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Quantum Theory of Laser Radiation. II. Statistical Aspects of Laser Light

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We have solved a master equation to obtain the diagonal elements of the density matrix for laser light. The master equation represents a generalization of a master equation derived earlier for a single radiation mode interacting with N stationary two-level atoms. The generalization takes into account the pumping scheme which characterizes a three-level laser. For positive temperatures and a lossless cavity, the equilibrium solution of the master equation gives the correct statistical-mechanical description of the atoms and radiation. For negative temperatures, the distribution function for the number of photons in the mode is either exponential or peaked, depending on whether the laser is operating below or above threshold. The calculated intensity fluctuations are in good agreement with semiclassical results for lasers operating slightly above threshold.

I. INTRODUCTION

EARLY speculation[1–3] on the qualitative differences between laser light and coherent light emanating from thermal sources has to a large extent been confirmed by recent photo-detection experiments.[4–8] The outcome of these experiments, briefly, is that above the threshold laser light behaves like an amplitude-stabilized oscillation while below it is characterized by a fluctuating wave amplitude having Gaussian statistics. This abrupt change in behavior at threshold can be justified by appealing to the nonlinear saturation properties of a van der Pol oscillator driven by a suitable noise source.[4,5,9,10] The present article is directed toward the objective of justifying these statistical differences by referring directly to the diagonal elements of the density matrix for laser light.

In a previous article,[11] hereafter referred to as I, a master equation was derived for a system of N stationary two-level atoms interacting with a single radiation mode. The effect of radiation loss was also included and the resulting model was applied numerically to the case of a Q-spoiled laser. In the present article the pumping effects which characterize a three-level laser are incorporated into the master equation of I. But some of our results will be independent of the pumping scheme. The resulting master equation is similar in varying respects to equations proposed by other authors.[12–14] Our approach resembles most closely that of Ref. 14, although the starting equations and the final results are somewhat different. The equilibrium solution for positive temperatures and a lossless cavity gives the correct statistical mechanical probability distribution for the system of atoms and radiation. When negative temperatures are allowed and radiation loss is included, the steady-state solutions lead to two distinct kinds of behavior for the distribution function of the number of photons in the mode. The forms which the distribution function takes may be characterized as "exponential" and "peaked" and the condition which defines the transition from one form to the other turns out to be the laser threshold condition. This is in basic agreement with experimental findings in Refs. 6–8. The calculated intensity fluctuations are also in agreement with those obtained using semiclassical theories.[9,10]

In Sec. II, the basic pumping scheme for a three-level laser is described in terms of rate equations. In Sec. III, terms describing the pumping process are added to the master equation in such a way that the rate equations of Sec. II obtain for the expected values of the upper and lower laser level populations. It is also shown in this section that the detailed solution of the resulting master equation at positive temperatures and for a lossless cavity gives the correct statistical thermodynamic distributions for both atoms and radiation. Sections IV–VI are devoted to the solution of the general steady-state master equation for negative temperatures. In Sec. VII, analytic solutions obtained in Secs. IV and V are compared with numerical solutions of the general master equation for a single atom.

II. RATE-EQUATION DESCRIPTION OF PUMPING PROCESS

Let us consider the energy-level scheme of Fig. 1 which characterizes a three-level laser. The upper and

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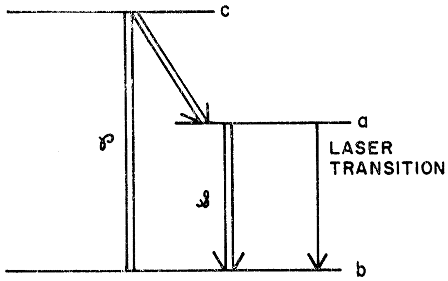

FIG. 1. Energy-level scheme for three-level laser as described by master equation. It is assumed that atoms are raised from the ground level b to the level c and in turn decay instantaneously to the upper laser level a. The rate constant for this process is $\mathcal{P}$. Atoms in turn make transitions to the ground level via the laser transition or through some other process like spontaneous emission into modes other than the laser mode. The rate constant for the latter process is $\mathcal{S}$.

lower laser levels are designated respectively as a and b, and the third level which participates in the pumping process is designated as c. If we consider but a single operating mode and the transition from c to a is sufficiently rapid compared to all other transitions, we may ignore the level c and describe the populations of the a and the b levels by means of the rate equations

$$dN_a/dt=-\langle n\rangle\kappa(N_a-N_b)-\kappa N_b+\mathcal{P}N_b-\mathcal{S}N_a, \quad (2.1a)$$

$$dN_b/dt=\langle n\rangle\kappa(N_a-N_b)+\kappa N_b-\mathcal{P}N_b+\mathcal{S}N_a, \quad (2.1b)$$

$$N_a+N_b=N. \quad (2.1c)$$

In Eqs. (2.1), κ is an absorption rate constant, $\langle n\rangle$ is the average number of photons in the laser mode, $\mathcal{P}$ is the rate at which ground-level atoms are pumped to the upper level, and $\mathcal{S}$ is the rate at which atoms are "de-excited" from the upper level due to all processes other than the laser transition. (By laser transition we mean a transition between levels a and b involving the laser mode only.) The first two terms in Eqs. (2.1a) and (2.1b) describe the rate at which the number of atoms in a level change owing to the laser transition. The second two terms in each equation describe the effect of the pumping process. The steady-state condition is

$$0=\langle n\rangle\kappa(N_a-N_b)+\kappa N_b +\mathcal{P}N_b-\mathcal{S}N_a, \quad (2.2a)$$

$$N_a+N_b=N. \quad (2.2b)$$

If thermodynamic equilibrium exists, for which we presume that $\mathcal{P}/\mathcal{S}<1$, then we expect the first and the second lines of Eq. (2.2a) to vanish separately. We may then define a temperature by means of the Boltzmann ratio

$$N_a/N_b=\mathcal{P}/\mathcal{S} =\exp(-\hbar\omega_0/kT), \quad (2.3)$$

where $\hbar\omega_0$ is the difference between the energy eigenvalues of the upper and lower laser levels.

We would expect that the rate equations (2.1) would hold for the expected values of the upper and lower level populations calculated from a detailed quantum description of the entire system. Such a detailed description is furnished by a master equation. In the following section we shall add terms which describe the pumping mechanism to the master equation derived in I. The resulting master equation determines differential equations for the expected values of the upper and lower state populations with pumping terms which are identical to those of Eqs. (2.1).

III. MASTER EQUATION FOR N-ATOM SYSTEM WITH PUMPING

Let us consider the following extension of Eq. (6.5a) in I:

$$\frac{d}{dt}P_m{}^n=-\kappa n[mP_m{}^n-(N-m+1)P_{m-1}{}^{n-1}] -\kappa(n+1)[(N-m)P_m{}^n-(m+1)P_{m+1}{}^{n+1}] +\mathcal{S}(N-m+1)P_{m-1}{}^n-\mathcal{P}mP_m{}^n +\mathcal{P}(m+1)P_{m+1}{}^n-\mathcal{S}(N-m)P_m{}^n. \quad (3.1)$$

In Eq. (3.1), terms proportional to $\mathcal{P}$ and $\mathcal{S}$ represent the effect of pumping and de-exciting. We examine first the behavior of $\langle n\rangle$ determined by Eq. (3.1). If we multiply Eq. (3.1) through by n and sum over m and n, the result is

$$d\langle n\rangle/dt=\kappa\langle(N-2m)n\rangle+\kappa\langle(N-m)\rangle. \quad (3.2)$$

Equation (3.2) is the N-atom transfer equation derived in I [Eq. (6.7)]. Its form is thus independent of the presence in the master equation of terms describing the exciting and de-exciting mechanisms. We next multiply through Eq. (3.1) by m and sum over m and n. The result is

$$d\langle m\rangle/dt=d\langle n\rangle/dt+\mathcal{S}\sum_{m,n}(m+1)(N-m)P_m{}^n -\mathcal{S}\sum_{m,n}m(N-m)P_m{}^n+\mathcal{P}\sum_{m,n}(m-1)mP_m{}^n -\mathcal{P}\sum_{m,n}m^2P_m{}^n$$
$$=\kappa\langle(N-2m)n\rangle+\kappa\langle(N-m)\rangle +\mathcal{S}\langle N-m\rangle-\mathcal{P}\langle m\rangle. \quad (3.3)$$

In a similar way we obtain

$$d\langle N-m\rangle/dt=-d\langle m\rangle/dt. \quad (3.4)$$

Thus the dependence of the expected values of the number of upper and lower state atoms on the pumping and de-exciting mechanisms is the same as that assumed in the rate equations (2.1).

We examine now the solution to the equilibrium form of Eq. (3.1), which is

$$0=-\kappa n[mP_m{}^n-(N-m+1)P_{m-1}{}^{n-1}]-\kappa(n+1)[(N-m)P_m{}^n-(m+1)P_{m+1}{}^{n+1}]+\mathcal{S}(N-m+1)P_{m-1}{}^n-\mathcal{P}mP_m{}^n+\mathcal{P}(m+1)P_{m+1}{}^n-\mathcal{S}(N-m)P_m{}^n. \quad (3.5)$$

Equation (3.5) will evidently be satisfied, provided $P_m{}^n$ satisfies simultaneously the following two difference equations:

$$\mathcal{P}(m+1)P_{m+1}{}^n-\mathcal{S}(N-m)P_m{}^n=0, \quad (3.6a)$$

$$(m+1)P_{m+1}{}^{n+1}-(N-m)P_m{}^n=0. \quad (3.6b)$$

A solution to Eq. (3.6a) will make the last two lines of Eq. (3.5) vanish separately; a solution of Eq. (3.6b) will cause the separate vanishing of the first two bracket terms of Eq. (3.5). The solution to Eq. (3.6a) may be written

$$P_m{}^n=\left(\frac{\mathcal{S}}{\mathcal{P}}\right)^m\frac{N!}{m!(N-m)!}P_0{}^n. \quad (3.7)$$

Let us put

$$\mathcal{P}/\mathcal{S}=\exp(-\hbar\omega_0/kT), \quad (3.8a)$$

$$g_m=N!/m!(N-m)!, \quad (3.8b)$$

then

$$P_{m_1}{}^n/P_{m_2}{}^n=(g_{m_1}/g_{m_2})\exp[(m_1-m_2)\hbar\omega_0/kT]. \quad (3.9)$$

But Eq. (3.9) represents the Boltzmann ratio of the probabilities of two configuration states in a system of N uncoupled spins in contact with a heat reservoir. Thus Eq. (3.7) gives the correct equilibrium distribution for the occupation of the atomic levels. We determine $P_0{}^n$ by substituting expression (3.7) into Eq. (3.6b). The result is

$$P_0{}^{n+1}=P_0{}^n\exp(-\hbar\omega_0/kT),$$
$$P_0{}^n=P_0{}^0\exp(-n\hbar\omega_0/kT), \quad (3.10)$$

which is a Bose-Einstein distribution for the occupation of the photon energy states. The complete normalized distribution may be written[15]

$$P_m{}^n=\frac{N!}{(N-m)!m!}p^m(1-p)^{N-m}\times[1-\exp(-\hbar\omega_0/kT)]\exp(-n\hbar\omega_0/kT), \quad (3.11)$$

where

$$p=\exp(\hbar\omega_0/kT)/[1+\exp(\hbar\omega_0/kT)]. \quad (3.12)$$

It will be noted that it is the atomic frequency ω_0, rather than the mode frequency ω (see I), which occurs in the Bose-Einstein distribution (3.11). This, however, is to be expected, since it is the atomic system which is in contact with the heat reservoir.

[15] Expression (3.11) satisfies the conditions of detailed balancing. The uniqueness of (3.11) as a solution to (3.5) follows by a well-known argument. See, for example, C. Kittel, *Elementary Statistical Physics* (John Wiley & Sons, Inc., New York, 1958), pp. 169–171.

By approximating the binomial distribution in Eq. (3.12), we obtain for the distribution of the atomic levels

$$P_m\cong[2\pi Np(1-p)]^{-1/2}\exp\left[-\frac{1}{2}\frac{(m-Np)^2}{Np(1-p)}\right]. \quad (3.13)$$

From (3.13), evidently

$$(N-\langle m\rangle)/\langle m\rangle=\exp(-\hbar\omega_0/kT), \quad (3.14a)$$

$$\sigma/\langle m\rangle=[(1-p)/pN]^{1/2}, \quad (3.14b)$$

where σ is the standard deviation in m. Because of the large value of N, the distribution in m/N is therefore exceedingly sharply defined and for all practical purposes the occupation of the atomic levels can be characterized by the Boltzmann factor (3.14a).

We have thus demonstrated that the master equation leads to the correct equilibrium description of the system of N atoms and radiation field if $\mathcal{P}/\mathcal{S}<1$ or, equivalently, for positive temperature. If $\mathcal{P}/\mathcal{S}\geq 1$, i.e., for infinite positive or for negative temperatures, the solution (3.11) is no longer finite. However, if terms describing radiation loss are added to Eq. (3.1), equilibrium solutions are again possible. We therefore adopt the following as the complete master equation describing the laser:

$$\frac{d}{dt}P_m{}^n=-\kappa n[mP_m{}^n-(N-m+1)P_{m-1}{}^{n-1}]-\kappa(n+1)[(N-m)P_m{}^n-(m+1)P_{m+1}{}^{n+1}]+\mathcal{S}(N-m+1)P_{m-1}{}^n-\mathcal{P}mP_m{}^n+\mathcal{P}(m+1)P_{m+1}{}^n-\mathcal{S}(N-m)P_m{}^n+(n+1)\gamma P_m{}^{n+1}-n\gamma P_m{}^n. \quad (3.15)$$

It should be remarked, finally, that the time-dependent form of Eq. (3.15) is approximate in the sense that it is based upon an approximate elimination of off-diagonal elements in terms of diagonal elements of the density matrix. All types of rate equations are approximate in this sense. However, this particular approximation becomes exact in the limit of equilibrium. For the further conditions of validity of the radiation-dependent parts of Eq. (3.15), the reader is referred to I.

IV. LASER STEADY-STATE SOLUTION OF MASTER EQUATION

We consider now the steady-state solution of Eq. (3.15) for negative temperatures. We must solve

$$0=-\kappa n[mP_m{}^n-(N-m+1)P_{m-1}{}^{n-1}]-\kappa(n+1)[(N-m)P_m{}^n-(m+1)P_{m+1}{}^{n+1}]+\mathcal{S}(N-m+1)P_{m-1}{}^n-\mathcal{P}mP_m{}^n+\mathcal{P}(m+1)P_{m+1}{}^n-\mathcal{S}(N-m)P_m{}^n+\gamma_c(n+1)P_m{}^{n+1}-\gamma_c nP_m{}^n. \quad (4.1)$$

Before proceeding, let us define the following moments:

$$\eta_a(n)=\sum_m P_m{}^n(N-m)/NP^n\,,$$
$$\eta_b(n)=\sum_m P_m{}^n m/NP^n\,, \tag{4.2}$$
$$P^n=\sum_m P_m{}^n\,.$$

The quantities $\eta_a(n)$ and $\eta_b(n)$ represent, respectively, the conditional probabilities of occupying the upper and lower levels, given that there are n photons in the mode. Let us now sum Eq. (4.1) over m. The terms proportional to $\mathcal{P}$ and $\mathcal{S}$ sum to 0, and the result is

$$\begin{aligned}(n+1)[\gamma_c+N\kappa\eta_b(n+1)]P^{n+1}\\ -(n+1)N\kappa[1-\eta_b(n)]P^n\\ -n[\gamma_c+N\kappa\eta_b(n)]P^n\\ +nN\kappa[1-\eta_b(n-1)]P^{n-1}=0\,.\end{aligned} \tag{4.3}$$

Equation (4.3) is satisfied by a solution of

$$P^{n+1}=\frac{N\kappa\eta_a(n)}{\gamma_c+N\kappa\eta_b(n+1)}P^n\,. \tag{4.4}$$

The solution of Eq. (4.4) may be expressed as[16]

$$P^n=\prod_{n'=0}^{n}\left[\frac{N\kappa\eta_a(n')}{\gamma_c+N\kappa\eta_b(n'+1)}\right]P^0\,. \tag{4.5}$$

Equation (4.5) is independent of the details of the pumping action and is therefore quite general and covers the cases both of thermodynamic equilibrium and nonequilibrium steady-state equally well. If, for example, it can be stipulated that η_a and η_b are determined by a Boltzmann ratio at positive temperature, and if $\gamma_c=0$, then Eq. (4.5) describes a Bose-Einstein distribution. If $\gamma_c>0$ and η_a and η_b are determined by a Boltzmann ratio at positive or, for that matter, negative temperature, so long as the general factor in Eq. (4.5) is less than unity, then Eq. (4.5) defines a Bose-Einstein distribution whose effective temperature depends on the temperature of the atoms, on the absorption-rate constant $N\kappa$, and on the loss-rate constant γ_c:

$$T_{\text{eff}}=-\hbar\omega_0/k\ln\left\{\frac{N\kappa\exp(-\hbar\omega_0/kT)}{\gamma_c[1+\exp(-\hbar\omega_0/kT)]+N\kappa}\right\}\,. \tag{4.6}$$

In the most general case, the density matrix of the complete system is not factorable as it is in the case of thermodynamic equilibrium, e.g., the case of Eq. (3.12), and a determination of $\eta_a(n)$ and $\eta_b(n)$ may require a detailed solution of the full master equation. This is precisely the situation with which we are confronted in the description of a laser.

To proceed with the determination of $\eta_a(n)$ and $\eta_b(n)$ let us assume that in Eq. (4.1) we can to a good approximation replace $P_m{}^{n+1}$ by $P_m{}^n$, $P_{m-1}{}^{n-1}$ by $P_{m-1}{}^n$, and $P_{m+1}{}^{n+1}$ by $P_{m+1}{}^n$. Let us also neglect the last two terms of Eq. (4.1),

$$\gamma_c(n+1)P_m{}^{n+1}-\gamma_c nP_m{}^n\cong\gamma_c P_m{}^n\,. \tag{4.7}$$

This neglect can be justified in the following way: For small values of n we expect the pumping terms in Eq. (4.1) to dominate, so the neglect will not be important in this case. For large values of n, on the other hand, the radiation terms will become important. But in that case the coefficients of the first four terms in Eq. (4.1) will be like $nN\kappa$ where $N\kappa\approx\gamma_c$, and the terms (4.7) would be small in comparison. We emphasize, however, that the terms proportional to γ_c have been taken into account without approximation in the determination of the n dependence of $P_m{}^n$ in Eq. (4.5).

As a consequence of the above approximations Eq. (4.1) becomes[17]

$$\begin{aligned}(m+1)(\mathcal{P}+\kappa n)P_{m+1}{}^n-(N-m)(\mathcal{S}+\kappa n)P_m{}^n\\ -m(\mathcal{P}+\kappa n)P_m{}^n+(N-m+1)\\ \times(\mathcal{S}+\kappa n)P_{m-1}{}^n=0\,.\end{aligned} \tag{4.8}$$

Equation (4.8) is satisfied by a solution of

$$(m+1)(\mathcal{P}+\kappa n)P_{m+1}{}^n-(N-m)(\mathcal{S}+\kappa n)P_m{}^n=0\,, \tag{4.9}$$

which may be expressed as

$$P_m{}^n=\left(\frac{\mathcal{S}+\kappa n}{\mathcal{P}+\kappa n}\right)^m\frac{N!}{m!(N-m)!}P_0{}^n\,. \tag{4.10}$$

Normalizing, we have

$$P_m{}^n=\frac{N!}{m!(N-m)!}\eta_b(n)^m\eta_a(n)^{N-m}P^n\,, \tag{4.11}$$

where

$$\eta_b(n)=(\mathcal{S}+\kappa n)/(\mathcal{S}+\mathcal{P}+2\kappa n)\,,$$
$$\eta_a(n)=(\mathcal{P}+\kappa n)/(\mathcal{S}+\mathcal{P}+2\kappa n)\,, \tag{4.12}$$

and P^n is the probability of there being n photons in the mode. For large N we may express Eq. (4.11) as

$$P_m{}^n\cong\frac{P^n}{[2\pi N\eta_b(n)\eta_a(n)]^{1/2}}\times\exp\left\{-\frac{1}{2}\frac{[m-N\eta_b(n)]^2}{N\eta_b(n)\eta_a(n)}\right\}\,. \tag{4.13}$$

From either Eq. (4.11) or (4.13) the mean values of $(N-m)/N$ and m/N are $\eta_a(n)$ and $\eta_b(n)$. From Eq.

[16] This solution satisfies detailed balancing. See footnote 15.

[17] Another way to derive Eq. (4.8) is to average Eq. (4.1) over some range of n about the particular value of n in question. The values of n which occur in Eq. (4.8) are, strictly speaking, suitable averages of n in this range. We have replaced these averages by n itself and have also ignored the difference between $n+1$ and n.

(4.13) the standard deviation in m/N is

$$\sigma=[\eta_b(n)\eta_a(n)]^{1/2}/N^{1/2}. \qquad (4.14)$$

For large N the distribution (4.13) in m is thus extremely sharply peaked at its mean value. It is interesting to note the saturation properties implied by Eqs. (4.5) and (4.7). For given n the ratio of the probabilities for occupying the upper and lower states is

$$\eta_a(n)/\eta_b(n)=(\mathcal{P}+\kappa n)/(\mathcal{S}+\kappa n). \qquad (4.15)$$

If we assume that $\kappa\ll\mathcal{P}$, $\kappa\ll\mathcal{S}$, then for small values of n the ratio (4.9) is determined solely by the pumping and de-excitation mechanisms. For increasing n, however, the ratio diminishes approaching unity.

V. PROPERTIES OF THE PHOTON DISTRIBUTION FUNCTION ABOVE THRESHOLD

The photon distribution function P^n defined in Eq. (4.5) displays two distinct types of behavior, depending on whether the ratio

$$N\kappa\eta_a(0)/[\gamma_c+N\kappa\eta_b(1)]\cong N\kappa\eta_a(0)/[\gamma_c+N\kappa\eta_b(0)] \qquad (5.1)$$

is less than or exceeds unity. In the former case, P^n decreases monotonically with n. In the latter case, P^n increases initially with n but reaches a maximum and then decreases with a further increase in n. The transition between the two types of behavior is defined by the condition

$$N\kappa[\eta_a(0)-\eta_b(0)]=\gamma_c. \qquad (5.2)$$

It will be noted that Eq. (5.2) is the Schalow-Townes threshold condition for laser oscillation. We have demonstrated, therefore, that the laser oscillation threshold is also the threshold for the changeover in the statistical behavior of laser radiation. Note that the threshold condition (5.2) is expressed in terms of the *unsaturated* inversion.

Let us assume now that the laser is operating above threshold or that the left-hand side of Eq. (5.2) exceeds the right-hand side. We determine now that value of n for which the distribution P^n peaks. From Eq. (4.5) we see that the maximum in P^n occurs for n such that

$$P^{n+1}=P^n, \qquad (5.3)$$

or, equivalently,

$$N\kappa[\eta_a(n)-\eta_b(n+1)]=\gamma. \qquad (5.4)$$

Equation (5.4) shows that the photon distribution peaks at that value of n for which the threshold condition is satisfied by the *saturated* inversion. Making use of Eqs. (4.12) for $\eta_a(n)$ and $\eta_b(n)$ and neglecting 1 compared with n, we now solve Eq. (5.4) for n. Calling this value n_0, we obtain

$$n_0=\frac{(\mathcal{S}+\mathcal{P})}{2\kappa}\left[\left(\frac{\mathcal{P}-\mathcal{S}}{\mathcal{P}+\mathcal{S}}\right)\frac{1}{\gamma(N\kappa)^{-1}}-1\right]. \qquad (5.5)$$

The first term in the bracket in Eq. (5.5) can be written as the ratio

$$\mathfrak{N}=[\eta_a(0)-\eta_b(0)]/[\eta_a(n_0)-\eta_b(n_0)], \qquad (5.6)$$

where $\mathfrak{N}$ is the ratio of the inversion which would exist if losses were large enough to prevent oscillation to the existing saturated inversion. [The numerator of (5.6) depends on pumping; the denominator is an intrinsic constant of the system and is independent of pumping.] Thus, Eq. (5.5) may be expressed in the form

$$n_0=[(\mathcal{S}+\mathcal{P})/2\kappa](\mathfrak{N}-1). \qquad (5.7)$$

The above value of n_0, which has been determined from the detailed solution to the master equation, can be checked by calculating it from the expected-value equations (3.2) and (3.4). Because $P_m{}^n$ is sharply peaked in m and, as we shall subsequently show, also in n, provided the laser is operating somewhat above threshold, we may write the steady-state form of Eqs. (3.2) and (3.3) in the following form after adding a radiation loss term to (3.2):

$$N\kappa[\eta_a(n_0)-\eta_b(n_0)]n_0+N\kappa\eta_a=\gamma_c n_0, \qquad (5.8a)$$

$$\gamma_c n_0+\mathcal{S}N\eta_a(n_0)-\mathcal{P}N\eta_b(n_0)=0. \qquad (5.8b)$$

If the spontaneous-emission term in (5.8a) is neglected, Eq. (5.8a) is identical with (5.4), and Eqs. (5.8) yield the value (5.7) for n_0.[18]

Let us now examine in greater detail the shape of the photon distribution P^n. Instead of using Eq. (4.5) we can also write P^n in the form

$$P^n=P^{n_0}\prod_{n'=n_0}^{n}\left[\frac{N\kappa\eta_a(n')}{\gamma_c+N\kappa\eta_b(n')}\right]^{\pm1}, \qquad (5.9)$$

where $n'+1$ in the argument of η_b has been replaced with n' and where the positive or negative exponent holds respectively in the cases where n is greater or less than n_0. Taking the logarithm of both sides of Eq. (5.9), we obtain

$$\ln P^n=\ln P^{n_0}\pm\sum_{n'=n_0}^{n}\ln\left[\frac{N\kappa\eta_a(n')}{\gamma_c+N\kappa\eta_b(n')}\right]. \qquad (5.10)$$

For the purpose of developing the logarithmic terms in Eq. (5.10) in a Taylor series about n_0, we define

$$f(n')=\ln\left[\frac{N\kappa\eta_a(n')}{\gamma_c+N\kappa\eta_b(n')}\right]. \qquad (5.11)$$

Making use of Eqs. (4.12) and (5.4) and neglecting 1

[18] The neglect of n compared with $n+1$ in Eq. (5.4) and the neglect of spontaneous emission in the rate equation (5.8a) are thus in a sense equivalent.

compared with n_0, we obtain

$$\left.\frac{df}{dn'}\right|_{n'=n_0}=\frac{1}{\eta_a(n_0)}[\eta_a'(n_0)-\eta_b'(n_0)]$$
$$=-\frac{2\gamma_c}{N(\mathcal{P}+n_0\kappa)}. \quad (5.12)$$

Thus,

$$\ln P^n=\mp\frac{2\gamma}{N(\mathcal{P}+n_0\kappa)}\sum_{n'=n_0}^{n}(n'-n_0)$$
$$\cong-\frac{2\gamma_c}{N(\mathcal{P}+n_0\kappa)}\frac{(n-n_0)^2}{2}. \quad (5.13)$$

We may therefore express P^n in the form

$$P^n\cong\left[\frac{\pi\gamma_c}{N(\mathcal{P}+n_0\kappa)}\right]^{-1/2}\times\exp\left[-\frac{\gamma_c}{N(\mathcal{P}+n_0\kappa)}(n-n_0)^2\right] \quad (5.14)$$

for a laser operating somewhat above threshold. The variance in n is

$$\sigma^2=N(\mathcal{P}+n_0\kappa)/2\gamma_c. \quad (5.15)$$

From Eq. (5.4), $N\kappa>\gamma_c$ and from Eq. (5.8b), $N\mathcal{P}>\gamma_c n_0$. Therefore, using (5.15), we may write the following inequality:

$$\sigma^2>n_0. \quad (5.16)$$

[For a Poisson distribution, which corresponds to a coherent state, $\sigma^2=n_0$.] The form (5.14) is valid also near and at threshold with a suitable normalization. Near threshold P^n takes the form of a truncated Gaussian. (See Sec. VII.)

With the help of Eqs. (4.5) and (5.4), we can also express Eq. (5.15) in the form

$$\frac{\sigma^2}{n_0}=\frac{\eta_a(n_0)}{\eta_a(n_0)-\eta_b(n_0)}\frac{\mathfrak{N}}{(\mathfrak{N}-1)}. \quad (5.17)$$

Expressing the output power P as

$$P=n_0\gamma_c \quad (5.18)$$

and the laser linewidth $\Delta\omega$ as

$$\Delta\omega=\frac{\gamma_c^2\hbar\omega}{2P}\frac{\eta_a(n_0)}{\eta_a(n_0)-\eta_b(n_0)}, \quad (5.19)$$

we can write Eq. (5.17) as

$$\sigma^2/n_0^2=[2\mathfrak{N}/(\mathfrak{N}-1)]\Delta\omega/\gamma_c. \quad (5.20)$$

An expression identical to (5.20) except for the factor $\mathfrak{N}$ has been derived from semiclassical theory.[19]

[19] See Ref. 10, especially Eq. (52).

It is possible to check the validity of our solution to the master equation upon which the variance formulas (5.17) and (5.20) are based by deriving the variance in another way. To do this we make use of Eqs. (4.5), (5.2), and (5.4). Let us attempt to find a linear approximation for

$$\ln\left[\frac{N\kappa\eta_a(n)}{\gamma_c+N\kappa\eta_b(n+1)}\right]\cong\frac{(n_0-n)}{\sigma^2}. \quad (5.21)$$

Such an approximation should be valid over the range of interest in the variation of n, provided the argument of the logarithm does not deviate significantly from 1 and provided $\eta_a(n_0)$ and $\eta_b(n_0)$ do not differ greatly from $\eta_a(0)$ and $\eta_b(0)$, which we can reasonably infer to be the unsaturated values of η_a and η_b. Both of these conditions in turn should be satisfied as long as the laser is not operating too far above the threshold, or if

$$\mathfrak{N}\approx 1. \quad (5.22)$$

If the approximation (5.21) is now substituted into Eq. (4.5), then

$$P^n\cong\frac{1}{\sigma(2\pi)^{1/2}}\exp\left[-\frac{(n-n_0)^2}{2\sigma^2}\right], \quad (5.23)$$

where, from (5.21),

$$\sigma^2=n_0\left[\ln\frac{N\kappa\eta_a(0)}{\gamma_c+N\kappa\eta_b(0)}\right]^{-1}. \quad (5.24)$$

Expanding the right-hand side of (5.24) in a Taylor series and making use of (5.4) to eliminate $\gamma_c(N\kappa)^{-1}$, we obtain

$$\frac{\sigma^2}{n_0}\cong\frac{\eta_a(n_0)+[\eta_b(0)-\eta_b(n_0)]}{\eta_a(n_0)-\eta_b(n_0)}\frac{1}{\mathfrak{N}-1}. \quad (5.25)$$

Both Eqs. (5.17) and (5.25) are in good agreement, provided condition (5.22) is met. The distribution (5.23) also holds in the immediate neighborhood of threshold, provided it is appropriately normalized. Therefore, we may be reasonably sure of the validity of the solution to the master equation developed in Sec. IV and the first part of Sec. V as long as the laser is operating slightly above threshold. This conclusion will also be borne out by numerical examples in Sec. VII. It will be noted that Eqs. (5.24) and (5.25) have been derived without specific reference to the pumping scheme.

To summarize, we have demonstrated that the basic character of the distribution function for the number of photons in the mode changes when the laser threshold condition is met and that above threshold the distribution is peaked, just as one would expect for an amplitude-stabilized oscillation and that, furthermore, the intensity fluctuations decrease with increasing power above threshold. These results are in basic agreement with the experimental findings.

VI. BEHAVIOR OF THE PHOTON DISTRIBUTION FUNCTION BELOW THRESHOLD

We consider now the case for which

$$N\kappa[\eta_a(0)-\eta_b(0)]<\gamma_c. \tag{6.1}$$

So long as $n\kappa \ll \mathcal{P}$, the photon distribution function behaves like

$$P^n=\text{const } q^n, \tag{6.2}$$

where q is defined by

$$q=N\kappa\eta_a(0)/[\gamma_c+N\kappa\eta_b(0)], \tag{6.3}$$

but deviates from (6.2) for $n\kappa\approx\mathcal{P}$. If, however, the laser is operating sufficiently below threshold, the contribution to $\langle n\rangle$ of the P^n which show saturation behavior will be negligible. Under such circumstances we may to a good approximation write P^n as

$$P^n=(1-q)q^n. \tag{6.4}$$

Thus, the effective temperature of a laser operating in a single mode somewhat below threshold is

$$T=-\hbar\omega_0\Big/k\ln\left[\frac{N\kappa\eta_a(0)}{\gamma_c+N\kappa\eta_b(0)}\right], \tag{6.5}$$

and the expected photon number is the Bose-Einstein average:

$$\langle n\rangle=1/(q^{-1}-1)$$
$$=\frac{N\kappa\eta_a(0)}{(\gamma_c-N\kappa[\eta_a(0)-\eta_b(0)])}. \tag{6.6}$$

But expression (6.6) is the steady-state solution of

$$d\langle n\rangle/dt=N\kappa[\eta_a(0)-\eta_b(0)]\langle n\rangle+N\kappa\eta_a(0)-\gamma_c\langle n\rangle, \tag{6.7}$$

which is precisely what one would expect on simple phenomenological grounds. However, the difficulty of understanding what happens as the denominator of Eq. (6.6) goes to 0 is no longer with us; for inversions sufficiently close to threshold we know that Eqs. (6.4) and (6.6) are simply inadequate and that account must be taken of the saturation behavior of $\eta_a(n)$ and $\eta_b(n)$.

VII. NUMERICAL EXAMPLES

The steady-state solutions developed in Sec. IV and at the end of Sec. V are applicable alike in the case of many atoms or in the case of a single atom. In the latter case it is possible to obtain, for comparison purposes, steady-state solutions of the master equation by truncating $P_m{}^n$ at a specific value of n. By adjusting the parameters $\mathcal{P}$, $\mathcal{S}$, κ, and γ_c it is possible to hold $\langle n\rangle$ and $n_{\max}$, the value above which $P_m{}^n$ is set equal to 0, to reasonable values and thus to solve simultaneously the differential equations (3.12). In Eqs. (3.11) the index n goes from $n=0$ to $n=n_{\max}$, and the index m goes from $m=0$ to 1. The initial value of $P_0{}^0$ is set equal to 1, all other probabilities are set equal to 0, and the system of equations is allowed to approach equilibrium.

Figures 2 and 3 apply to the case of a laser operating

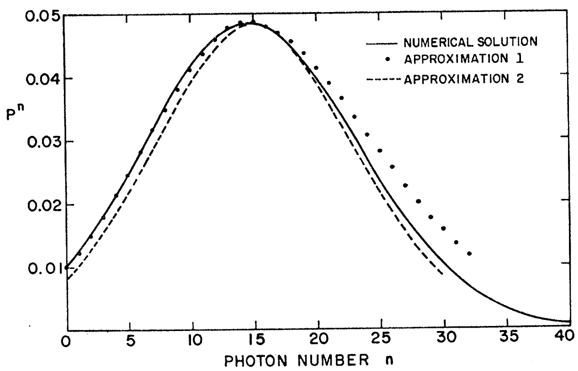

FIG. 2. Comparison between numerical solution of the master equation and approximate analytic solutions for single-atom laser operating slightly above threshold. Plotted is P^n, the normalized photon distribution as a function of photon number. By appropriate choice of the laser parameters, the expected number of photons has been held to a low value. Approximation 1 is based on Eqs. (4.12). Approximation 2 is based on Eq. (5.21). Both of these approximations give good agreement with semiclassical results.

slightly above threshold with $\mathcal{P}=75$, $\mathcal{S}=15$, $\kappa=1$, $\gamma_c=0.5$, and $n_{\max}=50$. From Eq. (5.5), $n_0=15$ and $\mathcal{N}=1.333$. The numerical solution of P^n is plotted in Fig. 2. Also plotted is the solution (4.5) calculated by (1) using $\eta_a(n)$ and $\eta_b(n)$ as given in Eqs. (4.12); and (2) using $\eta_a(n)$ and $\eta_b(n)$ calculated from Eqs. (5.21) and (5.24). In the latter case the value n_0 was calculated by Eq. (5.5). It will be remembered that Eq. (4.5) is exact, but both Eqs. (4.12) and Eq. (5.21) represent approximations. It is to be noted that the value of $n_0=15$, predicted by Eq. (5.5) which is in turn based on Eqs. (4.12), is in perfect agreement with the numerical solution. The solution based on Eqs. (4.12) provides a good fit to the numerical solution on the front side of P^n. The solution based on Eq. (5.21) provides perhaps a better over-all fit, but gives a distribution which is too narrow. Both cases (1) and (2) have been normalized so that the maximum value of P^n agrees with the numerical solution. The normalized conditional population inversion $\eta_a(n)-(n_b\eta)$ is plotted in Fig. 3 as a function of n. Again it is found that Eqs. (4.12) provide a good fit for $n<n_0$ but a poor fit for $n>n_0$. Equation (5.21) again provides a more consistent fit.

Figure 4 applies to the case of a laser operating well above threshold with $\mathcal{P}=17$, $\mathcal{S}=5$, $\kappa=1$, $\gamma_c=0.1$, and $n_{\max}=100$. In this case $\mathcal{N}=5.455$, and n_0 as calculated from Eq. (5.5) is 49. This value is in perfect agreement with the numerical calculation of P^n. However, due to the large value of $\mathcal{N}$ it might be expected that neither Eqs. (4.12) nor Eq. (5.21) should apply. The numerically calculated P^n is to a good approximation a Gaussian distribution with standard deviation $\sigma=9.1$. Formulas (5.15) or (5.17) give $\sigma=18.2$. Equation (5.24) on the other hand gives $\sigma=7.6$. This is a much better fit, but the discrepancy is obviously due to the fact that the logarithmic function in Eq. (5.21) cannot be expressed as a linear function of n over the range from 0 to n_0. In fact, if $\eta_a(n)-\eta_b(n)$ is calculated from (5.21) with $\sigma=9.1$, the dashed curve in Fig. 4 results. The successful prediction of n_0 by means of Eqs. (4.12) is explained by the fact that approximations like setting $P_m{}^{n+1}=P_m{}^n$ as were made in deriving (4.12) become exact for $n=n_0$. Thus we would expect expressions (4.11) through (4.13) to be quite accurate for $n=n_0$, even if the laser is operating well above threshold.

In summary, our numerical examples are perhaps somewhat inconclusive, since they do not correspond to actual physical laser systems. However, they lend credence to the intensity-fluctuation formulas developed earlier in the paper and by implication to semiclassical formulas, provided the laser is not operating too far above threshold. They also suggest that considerable care may have to be exercised in choosing equivalent quantum noise sources for semiclassical treatments of lasers operating well above threshold.

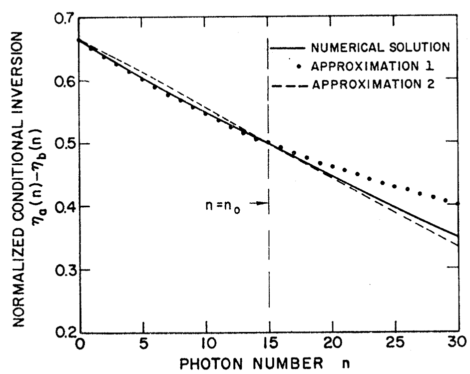

FIG. 3. Calculation for parameters of Fig. 2 of $\eta_a(n)-\eta_b(n)$ as a function of n, where $\eta_a(n)$ and $\eta_b(n)$ represent the conditional probabilities that an atom will occupy the upper and the lower laser levels, given that n photons are in the mode. For $n=0$, $\eta_a(n)-\eta_b(n)$ should very nearly equal the unsaturated normalized population inversion. For $n=n_0$, the most probable photon number, it should take the saturated value such that $N\kappa[\eta_a(n_0)-\eta_b(n_0)]=\gamma_c$.

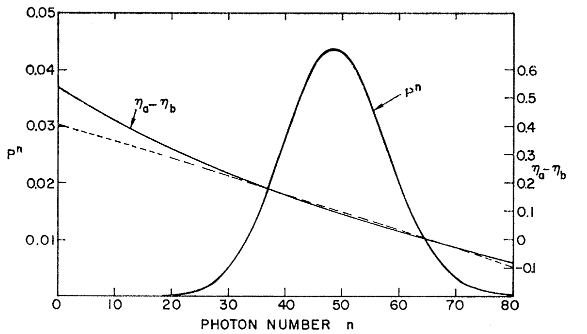

FIG. 4. Numerical solution of one-atom master equation for the case of a laser operating well above threshold with $\mathcal{N}=5.455$. Photon distribution is Gaussian, with standard deviation $\sigma=9.1$. Analytic expressions developed in text do not fit the numerical solution well. The dashed curve is a plot of $\eta_a-\eta_b$ fit to Eq. (5.21).

Note added in proof. Our photon distribution function above threshold can be expressed as

$$P^n=A\exp\left[-\frac{n^2}{2\sigma^2}+\frac{(\mathcal{S}+\mathcal{P})}{2\kappa\sigma^2}(\mathcal{N}-1)n\right],$$

where A is a normalization constant and σ^2 is defined by Eq. (5.15). All quantities appearing in this expression are slowly varying except $\mathcal{N}-1$ in the region just above threshold. Therefore, P^n may be determined by a single parameter which varies with the laser excitation. A distribution of this form has been fit to experimental data by A. W. Smith and J. A. Armstrong, Phys. Rev. Letters **16**, 1169 (1966). Similar expressions have been derived from a Fokker-Planck equation by H. Risken, Z. Physik **186**, 85 (1965) and by M. Lax and R. D. Hempstead, Bull. Am. Phys. Soc. **11**, 111 (1966).