0)} & a_{-2}^{(0)} & \cdots & a_{-N+3}^{(0)} & a_{-N+2}^{(0)} & a_{-N+1}^{(0)} \\
a_2^{(0)} & a_1^{(0)} & a_{-1}^{(0)} & \cdots & a_{-N+4}^{(0)} & a_{-N+3}^{(0)} & a_{-N+2}^{(0)} \\
a_3^{(0)} & a_2^{(0)} & a_0^{(0)} & \cdots & a_{-N+5}^{(0)} & a_{-N+4}^{(0)} & a_{-N+3}^{(0)} \\
\vdots & \vdots & \vdots & \vdots\vdots\vdots & \vdots & \vdots & \vdots \\
a_N^{(0)} & a_{N-1}^{(0)} & a_{N-3}^{(0)} & \cdots & a_2^{(0)} & a_1^{(0)} & a_0^{(0)}
\end{vmatrix}
$$

$$
+\tfrac{1}{4}\alpha_1^2\begin{vmatrix}
a_0^{(0)} & a_{-1}^{(0)} & a_{-2}^{(0)} & \cdots & a_{-N+3}^{(0)} & a_{-N+1}^{(0)} & a_{-N}^{(0)} \\
a_1^{(0)} & a_0^{(0)} & a_{-1}^{(0)} & \cdots & a_{-N+4}^{(0)} & a_{-N+2}^{(0)} & a_{-N+1}^{(0)} \\
a_2^{(0)} & a_1^{(0)} & a_0^{(0)} & \cdots & a_{-N+5}^{(0)} & a_{-N+3}^{(0)} & a_{-N+2}^{(0)} \\
\vdots & \vdots & \vdots & \vdots\vdots\vdots & \vdots & \vdots & \vdots \\
a_{N-1}^{(0)} & a_{N-2}^{(0)} & a_{N-3}^{(0)} & \cdots & a_2^{(0)} & a_0^{(0)} & a_{-1}^{(0)}
\end{vmatrix}. \tag{A5}
$$

The first term on the right-hand side of (A5) is just $S_N^{(0)}$. The fourth term, which is simply $-\frac{1}{4}\alpha_1^2 S_N^{(0)}$, appears for the following reason. By (A3), there are N terms, each of which is equal to $-\frac{1}{4}\alpha_1^2 S_N^{(0)}$. There are, in addition, $(N-1)$ terms of the form

$$
\begin{vmatrix}
a_0^{(0)} & \cdots & a_{-j+1}^{(0)} & a_{-j-1}^{(0)} & a_{-j}^{(0)} & a_{-j-2}^{(0)} & \cdots & a_{-N+1}^{(0)} \\
a_1^{(0)} & \cdots & a_{-j+2}^{(0)} & a_{-j}^{(0)} & a_{-j+1}^{(0)} & a_{-j-1}^{(0)} & \cdots & a_{-N+2}^{(0)} \\
a_2^{(0)} & \cdots & a_{-j+3}^{(0)} & a_{-j+1}^{(0)} & a_{-j+2}^{(0)} & a_{-j}^{(0)} & \cdots & a_{-N+3}^{(0)} \\
\vdots & \vdots\vdots\vdots & \vdots & \vdots & \vdots & \vdots & \vdots\vdots\vdots & \vdots \\
a_{N-1}^{(0)} & \cdots & a_{-j+N}^{(0)} & a_{-j+N-2}^{(0)} & a_{-j+N-1}^{(0)} & a_{-j+N-3}^{(0)} & \cdots & a_0^{(0)}
\end{vmatrix}, \tag{A6}
$$

where $0 \le j \le N-2$. Since (A6) is equal to $\frac{1}{4}\alpha_1^2 S_N^{(0)}$, we get the fourth term on the right-hand side of (A5). It is a *consequence* of (A2) that the coefficient of this fourth term is *independent* of N.

The rest of the calculation involves the explicit evaluation of these eleven terms of (A5) by the method of Sec. 4. This is straightforward but tedious:

$$
\begin{aligned}
S_N \sim S_N^{(0)}\{&1+\tfrac{1}{2}\alpha_1 N(2N+1)^{-1}-\tfrac{1}{2}\alpha_1(-N)(2N-1)^{-1}-\tfrac{1}{4}\alpha_1^2 \\
&-\tfrac{1}{4}\alpha_1^2(N+1)(N-1)(2N-1)^{-1}(2N+1)^{-1}+\tfrac{3}{8}\alpha_1^2\tfrac{3}{2}N(N+1)(2N+1)^{-1}(2N+3)^{-1} \\
&-\tfrac{1}{8}\alpha_1^2(-\tfrac{1}{2})N(N+1)(2N-1)^{-1}(2N+1)^{-1}+\tfrac{3}{8}\alpha_1^2(\tfrac{1}{2})N(N-1)(2N-1)^{-1}(2N+1)^{-1} \\
&-\tfrac{1}{8}\alpha_1^2(-\tfrac{3}{2})N(N-1)(2N-1)^{-1}(2N-3)^{-1}+\tfrac{1}{4}\alpha_1^2(-\tfrac{1}{2})N(N-1)(2N-1)^{-1}(2N+1)^{-1} \\
&+\tfrac{1}{4}\alpha_1^2(\tfrac{3}{2})N(N-1)(2N-1)^{-1}(2N-3)^{-1}\} \\
&= S_N^{(0)}\{1+2\alpha_1 N^2/(4N^2-1)+\tfrac{1}{8}\alpha_1^2[1-\tfrac{1}{8}(4N^2-1)^{-1}+81(4N^2-9)^{-1}/8]\}.
\end{aligned} \tag{A7}
$$

This is the required answer.

It is of some interest to rewrite (A7) in the following form, in view of (5.6):

$$S_N \sim S_N^{(0)}(1+\tfrac{1}{2}\alpha_1+\tfrac{1}{8}\alpha_1^2)[1+\tfrac{1}{2}\alpha_1(4N^2-1)^{-1}+\alpha_1^2(4N^2+9/8)(4N^2-1)^{-1}(4N^2-9)^{-1}]$$
$$=S_N^{(0)}(1+\tfrac{1}{2}\alpha_1+\tfrac{1}{8}\alpha_1^2)\{1+\tfrac{1}{2}\alpha_1(4N^2-1)^{-1}+\alpha_1^2[(4N^2-1)^{-1}+81(4N^2-1)^{-1}(4N^2-9)^{-1}/8]\}. \quad \text{(A8)}$$

In particular, if we further let N be larger, we get

$$S_N \sim S_N^{(0)}(1+\tfrac{1}{2}\alpha_1+\tfrac{1}{8}\alpha_1^2)\{1+\tfrac{1}{8}\alpha_1(1+2\alpha_1)(N^{-2}+\tfrac{1}{4}N^{-4})+(81/128)\alpha_1^2N^{-4}\}. \quad \text{(A9)}$$

APPENDIX B

In this Appendix, we consider the Wiener-Hopf sum equation of Sec. 5. Let

$$\sum_{m=0}^{\infty} c_{n-m}x_m = y_n \quad \text{(B1)}$$

hold for all $n \geq 0$. Let

$$\varphi(\theta) = \sum_{n=-\infty}^{\infty} c_n e^{in\theta}, \quad \text{(B2)}$$

for $0 \leq \theta \leq 2\pi$, be such that $\varphi(\theta) \neq 0$ and

$$\sum_{n=-\infty}^{\infty} |d_n| < \infty, \quad \text{(B3)}$$

where

$$\sum_{n=-\infty}^{\infty} d_n e^{in\theta} = e^{i\alpha(\theta-\pi)}\varphi(\theta). \quad \text{(B4)}$$

For the purposes of Sec. 5, $\alpha = \frac{1}{2}$. Without loss of generality, we can choose α such that $\ln\varphi(\theta)+i\alpha\theta$ is continuous and periodic. We also assume that $\sum_{n=0}^{\infty}|y_n|$ converges.

By the Wiener-Levy theorem,[22] we can find $G_+(\xi)$ and $G_-(\xi)$ such that, for $|\xi|=1$,

$$-\ln \sum_{n=-\infty}^{\infty} d_n \xi^n = G_+(\xi)+G_-(\xi), \quad \text{(B5)}$$

where $G_+(\xi)$ is analytic for $|\xi|<1$ and continuous for $|\xi|\leq 1$, while $G_-(\xi)$ is analytic for $|\xi|>1$ and continuous for $|\xi|\geq 1$. By the procedure of Wiener and Hopf, let

$$z_n = \sum_{m=0}^{\infty} c_{n-m}x_m \quad \text{(B6)}$$

for $n \leq -1$, and define

$$X(\xi) = \sum_{n=0}^{\infty} x_n \xi^n,$$

$$Y(\xi) = \sum_{n=0}^{\infty} y_n \xi^n,$$

and

$$Z(\xi) = \sum_{n=-\infty}^{-1} z_n \xi^n.$$

[22] See, for example, Chap. VI of Ref. 10.

We look for solutions that satisfy $\sum_{n=0}^{\infty}|x_n| < \infty$. Thus both $X(\xi)$ and $Y(\xi)$ are analytic for $|\xi|<1$ and continuous for $|\xi|\leq 1$. Because of (B3), $\lim_{n\to-\infty} nz_n$ exists and $Z(\xi)$ is analytic for $|\xi|>1$, and continuous for $|\xi|\geq 1$ but $\xi\neq 1$, the singularity at $\xi=1$ being logarithmic. It then follows from (B1) and (B6) that, for $|\xi|=1$,

$$C(\xi)X(\xi) = Y(\xi)+Z(\xi), \quad \text{(B7)}$$

where

$$C(e^{i\theta}) = \varphi(\theta). \quad \text{(B8)}$$

By (B5), (B7) can be rewritten in the form, again for $|\xi|=1$.

$$X(\xi)(1-\xi)^{-\alpha}\exp[-G_+(\xi)] = Y(\xi)(1-\xi^{-1})^{-\alpha}\exp G_-(\xi)+Z(\xi)(1-\xi^{-1})^{-\alpha} \times \exp G_-(\xi), \quad \text{(B9)}$$

or

$$X(\xi)(1-\xi)^{-\alpha}\exp[-G_+(\xi)] - [Y(\xi)(1-\xi^{-1})^{-\alpha}\exp G_-(\xi)]_+ = [Y(\xi)(1-\xi^{-1})^{-\alpha}\exp G_-(\xi)]_- + Z(\xi)(1-\xi^{-1})^{-\alpha} \times \exp G_-(\xi) \quad \text{(B10)}$$

provided that $\alpha<1$. The right- and left-hand sides of (B10) are thus analytical continuations of each other and they together define a function that is analytic everywhere except possibly at $\xi=1$ and approaches zero at infinity. Moreover, since, for ξ near 1, this function is bounded by $-|1-\xi|^{-\alpha}\ln|1-\xi|$ multiplied by a constant, there is actually no singularity at $\xi=1$ for $\alpha<1$. Therefore,

$$X(\xi)(1-\xi)^{-\alpha}\exp[-G_+(\xi)] - [Y(\xi)(1-\xi^{-1})^{-\alpha}\exp G_-(\xi)]_+ = 0, \quad \text{(B11)}$$

or

$$X(\xi) = (1-\xi)^{\alpha}[\exp G_+(\xi)] \times [Y(\xi)(1-\xi^{-1})^{-\alpha}\exp G_-(\xi)]_+. \quad \text{(B12)}$$

Consequently, for $0\leq\alpha<1$, the Wiener-Hopf equation (B1) has a unique solution given by (B12). This is the case relevant to Sec. 5. For $\alpha<0$, (B1) does not have a solution that satisfies $\sum_{n=0}^{\infty}|x_n| < \infty$ unless

$$(d/d\xi)^n[Y(\xi)(1-\xi^{-1})^{-\alpha}\exp G_-(\xi)]_+|_{\xi=1} = 0 \quad \text{(B13)}$$

for all integers n such that $0\leq n<-\alpha$; if (B13) is satisfied, the solution is unique. For $\alpha\geq 1$, the solution of (B1) is not unique.