Quantum Theory of Laser Radiation. I. Many-Atom Effects*

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The interaction is considered between N stationary two-level atoms and a radiation field described by a single cavity mode. The state vector for the complete system of atoms plus radiation is expressed as a linear superposition of states constructed from a product of photon states in the n representation and products of Pauli spin eigenstates describing all combinations of atoms in the lower and upper energy levels. Equations for the corresponding probability amplitudes are derived by substituting this superposition into the Schrödinger equation. The resulting equations are combined into bilinear form and phenomenological damping contributions are added. After the neglect of certain of the bilinear quantities, a master equation is derived which governs the probability $P_m{}^n$ of having m atoms in the lower level and n photons in the mode. This master equation takes account of multiple single-quantum absorption and emission processes but not of simultaneous multiple processes involving two or more atoms at a time. The equation which governs the expected number of photons $\langle n \rangle$ derived from the master equation bears a close resemblance to a rate equation. The effect of radiation loss from the cavity is incorporated into the master equation. Numerical calculations for a Q-spoiled laser show that the statistics of the number of photons in the mode bear a qualitative resemblance to Poisson statistics.

I. INTRODUCTION

THE theory of maser and laser devices which has paralleled the extensive experimental advances in the field has, to a large extent, been based on a single model. This model,[1–12] known variously as the phenomenological semiclassical, the neoclassical, or the SCFA (self-consistent-field approximation) model, has made possible the qualitative understanding of many laser phenomena and in some cases has enabled the quantitative description as well. Despite these successes, this model is inadequate in several respects. First of all, the SCFA model embodies the assumption that the density matrix for the complete laser system can be represented as the product of density matrices for the radiation field and for each individual atom.[8,10] As such it is not appropriate for understanding the collective radiative phenomena which have been described by Dicke[13] and Senitzky.[14] Second, it is not fully clear from the usual SCFA derivations just how the effects of spontaneous emission should be taken into account. These effects have of course an important bearing on such coherence properties of the laser as linewidth and amplitude and intensity fluctuations. The traditional approach to this problem has been the phenomenological one of adding suitable noise sources based on equilibrium or other considerations to the SCFA equations of motion.[15–17] Finally, the SCFA model sheds little or no light on the important question of the statistical nature of laser radiation.[18,19]

A satisfactory program for enlarging upon the SCFA model to the point where these deficiencies are removed should involve proceeding from an N-atom Hamiltonian, carrying along a fully quantized electromagnetic field from the start, and treating the density matrix for the complete system in unfactored form. Nevertheless, a certain amount of phenomenology remains unavoidable in treating the effects of pumping, damping, and radiation loss. The success of the model will depend in large measure on the degree of realism which can be achieved in this phenomenology. In the present as well as in two succeeding articles, we attempt to carry out this program to determine the density matrix, or at least those elements of it which bear upon the statistical and coherence properties of laser radiation.

The present article is concerned with the derivation of those equations which determine the behavior of the diagonal elements of the density matrix for a single cavity mode interacting with N stationary two-level atoms. As such these equations do not furnish us with

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any phase-dependent information. However, they govern the dynamics and steady-state behavior of the radiation intensity emitted from a laser, as well as the statistical distribution of the number of photons to be found in the cavity mode. The effect of radiation loss from the cavity is incorporated into these equations in the present article. The resulting equations are adequate for a detailed discussion of the emission properties of a Q-spoiled laser. In the second article in the series, the effects of pumping and spontaneous-emission loss will be included. This will enable a discussion of the emission properties and the statistical distribution governing the photon number for a single-mode cw laser. The third article in the series (to be published), will be devoted to those elements of the density matrix which govern the phase-dependent properties of laser radiation.

There are two ways to set about the determination, or, more appropriately, the approximation of the density matrix. The first is to proceed from the general Liouville equation which is satisfied by the density operator.[8,20] The second equivalent way is to expand the state vector in an appropriate complete set of states and by substitution into Schrödinger equation to determine the equations of motion of the corresponding amplitudes. From these one can derive equations governing the various bilinear combinations of the probability amplitudes which make up the density matrix. In view of the complexity of the problem under discussion, it is felt that the second way has definite advantages over the first. In any case, it is the one that is followed here.

A discussion of the Hamiltonian and the basic states along with other preliminaries is given in Sec. II. By way of illustration, the equations governing the diagonal density matrix elements for a radiation field interacting with a single atom are derived and discussed in Sec. III. Sections IV and V are devoted to the derivation of bilinear equations in the N-atom case. In Secs. VI and VII, as well as in the Appendix, these equations are discussed in two limiting cases: the so-called "superradiant" case, first described by Dicke,[13] in which the atoms emit at a rate proportional to N^2, and the normal case in which the radiation rate is proportional to N. It is concluded that the latter is the only case of physical interest where lasers are concerned. Section VIII deals with variable coupling between atoms and the field. In Sec. IX the radiation-loss mechanism is introduced. The basic master equation governing the diagonal elements of the density matrix is found at the end of this section. It resembles equations proposed by Shimoda, Takahashi, and Townes[21] and by McCumber.[22] It differs, however, by the inclusion of additional terms. In Secs. X and XI numerical examples are discussed which illustrate the emission and statistical properties of a Q-spoiled laser.

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II. HAMILTONIAN FOR THE LASER SYSTEM

We begin by considering an expansion of the electromagnetic field in the laser cavity in terms of a set of normal-mode eigenfunctions. The vector potential and electromagnetic field are given by

$$\begin{aligned}\mathbf{A}&=c(4\pi)^{1/2}\textstyle\sum_k \boldsymbol{\varepsilon}_k q_k(t)E_k(\mathbf{X}),\\ \mathbf{E}&=-(4\pi)^{1/2}\textstyle\sum_k \boldsymbol{\varepsilon}_k p_k(t)E_k(\mathbf{X}),\\ \mathbf{H}&=-c(4\pi)^{1/2}\textstyle\sum_k(\boldsymbol{\varepsilon}_k\times\boldsymbol{\nabla})q_k(t)E_k(\mathbf{X}),\end{aligned}\tag{2.1}$$

where the $E_k(\mathbf{X})$ are the normalized cavity eigenfunctions, and the $\boldsymbol{\varepsilon}_k$ are unit polarization vectors. We restrict our attention to a single cavity mode of circular frequency ω and describe the quantized field by means of the creation and annihilation operators $a^\dagger$, and a where

$$\begin{aligned}&[a,a^\dagger]=1,\\ &a=(2\hbar\omega)^{-1/2}(p-i\omega q),\\ &a^\dagger=(2\hbar\omega)^{-1/2}(p+i\omega q).\end{aligned}\tag{2.2}$$

The Hamiltonian for the uncoupled electromagnetic field is thus

$$H_0=\hbar\omega a^\dagger a,\tag{2.3}$$

if we leave off the zero-point field energy.

We consider next a system of N stationary two-level atoms with energy levels $E_a=-\hbar\omega_a$ and $E_b=-\hbar\omega_b$, where E_b is assumed to be the lower level. The atomic transition frequency is given by $\omega_0=\omega_b-\omega_a$. Negligible overlap of the atomic wave functions is assumed so that symmetry effects may be neglected. The atomic system may be represented in terms of products of Pauli spin functions for each individual atom. For a single atom these states may be represented in matrix form as

$$\psi_a=\begin{pmatrix}1\\0\end{pmatrix},\quad \psi_b=\begin{pmatrix}0\\1\end{pmatrix}.\tag{2.4}$$

A suitable representation for the Hamiltonian of the noninteracting system of atoms is

$$H_a=-\hbar\omega_a\textstyle\sum_j\sigma_j^+\sigma_j^- -\hbar\omega_b\sum_j\sigma_j^-\sigma_j^+,\tag{2.5}$$

where σ_j^+ and σ_j^- are, respectively, the raising and lowering operator for the jth atom, defined such that[23]

$$\begin{aligned}&\sigma^+\psi_b=\psi_a,\quad \sigma^+\psi_a=0;\\ &\sigma^-\psi_a=\psi_b,\quad \sigma^-\psi_b=0.\end{aligned}\tag{2.6}$$

A basis state vector for the N-atom system may be represented as

$$|\psi\rangle=|\{m\}\rangle,\tag{2.7}$$

where the curly bracket stands for a particular configuration of occupied states in which exactly m atoms occupy the lower state. The state vector (2.7) is constructed as a product of N vectors of the type (2.4).

[23] See Ref. 11, p. 89.

An electric dipole interaction is assumed between the electromagnetic field and the atoms,

$$H_{\rm int}=-\mathbf{M}\cdot\mathbf{E}, \tag{2.8}$$

where $\mathbf{M}$ is the electric dipole moment of the entire atomic system. In terms of the operators (2.2) and (2.6), Eq. (2.8) can be expressed as

$$H_{\rm int}=(a+a^\dagger)\textstyle\sum_j \hbar\alpha_j(\sigma_j^+ +\sigma_j^-), \tag{2.9}$$

where

$$\alpha_j=(2\pi\omega/\hbar)^{1/2}\mu E_l(\mathbf{X}_j). \tag{2.10}$$

In Eq. (2.10), μ is the atomic dipole moment matrix element projected along the direction of the electric field polarization, and $\mathbf{X}_j$ represents the position of the jth atom.

We may thus write the complete Hamiltonian of the system of interacting radiation and atoms as

$$H=\hbar\omega a^\dagger a-\hbar\omega_a\textstyle\sum_j\sigma_j^+\sigma_j^- -\hbar\omega_b\sum_j\sigma_j^-\sigma_j^+ +(a+a^\dagger)\sum_j\hbar\alpha_j(\sigma_j^+ +\sigma_j^-). \tag{2.11}$$

We shall not follow the frequent practice of including in the Hamiltonian the mechanisms of pumping and radiation loss. Instead we shall take account of them by making additions to the equations for the elements of the density matrix, which are consistent with both the required properties of the density matrix and the desired effects of the mechanism to be reproduced.

Finally, a set of basis state vectors appropriate for the description of the complete system may be written in the form

$$|\psi\rangle=|\{m\}n\rangle\equiv|\{m\}\rangle|n\rangle, \tag{2.12}$$

where n signifies the state of the radiation field in the n representation.

III. INTERACTION BETWEEN A SINGLE QUANTIZED RADIATION MODE AND A SINGLE ATOM

A nonperturbative treatment of the problem of a single atom interacting with a single quantized mode in a lossless cavity has been given by Jaynes and Cummings.[4] This case is briefly considered here in somewhat greater generality to illustrate in a simple way the manner in which the N-atom problem can be treated. For a single atom, the eigenkets may be designated as $|an\rangle$ and $|bn\rangle$ and the state vector as a linear combination of them,

$$|\psi(t)\rangle=\textstyle\sum_n a_n(t)|an\rangle+\sum_n b_n(t)|bn\rangle. \tag{3.1}$$

The Hamiltonian is

$$H=\hbar\omega a^\dagger a-\hbar\omega_a\sigma^+\sigma^- -\hbar\omega_b\sigma^-\sigma^+ +\hbar\alpha(a+a^\dagger)(\sigma^+ +\sigma^-). \tag{3.2}$$

Expression (3.1) is now substituted into the Schrödinger equation

$$i\hbar\frac{\partial}{\partial t}|\psi(t)\rangle=H|\psi(t)\rangle. \tag{3.3}$$

If the scalar product of both sides of the resulting equations is taken successively with $\langle an|$ and $\langle bn|$ and the orthogonality of the basis vectors made use of, the following differential equations for the amplitudes are obtained:

$$i\dot{b}_n=(n\omega-\omega_b)b_n+\alpha(n)^{1/2}a_{n-1}+\alpha(n+1)^{1/2}a_{n+1}, \tag{3.4a}$$

$$i\dot{a}_n=(n\omega-\omega_a)a_n+\alpha(n+1)^{1/2}b_{n+1}+\alpha(n)^{1/2}b_{n-1}. \tag{3.4b}$$

In deriving Eqs. (3.4), we have made use of the following properties of the operators a and $a^\dagger$:

$$a|n\rangle=(n)^{1/2}|n-1\rangle,\qquad a^\dagger|n\rangle=(n+1)^{1/2}|n+1\rangle. \tag{3.5}$$

The last terms on the right-hand sides of Eqs. (3.4) come from the nonresonant terms $a\sigma^-$ and $a^\dagger\sigma^+$ in the interaction part of the Hamiltonian (3.2) and can to an accurate approximation be neglected. When this rotating-wave approximation is made, the amplitude equations can be grouped into self-contained pairs

$$i\dot{a}_{n-1}=[(n-1)\omega-\omega_a]a_{n-1}+\alpha n^{1/2}b_n,\qquad i\dot{b}_n=(n\omega-\omega_a)b_n+\alpha n^{1/2}a_{n-1}, \tag{3.6}$$

wherein an amplitude for an upper state and a given photon number is coupled to the amplitude for a lower state with one additional photon. The equations (3.6) are easily solved for given initial conditions of the amplitudes, and the resulting solution is equivalent to that obtained by Jaynes and Cummings[4] in a somewhat different way. However, we shall be interested in expressing equations like (3.6) in bilinear form. After multiplying through Eqs. (3.6) successively by a_{n-1}^*, b_n^*, taking complex conjugates and combining equations, one obtains

$$\frac{d}{dt}|a_{n-1}|^2=i\alpha(n)^{1/2}(a_{n-1}b_n^*-a_{n-1}^*b_n),$$

$$\frac{d}{dt}|b_n|^2=-i\alpha(n)^{1/2}(a_{n-1}b_n^*-a_{n-1}b_n^*), \tag{3.7}$$

$$\frac{d}{dt}a_{n-1}b_n^*=i(\omega-\omega_0)a_{n-1}b_n^* +i\alpha(n)^{1/2}(|a_{n-1}|^2-|b_n|^2).$$

If, for example, one assumes that initially $|a_n|^2=1$ and $|b_{n+1}|^2=a_nb_{n+1}^*=0$, the solution of Eqs. (3.7) gives

$$|b_{n+1}|^2=\frac{4\alpha^2(n+1)}{(\omega-\omega_0)^2+4\alpha^2(n+1)}\times\sin^2\{\tfrac{1}{2}t[(\omega-\omega_0)^2+4\alpha^2(n+1)]^{1/2}\}. \tag{3.8}$$

The obvious interpretation of Eq. (3.8) is that the photon is periodically absorbed and re-emitted. But this does not correspond to a real physical situation because the radiation is not allowed to come into equilibrium

with the atom. The important ingredient which is missing and which was not included in the Hamiltonian (3.2) is a damping mechanism. Such damping could arise from spontaneous emission into modes other than the lossless mode considered here or from collisions, etc. However, we are not interested in details, so we assume that the statistical effect of the damping mechanism is to introduce a finite lifetime for the dephasing of the amplitudes a_{n-1} and b_n^*. The last of Eqs. (3.7) is thus altered to read

$$\frac{d}{dt}a_{n-1}b_n^* = -(\gamma/2)a_{n-1}b_n^* + i(\omega-\omega_0)a_{n-1}b_n^* + i\alpha(n)^{1/2}(|a_{n-1}|^2 - |b_n|^2). \quad (3.9)$$

This is quite analogous to the introduction of damping into the usual form of the SCFA. The equation (3.9) and the first two of Eqs. (3.7) also bear a close resemblance to the standard SCFA equations, the only innovation being the lower photon indices.[1,3,7]

If the damping is sufficiently strong, the solution of Eq. (3.9) may, to a good approximation, be written as

$$a_{n-1}b_n^* = \frac{i\alpha(n)^{1/2}}{-i(\omega-\omega_0)+\gamma/2}(|a_{n-1}|^2 - |b_n|^2). \quad (3.10)$$

When this expression is substituted into the first two of Eqs. (3.7), the following equations for $|a_{n-1}|^2$ and $|b_n|^2$ result:

$$\frac{d}{dt}|a_{n-1}|^2 = -\kappa n(|a_{n-1}|^2 - |b_n|^2), \quad (3.11a)$$

$$\frac{d}{dt}|b_n|^2 = \kappa n(|a_{n-1}|^2 - |b_n|^2), \quad (3.11b)$$

where

$$\kappa = \gamma\alpha^2/[(\omega-\omega_0)^2+\gamma^2/4]. \quad (3.12)$$

If we now multiply Eq. (3.11a) by $n-1$, Eq. (3.11b) by n, add the two and sum over n, the result is

$$\frac{d}{dt}\langle n\rangle = \kappa\sum_n n^2(|a_{n-1}|^2 - |b_n|^2) - \kappa\sum_n n(n-1)(|a_{n-1}|^2 - |b_n|^2), \quad (3.13)$$

where $\langle n\rangle$ is the expected value of the photon number. After combining terms and rearranging summations, we obtain

$$\frac{d}{dt}\langle n\rangle = \kappa\sum_n |a_n|^2 + \kappa\sum_n n(|a_n|^2 - |b_n|^2). \quad (3.14)$$

This equation has the form of an equation of transfer for photons in the mode.[24] The first term represents spontaneous emission and the second term induced emission and absorption. However, the correspondence is not exact because the n cannot in general be removed from the second summation and replaced by $\langle n\rangle$.

In the case of equilibrium, Eq. (3.9) is rigorously satisfied by expression (3.10), and the equilibrium solutions of Eqs. (3.11) are

$$|a_{n-1}|^2 = |b_n|^2. \quad (3.15)$$

Beyond satisfying conditions (3.15), the equilibrium values of $|a_n|^2$ and $|b_n|^2$ are arbitrary and depend on the starting condition of the system. If we choose conditions such that

$$|a_n|^2/|b_n|^2 = \exp[-\hbar\omega/kT], \quad (3.16)$$

independent of n, then condition (3.15) gives

$$|a_n|^2/|a_{n-1}|^2 = |b_n|^2/|b_{n-1}|^2 = \exp[-\hbar\omega/kT], \quad (3.17)$$

i.e., the photons satisfy a Bose-Einstein distribution. Normalizing we obtain

$$|a_n|^2 = \exp[-n\hbar\omega/kT]\frac{1-\exp[-\hbar\omega/kT]}{\exp[\hbar\omega/kT]+1}, \qquad |b_n|^2 = \exp[-n\hbar\omega/kT]\frac{\exp[\hbar\omega/kT]-1}{\exp[\hbar\omega/kT]+1}. \quad (3.18)$$

Equation (3.14) for equilibrium can be written

$$0 = \kappa\sum_n |a_n|^2 + \kappa\sum_n n(|a_n|^2 + |b_n|^2) \times \frac{(|a_n|^2 - |b_n|^2)}{(|a_n|^2 + |b_n|^2)}. \quad (3.19)$$

Because of (3.16), the fraction

$$(|a_n|^2 - |b_n|^2)/(|a_n|^2 + |b_n|^2)$$

is independent of n, and Eq. (3.19) can be written

$$\kappa\eta_b + \kappa\langle n\rangle(\eta_a - \eta_b) = 0, \quad (3.20)$$

where $\langle n\rangle$ is the usual Bose-Einstein average

$$\langle n\rangle = 1/\{\exp[\hbar\omega/kT]-1\} \quad (3.21)$$

and η_a and η_b are, respectively, the probabilities of the atom being in the upper and lower state. One other case in which the n could be removed from the summation in Eq. (3.14) and replaced by its average value is that in which both $|a_n|^2$ and $|b_n|^2$ are very sharply peaked about some particular value.

In view of the form of Eq. (3.14) we are led to relate the constant κ to the radiation absorption cross section σ through

$$\kappa = V^{-1}\sigma c, \quad (3.22)$$

where V is the volume of the cavity. The factor V^{-1} enters through α^2 which contains the volume normalization of the mode eigenfunctions.

[24] S. Chandrasekhar, *Introduction to the Study of Stellar Structure* (University of Chicago Press, Chicago, Illinois, 1939), p. 199.

IV. INTERACTION BETWEEN A SINGLE QUANTIZED MODE AND A SYSTEM OF N ATOMS

The method applied in the previous section to the case of a single atom can be extended without difficulty to the case involving N atoms. Here the Hamiltonian of Eq. (2.11) is the one to be used, and the state vector is expressed as a linear combination of the basis vectors $|\{m\}n\rangle$,

$$|\psi(t)\rangle=\sum_n\sum_{\{m\}} C_{\{m\}}{}^n(t)\,|\{m\}n\rangle, \tag{4.1}$$

where the summation extends over all configurations and values of m. When the state vector (4.1) is substituted into the Schrödinger equation (3.3), the result is

$$\begin{aligned} i\sum_n\sum_{\{m\}}\dot{C}_{\{m\}}{}^n|\{m\}n\rangle &=\sum_n\sum_{\{m\}} C_{\{m\}}{}^n[n\omega-m\omega_b-(N-m)\omega_a]|\{m\}n\rangle+\sum_n\sum_{\{m\}}\sum_{\{m-1\}}^{\{m\}} C_{\{m\}}{}^n\alpha(n)^{1/2}|\{m-1\}n-1\rangle \\ &+\sum_n\sum_{\{m\}}\sum_{\{m+1\}}^{\{m\}} C_{\{m\}}{}^n\alpha(n+1)^{1/2}|\{m+1\}n+1\rangle+\sum_n\sum_{\{m\}}\sum_{\{m-1\}}^{\{m\}} C_{\{m\}}{}^n\alpha(n+1)^{1/2}|\{m-1\}n+1\rangle \\ &+\sum_n\sum_{\{m\}}\sum_{\{m+1\}}^{\{m\}} C_{\{m\}}{}^n\alpha(n)^{1/2}|\{m+1\}n-1\rangle. \end{aligned} \tag{4.2}$$

The summations in Eq. (4.2) with both upper and lower configuration indices signify a summation over all configurations (represented by the lower index) which can be obtained from a given configuration (represented by the upper index) by means of a single spin flip. For example, the first such summation in Eq. (4.2) is over all configurations $\{m-1\}$ which can be obtained by flipping upward a single spin of the basic configuration $\{m\}$. This notation form will be used repeatedly in what follows. No attempt has been made to index α, but it should be regarded as having an index appropriate to the spin which has been flipped. The last two summations in Eq. (4.2) come from nonresonant operators of the form $a^\dagger\sigma_j{}^+$ and $a\sigma_j{}^-$, and as such they will usually be eliminated from consideration. However, for the sake of consistency they are retained at this stage.

A differential equation for a specific amplitude $C_{\{m\}}{}^n$ is obtained by taking the scalar product of Eq. (4.2) with the appropriate bra vector $\langle n\{m\}|$ and making use of the orthogonality of the basic states. The result is

$$\begin{aligned} i\dot{C}_{\{m\}}{}^n &=[n\omega-m\omega_b-(N-m)\omega_a]C_{\{m\}}{}^n+\sum_{\{m+1\}}^{\{m\}} C_{\{m+1\}}{}^{n+1}\alpha(n+1)^{1/2}+\sum_{\{m-1\}}^{\{m\}} C_{\{m-1\}}{}^{n-1}\alpha(n)^{1/2} \\ &+\sum_{\{m+1\}}^{\{m\}} C_{\{m+1\}}{}^{n-1}\alpha(n)^{1/2}+\sum_{\{m-1\}}^{\{m\}} C_{\{m-1\}}{}^{n+1}\alpha(n+1)^{1/2}. \end{aligned} \tag{4.3}$$

We shall for the moment restrict our attention to the simpler case where all of the α's are equal, and we shall neglect the last two summations. The equality of the α's implies that the atoms are clustered at positions having the periodicity of the mode eigenfunction. We shall, however, relax the condition of the constancy of the α's at a later point of the development. Equation (4.3) now becomes

$$i\dot{C}_{\{m\}}{}^n=[n\omega-m\omega_b-(N-m)\omega_a]C_{\{m\}}{}^n+\alpha\sum_{\{m+1\}}^{\{m\}} C_{\{m+1\}}{}^{n+1}(n+1)^{1/2}+\alpha\sum_{\{m-1\}}^{\{m\}} C_{\{m-1\}}{}^{n-1}(n)^{1/2}. \tag{4.3a}$$

This is the analog of Eqs. (3.4) and couples the amplitude for a given photon number and spin configuration with amplitudes for photon number one greater and for one additional spin down; and with amplitudes for photon number one less and for one additional spin up. If, for example, the radiation field is initially in the vacuum state, the amplitude equations for atoms fall, in general, into independent groups of equations containing the following sets of amplitudes:

$$\begin{aligned} &C_0{}^0,\ C_{\{1\}}{}^1,\ \cdots,\ C_{\{m\}}{}^m,\ \cdots,\ C_N{}^N;\ \cdots \\ &C_{\{m\}}{}^0,\ C_{\{m+1\}}{}^1,\ \cdots,\ C_N{}^m;\ \cdots \\ &C_N{}^0. \end{aligned} \tag{4.4}$$

If initially the field is in the vacuum state and all N atoms are excited, the system of equations greatly simplifies and we need consider only the first of the sets (4.4). A further simplification results because all of the amplitudes $C_{\{m\}}{}^m$ for different configurations and the same m satisfy the same initial conditions and differential equations, and hence are equal. For this case, Eqs. (4.3a) may be written

$$\begin{aligned} i\dot{C}_m &=[n\omega-m\omega_b-(N-m)\omega_a]C_m \\ &+\alpha(N-m)(n+1)^{1/2}C_{m+1} \\ &+\alpha m(n)^{1/2}C_{m-1},\quad m=0,1,\cdots,N, \\ &C_N(t=0)=1,\quad C_m(t=0)=0,\quad m\neq N, \end{aligned} \tag{4.5}$$

in which the photon index and the configuration sign have been dropped. The specific case of Eq. (4.5) has been derived and studied numerically by Abate and Haken.[25] Their results show that, as in the case of a single atom, the behavior of $\langle n\rangle$ is oscillatory and shows no tendency to come to equilibrium. Our interest will again be directed toward expressing the amplitude equations in bilinear form and adding damping so that the system can reach equilibrium.

Another case of interest is that in which all of the atoms are initially in the ground state and the radiation field is in a specific n state with $n<N$. Here the required set of amplitudes is

$$C_{\{N-n\}}{}^{0},\ \cdots,\ C_{\{m\}}{}^{n-m},\ \cdots,\ C_{N}{}^{n}, \tag{4.6}$$

and the governing equations are

$$\begin{aligned} i\dot{C}_{m}{}^{n-m}=&[(n-m)\omega-m\omega_b-(N-m)\omega_a]C_{m}{}^{n-m}\\ &+\alpha(N-m)(n-m+1)^{1/2}C_{m}{}^{n-m+1}\\ &+\alpha m(n-m)^{1/2}C_{m-1}{}^{n-m-1},\quad m=0,1,\cdots,n;\\ &C_{N}{}^{n}(t=0)=1,\ C_{m}{}^{n-m}(t=0)=0,\ m\neq N. \end{aligned} \tag{4.7}$$

Here again the configuration symbol has been dropped because the amplitudes for given n and m are all equal.

Finally, if the radiation field is in a general state, there will be a closed amplitude set for each value of n

$$C_{0}{}^{n},\ \cdots,\ C_{\{m\}}{}^{n+m},\ \cdots,\ C_{N}{}^{n+N}. \tag{4.8}$$

Here n takes on the values $-N$ to ∞, and an amplitude is considered to vanish if its upper index is negative. Equations similar to (4.5) and (4.7) would govern these amplitude sets.

V. BILINEAR EQUATIONS IN THE N-ATOM CASE

An equation for $|C_{\{m\}}{}^{n}|^2$ can be obtained from Eq. (4.3a) in a manner similar to the derivation of the bilinear Eqs. (3.7) for the one-atom case. The result is

$$\begin{aligned} \frac{d}{dt}|C_{\{m\}}{}^{n}|^2=i\alpha\{&\sum_{\{m+1\}}^{\{m\}}C_{\{m+1\}}{}^{n+1*}C_{\{m\}}{}^{n}(n+1)^{1/2}\\ &+\sum_{\{m-1\}}^{\{m\}}C_{\{m-1\}}{}^{n-1*}C_{\{m\}}{}^{n}(n)^{1/2}\}+\text{c.c.} \end{aligned} \tag{5.1}$$

The terms on the right-hand side of Eq. (5.1) are governed by the following equation which is obtained in a similar way,

$$\begin{aligned} \frac{d}{dt}C_{\{m+1\}}{}^{n+1*}C_{\{m\}}{}^{n}=&[i(\omega-\omega_0)-\gamma/2]C_{\{m+1\}}{}^{n+1*}C_{\{m\}}{}^{n}+i\alpha\sum_{\{m+2\}}^{\{m+1\}}C_{\{m+2\}}{}^{n+2*}C_{\{m\}}{}^{n}(n+2)^{1/2}\\ &+i\alpha\sum_{\{m\}'}^{\{m+1\}}C_{\{m\}'}{}^{n*}C_{\{m\}}{}^{n}(n+1)^{1/2}-i\alpha\sum_{\{m+1\}'}^{\{m\}}C_{\{m+1\}}{}^{n+1*}C_{\{m+1\}'}{}^{n+1}(n+1)^{1/2}\\ &-i\alpha\sum_{\{m-1\}}^{\{m\}}C_{\{m+1\}}{}^{n+1*}C_{\{m-1\}}{}^{n-1}(n)^{1/2}. \end{aligned} \tag{5.2}$$

A phenomenological damping term has been added to the right-hand side of Eq. (5.2) to account for the statistical effect of dephasing between $C_{\{m+1\}}{}^{n+1*}$ and $C_{\{m\}}{}^{n}$. In Eq. (5.2), primes are used to indicate that the summation configurations are not necessarily the same as the configurations $\{m\}$ and $\{m+1\}$. On the right-hand side of Eq. (5.2) there appear, in addition to terms with two like upper indices, terms wherein the upper indices differ by 2. These terms are governed by the equation

$$\begin{aligned} \frac{d}{dt}C_{\{m+2\}}{}^{n+2*}C_{\{m\}}{}^{n}=&2[i(\omega-\omega_0)-\gamma/2]C_{\{m+2\}}{}^{n+2*}C_{\{m\}}{}^{n}+i\alpha\sum_{\{m+3\}}^{\{m+2\}}C_{\{m+3\}}{}^{n+3*}C_{\{m\}}{}^{n}(n+3)^{1/2}\\ &+i\alpha\sum_{\{m+1\}}^{\{m+2\}}C_{\{m+1\}}{}^{n+1*}C_{\{m\}}{}^{n}(n+2)^{1/2}-i\alpha\sum_{\{m+1\}}^{\{m\}}C_{\{m+2\}}{}^{n+2*}C_{\{m+1\}}{}^{n+1}(n+1)^{1/2}\\ &-i\alpha\sum_{\{m-1\}}^{\{m\}}C_{\{m+2\}}{}^{n+2*}C_{\{m-1\}}{}^{n-1}n^{1/2}. \end{aligned} \tag{5.3}$$

The damping constant in Eq. (5.3) is taken as twice that in Eq. (5.2) because *two* independent dephasing processes are going on: between $C_{\{m\}}{}^{n}$ and $C_{\{m+1\}}{}^{n+1}$ and between $C_{\{m+1\}}{}^{n+1}$ and $C_{\{m+2\}}{}^{n+2}$. However, the matter need not be pursued here. It is seen from Eqs. (5.1) to (5.3) that an exact description of the system requires an infinite sequence of differential equations connecting the products of amplitudes differing in the n indices by successively larger integers. It is not difficult to show that if the variables on the left-hand side of Eqs. (5.2) and (5.3) are expanded in

[25] E. Abate and H. Haken, Z. Naturforsch. **19a**, 857 (1964).

a perturbation series in powers of the dimensionless parameter α/γ, then to first order in α/γ,

$$\frac{d}{dt}C_{\{m+1\}}{}^{n+1*}C_{\{m\}}{}^{n(1)}=[i(\omega-\omega_0)-\gamma/2]C_{\{m+1\}}{}^{n+1*}C_{\{m\}}{}^{n(1)}+i\alpha\sum_{\{m\}'}^{\{m+1\}}C_{\{m\}'}{}^{n*}C_{\{m\}}{}^{n}(n+1)^{1/2}$$
$$-i\alpha\sum_{\{m+1\}'}^{\{m\}}C_{\{m+1\}}{}^{n+1*}C_{\{m+1\}}{}^{n+1}(n+1)^{1/2}. \quad (5.4)$$

If this equation is solved and the result substituted into Eq. (5.1), then the rate of change of $|C_{\{m\}}{}^{n}|^2$ is proportional to $(\alpha/\gamma)^2\times$diagonal elements. Similarly, to second order in α/γ

$$\frac{d}{dt}C_{\{m+2\}}{}^{n+2*}C_{\{m\}}{}^{n(2)}=2[i(\omega-\omega_0)-\gamma/2]C_{\{m+2\}}{}^{n+2*}C_{\{m\}}{}^{n}+i\alpha\sum_{\{m+1\}}^{\{m+2\}}C_{\{m+1\}}{}^{n+1}C_{\{m\}}{}^{n(1)}(n+2)^{1/2}$$
$$-i\alpha\sum_{\{m+1\}}^{\{m\}}C_{\{m+2\}}{}^{n+2*}C_{\{m+1\}}{}^{n+1(1)}(n+1)^{1/2}. \quad (5.5)$$

If this equation is solved and the result substituted into Eq. (5.2), the next higher order correction to $C_{\{m+1\}}{}^{n+1*}C_{\{m\}}{}^{n}$ will be of order $(\alpha/\gamma)^3$ and this in turn will contribute to the right-hand side of Eq. (5.4) a term of order $(\alpha/\gamma)^4$.

If only the first-order approximation embodied in Eq. (5.4) is retained, a coupling between states differing in n by only 1 is implied. This is interpreted to mean that the various states are coupled through the emission or absorption of at most a single quantum by one atom. If the next higher order approximation is retained, as embodied in Eq. (5.5), a coupling between states differing in n by 2 is implied, and this is interpreted to mean that the various states are coupled as well by the simultaneous absorption or emission of 2 quanta by 2 atoms. The above interpretation is also borne out by the nature of the resonant terms in Eqs. (5.4) and (5.5), as well as by the powers of α/γ which each approximation introduces into Eq. (5.1). We conclude, finally, that the nth-order approximation would involve the simultaneous absorption or emission of n quanta by n atoms.

We shall now proceed with the development, assuming that the perturbation procedure just described converges, and retaining only the first approximation. Thus the basic equations on which our further discussion is based are Eqs. (5.1) and (5.4). We remark, however, that these equations still constitute a "nonperturbative" formulation of the N-atom problem in the sense that solutions to these equations describe the system for arbitrary time.

VI. DISTINCTION BETWEEN "SUPER-RADIANT" AND NORMAL EMISSION

Equations (5.1) and (5.4) still do not constitute a closed system because the latter contains terms which involve $C_{\{m\}'}{}^{n*}C_{\{m\}}{}^{n}$ with $\{m\}'\neq\{m\}$. These terms can be disposed of in two limiting cases. From the symmetry of Eq. (4.3a) one would expect that on a statistical basis the magnitude of the amplitudes for a given n and m would be independent of configuration, although the phase is another matter. One limiting case is that for which the amplitudes for different configurations but like values of m and n are all in perfect phase; the other is the case in which the phases of these amplitudes are completely random. The first case leads to radiation rates which are proportional to the square of the number of atoms present. This type of emission has been given the name "super-radiance" by Dicke.[13] The latter case leads to radiation rates which are proportional to the number of atoms present, and will be referred to as normal emission. The applicability of these two phasing conditions will be discussed in Sec. VII.

We consider the "super-radiant" case first. The quantities $C_{\{m\}'}{}^{n*}C_{\{m\}}{}^{n}$ are now all equal, and Eqs. (5.1) and (5.4) become

$$\frac{d}{dt}|C_{\{m\}}{}^{n}|^2=i\alpha\{(N-m)(n+1)^{1/2}C_{m+1}{}^{n+1*}C_{m}{}^{n}$$
$$+mn^{1/2}C_{m-1}{}^{n-1*}C_{m}{}^{n}\}+\text{c.c.}, \quad (6.1)$$

$$\frac{d}{dt}C_{m+1}{}^{n+1*}C_{m}{}^{n}=[i(\omega-\omega_0)-\gamma/2]C_{m+1}{}^{n+1*}C_{m}{}^{n}$$
$$+i\alpha\{(m+1)(n+1)^{1/2}|C_{m}{}^{n}|^2$$
$$-(N-m)(n+1)^{1/2}|C_{m+1}{}^{n+1}|^2\}, \quad (6.2)$$

where the configuration signs have been dropped. If the derivative in Eq. (6.2) is neglected in comparison with the resonant and damping terms,

$$C_{m+1}{}^{n+1*}C_{m}{}^{n}\cong\frac{i\alpha(n+1)^{1/2}}{-i(\omega-\omega_0)+\gamma/2}$$
$$\times\{(m+1)|C_{m}{}^{n}|^2-(N-m)|C_{m+1}{}^{n+1}|^2\}. \quad (6.3)$$

As the system approaches equilibrium, the approximation (6.3) becomes exact, of course. There are in all

$$N!/(N-m)!m! \quad (6.4)$$

different configurations for a given value of m; hence the

total probability for all states of given n and m is

$$P_m{}^n=\frac{N!}{m!(N-m)!}|C_m{}^n|^2. \tag{6.4a}$$

If the expression (6.3) is substituted into Eq. (6.1) and the resulting equation multiplied through by the factor (6.4), the result is

$$\frac{d}{dt}P_m{}^n=-\kappa(N-m)(m+1)(n+1)(P_m{}^n-P_{m+1}{}^{n+1}) -\kappa(N-m+1)mn(P_m{}^n-P_{m-1}{}^{n-1}), \tag{6.5}$$

where κ is defined as in the single-atom case by

$$\kappa=\frac{\gamma\alpha^2}{(\omega-\omega_0)^2+\gamma^2/4}. \tag{6.6}$$

The γ which appears in Eq. (6.6) has, however, a somewhat more specialized meaning than the one introduced in the single-atom case.

Equation (6.5) has the form of a master equation.[21,22,24,26] The rate of change of the probability in the state (m,n) is equated to the rates of gain by emission from the state $(m+1, n+1)$ and absorption from the state $(m-1, n-1)$, minus the rates of loss by emission to state $(m+1, n+1)$ and absorption to the state $(m-1, n-1)$. In Eq. (6.5) the negative terms proportioned to $(n+1)$ and n represent, respectively, the emission and absorption loss rates.

In the case of normal emission the only terms which appear in the summations in Eq. (5.4) are $|C_m{}^n|^2$ and $|C_{m+1}{}^{n+1}|^2$, the other terms giving no contribution because of their random phasing. The counterpart to Eq. (6.2) becomes

$$\frac{d}{dt}C_{m+1}{}^{n+1*}C_m{}^n=[i(\omega-\omega_0)-\gamma/2]C_{m+1}{}^{n+1*}C_m{}^n +i\alpha(n+1)^{1/2}(|C_m{}^n|^2-|C_{m+1}{}^{n+1}|^2), \tag{6.2a}$$

and the relation corresponding to Eq. (6.3) is

$$C_{m+1}{}^{n+1*}C_m{}^n=\frac{i\alpha(n+1)^{1/2}}{-i(\omega-\omega_0)+\gamma/2} \times(|C_m{}^n|^2-|C_{m+1}{}^{n+1}|^2). \tag{6.3a}$$

If (6.3a) is substituted into (6.1) and the resulting equation is multiplied by (6.4a), the result is

$$\frac{d}{dt}P_m{}^n=-\kappa(n+1)[(N-m)P_m{}^n-(m+1)P_{m+1}{}^{n+1}] -\kappa n[mP_m{}^n-(N-m+1)P_{m-1}{}^{n-1}]. \tag{6.5a}$$

The interpretation of Eq. (6.5a) is the same as that of Eq. (6.5).

It will be noted in Eq. (6.5) that the maximum coefficient corresponding to emission occurs for $m=N/2$ and implies a transition probability $(N/2)(N/2+1)$ times that which would be expected for a single atom. This is in complete agreement with the result of Dicke,[13] which is based on perturbation theory. Equation (6.5), however, contains the effects of field quantization as well as multiple absorption and emission. The maximum-emission coefficient in Eq. (6.5a), on the other hand, occurs for $m=0$. The corresponding transition probability is N times that for a single atom, hence the appellation normal emission.

It is possible to obtain a better understanding of Eqs. (6.5) and (6.5a) by calculating the expected emission rates implied by these equations. If Eq. (6.5) is multiplied through by n and the resulting equation is then summed over m and n, the result can be written

$$\frac{d}{dt}\langle n\rangle=-\kappa\sum_{m,n}(N-m)(m+1)(n+1)nP_m{}^n+\kappa\sum_{m,n}(N-m+1)mn(n-1)P_m{}^n -\kappa\sum_{m,n}(N-m+1)mn^2P_m{}^n+\kappa\sum_{m,n}(N-m)(m+1)(n+1)^2P_m{}^n, \tag{6.7}$$

where the summations in (6.7) taken in order represent summations over the four right-hand members of Eq. (6.5). When the coefficients of $P_m{}^n$ are combined, the result is

$$\frac{d}{dt}\langle n\rangle=\kappa\sum_{m,n}(N-2m)nP_m{}^n+\kappa\sum_{m,n}(N-m)(m+1)P_m{}^n =\kappa\langle(N-2m)n\rangle+\kappa\langle(N-m)(m+1)\rangle. \tag{6.7a}$$

The first right-hand member of Eq. (6.7a) represents the combined contributions of absorption and stimulated emission. The second term represents the contribution of spontaneous emission. The presence of m in the latter is due to the phasing of the different configuration amplitudes.

[26] The master or Pauli equation has been derived by a number of authors under varying assumptions. For further discussion the reader is referred to *Fundamental Problems in Statistical Mechanics*, edited by E. G. D. Cohen (North-Holland Publishing Company, Amsterdam, 1962).

In a similar manner, one obtains for Eq. (6.5a)

$$\frac{d}{dt}\langle n\rangle=-\kappa\sum_{m,n}(N-m)(n+1)nP_m{}^n+\kappa\sum_{m,n}mn(n-1)P_m{}^n-\kappa\sum_{m,n}mn^2P_m{}^n+\kappa\sum_{m,n}(N-m)(n+1)^2P_m{}^n. \quad (6.8)$$

After combining the coefficients of $P_m{}^n$, the result is

$$\frac{d}{dt}\langle n\rangle=\kappa\sum_{m,n}(N-2m)nP_m{}^n+\kappa\sum_{m,n}(N-m)P_m{}^n$$
$$=\kappa\langle(N-2m)n\rangle+\kappa\langle(N-m)\rangle. \quad (6.9)$$

The interpretation of the right-hand members of Eq. (6.9) is the same as for Eq. (6.7); the only difference is that the spontaneous emission term proportional to m, as expected, does not occur.

It is a simple matter to show in the case of both Eqs. (6.5) and (6.5a) that

$$-\frac{d}{dt}\langle(N-m)\rangle=\frac{d}{dt}\langle n\rangle. \quad (6.10)$$

If it is assumed that the first right-hand member of Eqs. (6.7a) and (6.9) can be written as the product $\langle N-2m\rangle\langle n\rangle$ and that the second right-hand member of (6.7a) can be written as the product $\langle N-m\rangle\langle m+1\rangle$, then Eqs. (6.7a), (6.9) and (6.10) assume the form of the simple rate equations,[27] which are often used to describe the intensity of laser emission, but with the inclusion of the appropriate spontaneous emission rates. In the case of Eq. (6.9) the rate equation is

$$\frac{d}{dt}\langle n\rangle=\kappa\langle N-2m\rangle\langle n\rangle+\kappa\langle N-m\rangle. \quad (6.11)$$

This close correspondence with the rate equation approach is in part due to the approximations (6.3) and (6.3a).[28]

VII. DISCUSSION OF PHASING CONDITIONS

The condition of perfect phasing between the amplitudes of different configurations would be impossible to achieve in practice, although Dicke has suggested two ways in which it can be achieved in principle.[13] In the first way, all the atoms are initially placed in the excited state and the radiation field is left in the vacuum state. In the second way, all the atoms are placed in the ground state and the radiation field is initially in a specific n state. These are precisely the cases considered at the end of Sec. IV and are governed by Eqs. (4.5) and (4.7). A mixed radiation state would serve equally well in the second method, although the description of this case in terms of the amplitudes (4.8) would be considerably more involved. The important thing is that when all of the atoms start in either of the configurations $\{N\}$ or $\{0\}$ the dynamics determines that all amplitudes with like values of n and m will be in phase. Under usual circumstances, however, the starting state of the system can be expected to be much more complicated and chaotic than either of the ones just mentioned.

In general the initial wave function of the atoms can be represented as

$$\psi=\prod_{j=1}^{N}[A_j\psi_a(j)+B_j\psi_b(j)]. \quad (7.1)$$

For an equilibrium system or for a system pumped by any conceivable pump source, one would expect an initial configuration amplitude to be given by

$$C_{\{m\}}=|A|^{(N-m)}|B|^m\exp\{-i\phi_{\{m\}}\}, \quad (7.2)$$

where $|A|^2$ and $|B|^2$ are the proportions of atoms in the upper and lower states and where $\phi_{\{m\}}$ is a randomly distributed amplitude. The description of the subsequent behavior of the system will generally involve all of the closed sets (4.8). If the radiation field is initially in a mixed state, all the amplitudes have initial values. In any case, because of Eq. (7.2), there will be a high degree of phase randomness present initially, and it is shown in the Appendix that this randomness will tend to be preserved. The initial values of $C_0{}^n$ and $C_N{}^n$ will tend to initiate phase correlations. This could be important at low temperatures. But it must be remembered that for all amplitudes of given n and m to be strongly correlated the coupling constants α must be the same, and this would require that all of the atoms be located in a volume of dimension small compared with a radiation wavelength. We conclude that under conceivable practical circumstances the assumption of dephasing among amplitudes for configurations with like m and n is the reasonable one to make for an operating laser.

A natural question to raise at this point is whether this lack of phasing is in any way inconsistent with the existence of coherent radiation and of macroscopic dipole moments. The answer is, of course, no. The connection between macroscopic dipole moments and coherent radiation and related questions will be ex-

[27] See, for example, R. Dunsmuir, J. Electron. Control **10**, 453 (1961). Also W. G. Wagner and O. A. Lengyel, J. Appl. Phys. **34**, 2040 (1963).

[28] For a derivation of the rate equations from the SCFA model, the reader is referred to Refs. 3 and 5.

amined in detail in the third paper in this series (to be published). However, we anticipate later results and justify our negative answer as follows: The expected value for the electric field operator

$$\langle E\rangle = \text{const}\times \mathrm{Tr}\rho(a+a^\dagger) \tag{7.3}$$

can be expressed using the notation of the present article as

$$\langle E\rangle = \text{const}\times \sum_n \sum_{\{m\}} C_{\{m\}}{}^{n+1*} C_{\{m\}}{}^n + \text{c.c.}, \tag{7.4}$$

while the expected value of the component of the electric dipole moment along the direction of the field polarization

$$\langle M_z\rangle = \text{const}\times \mathrm{Tr}\rho \sum_j (\sigma_j{}^+ + \sigma_j{}^-) \tag{7.5}$$

can be expressed as

$$\langle M_z\rangle = \text{const}\times \sum_n \sum_{\{m\}} C_{\{m+1\}}{}^{n*} C_{\{m\}}{}^n + \text{c.c.} \tag{7.6}$$

The summation in Eq. (7.6) is over configuration pairs $\{m\}$, $\{m+1\}$, which are related by a spin flip. It will be noted first of all that the bilinear quantities which appear in Eqs. (7.4) and (7.6) have not appeared in the discussion so far. This is because our analysis thus far has dealt with only those elements of the density matrix which deal with energy and not phase. Secondly, the existence of nonvanishing $\langle E\rangle$ and $\langle M_z\rangle$ depends only on the phase differences between $C_{\{m\}}{}^{n+1}$, $C_{\{m\}}{}^n$ and $C_{\{m+1\}}{}^n$, $C_{\{m\}}{}^n$, respectively; otherwise the relative phases of the $C_{\{m\}}{}^n$ for constant n and m may be arbitrary. It can be shown that such expectation values will be multiplied by slow time-decaying exponentials.

VIII. EFFECT OF VARIABLE COUPLING BETWEEN ATOMS AND THE RADIATION FIELD

If the atoms are distributed uniformly over the standing wave pattern of the radiation mode, the coupling constants α_j vary from 0 at the nodes to a maximum value at the crests. In this case it is not difficult to show that the counterpart of Eq. (6.5a) is

$$\begin{aligned}\frac{d}{dt}|C_{\{m\}}{}^n|^2 &= -\frac{\gamma}{(\omega-\omega_0)^2+\gamma^2/4}\{(n+1)\sum_{\{m+1\}}^{\{m\}}(\alpha_{\{m+1\}\{m\}})^2 \\ &\times(|C_{\{m\}}{}^n|^2-|C_{\{m+1\}}{}^{n+1}|^2)+n\sum_{\{m-1\}}^{\{m\}}(\alpha_{\{m-1\}\{m\}})^2 \\ &\times(|C_{\{m\}}{}^n|^2-|C_{\{m-1\}}{}^{n-1}|^2)\}. \end{aligned}\tag{8.1}$$

In Eq. (8.1) $\alpha_{\{m+1\}\{m\}}$ stands for the α_j associated with the atom whose spin must be flipped in order to get from $\{m\}$ to $\{m+1\}$. If initially $|C_{\{m\}}{}^n|^2$ is independent of $\{m\}$ for given m and n, it will remain so. Since in the summations in Eq. (8.1) a uniform sampling of the α_j is represented, we may in our single-mode case replace the $\alpha_j{}^2$ by

$$\begin{aligned}\langle\alpha^2\rangle_{\text{av}} &= (N-m)^{-1}\sum_{\{m+1\}}(\alpha_{\{m+1\}\{m\}})^2 \\ &= m^{-1}\sum_{\{m\}}(\alpha_{\{m-1\}\{m\}})^2 = \frac{1}{N}\sum_{j=1}^N \alpha_j{}^2. \end{aligned}\tag{8.2}$$

Consequently, the right-hand side of Eq. (8.1) is independent of configuration, and all $|C_m{}^n|^2$ for given m and n satisfy the same differential equation and initial condition. Furthermore, because of the randomness of the couplings represented in Eq. (8.1) there is no reason to favor any configuration over any other, regardless of initial conditions. Exceptions may occur for $m=0$, N, but for large numbers of atoms these cases are unimportant. We conclude, therefore, that the effect of distributing the atoms uniformly over the laser cavity can be accounted for by replacing the α^2 in the formulas already developed by

$$\begin{aligned}\alpha^2 &= \langle\alpha^2\rangle_{\text{av}} \\ &= \frac{2\pi\omega}{\hbar V}\mu^2\int E_l{}^2(\mathbf{X})d^3X = \frac{2\pi\omega}{\hbar V}\mu^2. \end{aligned}\tag{8.3}$$

IX. RADIATION LOSS FROM THE LASER CAVITY

Thus far we have regarded the radiating system as though it were contained in a hermetically sealed cavity. We now incorporate in our treatment the effect of radiation losses from this cavity. The usual method for accomplishing this is to include in the quantum-mechanical system from the start a set of loss oscillators which are coupled to the radiation field.[29] As explained earlier, we shall employ a somewhat different phenomenological method which leads to results that are consistent with the correct classical behavior of a loss mechanism while retaining the required properties of the density matrix. In any case, the results of this method can be shown to be equivalent to those of the more usual method.

Let us for a moment consider the radiation system alone. We then assert that the following equations involving diagonal elements of the radiation density matrix afford a suitable description of the loss mechanism in the sense just mentioned:

$$\frac{d}{dt}P^n = \gamma_c(n+1)P^{n+1} - \gamma_c n P^n, \tag{9.1}$$

where γ_c is the cavity decay constant and $P^n = \sum_m P_m{}^n$. First of all, Eq. (9.1) conserves probability, or $\mathrm{Tr}\rho = 1$. This is easily seen by summing Eq. (9.1) over n. Secondly, probability lost by a given n state appears in the next lower n state. This leads to an accumulation of probability in the vacuum state, as it should when no radiation sources are present. Finally, we consider the

[29] Reference 11, p. 255.

equation satisfied by $\langle n\rangle$. Multiplying Eq. (9.1) through by n and summing over n, we have

$$\begin{aligned}\langle n\rangle &= \sum_n nP^n,\\ \frac{d}{dt}\langle n\rangle &= \gamma_c\{\sum_n n(n+1)P^{n+1} - \sum_n n^2P^n\}\\ &= -\gamma_c\langle n\rangle.\end{aligned} \tag{9.2}$$

In particular, if the radiation system is in a coherent state,[19,30] described by

$$P^n = e^{-x^2}x^{2n}/n!, \tag{9.3}$$

the corresponding classical electric field, i.e. the expectation of the electric field operator, executes a pure harmonic oscillation with amplitude and intensity proportional to x and x^2. Substituting from Eq. (9.3) into Eq. (9.1), we obtain

$$\begin{aligned}\frac{d}{dt}\frac{e^{-x^2}x^{2n}}{n!} &= \left[-\frac{e^{-x^2}x^{2n}}{n!} + \frac{e^{-x^2}x^{2(n-1)}}{(n-1)!}\right]\frac{dx^2}{dt}\\ &= \gamma_c\left[\frac{e^{-x^2}x^{2(n+1)}}{n!} - \frac{e^{-x^2}x^{2n}}{(n-1)!}\right],\end{aligned} \tag{9.4}$$

or

$$dx^2/dt = -\gamma_c x^2, \tag{9.5}$$

independent of n. In view of Eqs. (9.2) and (9.5) we conclude that the expectation values determined by (9.1) have the correct classical behavior.

If we now couple the formulation of the loss mechanism embodied in Eq. (9.1) with the formulation of the N-atom radiation problem embodied in Eq. (6.5a), the result is

$$\begin{aligned}\frac{d}{dt}P_m{}^n &= -\kappa(n+1)[(N-m)P_m{}^n - (m+1)P_{m+1}{}^{n+1}]\\ &\quad -\kappa n[mP_m{}^n - (n-m+1)P_{m-1}{}^{n-1}]\\ &\quad +\gamma_c(n+1)P_m{}^{n+1} - \gamma_c nP_m{}^n.\end{aligned} \tag{9.6}$$

Equation (9.6) constitutes the basic result of the entire article. It is still lacking in terms which describe the effect of a pumping mechanism. These, however, will be supplied in the following article in this series. It is adequate, in any case, for describing the details of photon emission, including photon statistics, from a Q-spoiled laser, wherein pumping details are of no importance during the period of emission.

X. NUMERICAL EXAMPLES

In this section we discuss two examples for which Eqs. (9.6) have been integrated numerically. We consider a system of 30 atoms, initially in a specific state of inversion, and we assume the electromagnetic field to be initially in the vacuum state. This situation describes qualitatively the emission by a Q-spoiled laser. The initial wave function for the atoms is assumed to be of the form (7.1) with

$$|A_j| = |A|, \quad |B_j| = |B|; \quad j = 1, 2, \cdots, N. \tag{10.1}$$

Thus

$$\begin{aligned}P_m{}^0(t=0) &= \frac{N!}{(N-m)!m!}|A|^{2(N-m)}|B|^{2m};\\ P_m{}^n(t=0) &= 0, \quad n\neq 0.\end{aligned} \tag{10.2}$$

The coupling between the $P_m{}^n$ in Eq. (9.7) is such that both m and n run from 0 to a maximum value of N. But only those $P_m{}^n$ differ from 0 for which $n\leq m$. Thus the required number of equations is $(N+1)(N+2)/2$. In the case of 30 atoms this amounts to 496. We assume that 75% of the atoms are initially in the upper state or that $|A|^2=0.75$ and $|B|^2=0.25$. We take $\kappa=1$; in case (a) we take $\gamma_c=0$ and in case (b) $\gamma_c=1$. Comparison is made between $\langle n\rangle$ and $[N-2\langle m\rangle]/N$, i.e., the inversion, as calculated using Eqs. (9.6) and using the rate equations (6.10) and (6.11).

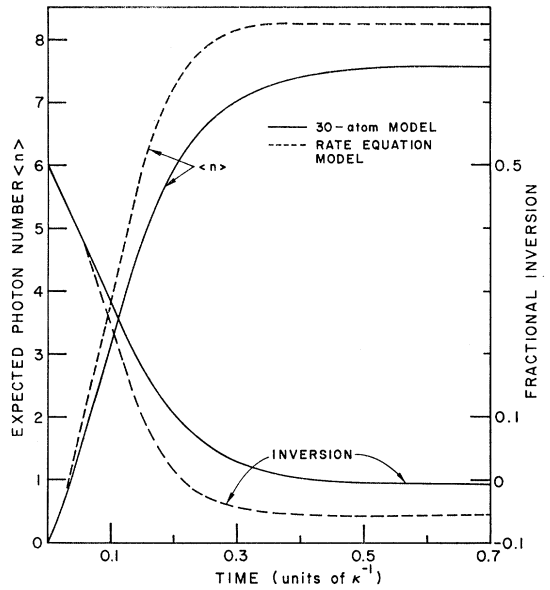

FIG. 1. Comparison of master-equation and rate-equation calculation for emission by 30 atoms in a lossless cavity. It is assumed that initially 75% of the atoms are in the upper state. The number of simultaneous equations required in the master-equation calculation is 463. Time is measured in units of the absorption lifetime for a single atom.

Exhibited in Fig. 1 are the results of case (a) corresponding to a hermetically sealed cavity. The interesting thing here is that the final equilibrium values of $\langle n\rangle$ are different for the two methods of calculation. In Fig. 2 are displayed the results for case (b) in which the radiation is allowed to escape from the laser cavity. In view of the results of case (a) it is not surprising that the mean photon number peaks at a higher value and the inversion drops lower in the rate equation calculation than in the calculation using Eq. (9.6). However, it is

[30] Reference 11, p. 126.

interesting to note the close correspondence between the two methods of calculation for the exponential dropoff in $\langle n \rangle$ following the peaks. The discrepancy between the rate equation and the master equation calculations is to be attributed to the importance of n-m correlations. The importance of such correlations in cases involving a more realistic number of atoms will be the subject of a later publication.

XI. PHOTON STATISTICS FOR A Q-SPOILED LASER

There has been considerable discussion as to which photon distribution function should best apply to the light from a laser. While for light from thermal sources, the number of quanta in the mode is distributed according to the well-known exponential distribution

$$P^n = [1 - e^{-\hbar\omega/kT}]e^{-n\hbar\omega/kT}, \qquad (11.1)$$

it has been proposed that for laser light the distribution function most applicable is the Poisson distribution (9.3).[18,19] A detailed discussion of this question with reference to cw lasers will be given in the second paper in this series. However, the results of problem (b) described in the previous section give us some qualitative insight into the photon statistics to be expected in the case of a Q-spoiled laser. The photon distribution P^n at the time of the peak value of $\langle n \rangle$ is exhibited in Fig. 3 where it is compared with a Poisson distribution with the same mean. The two distributions are qualitatively similar in that they are both peaked about their mean values, unlike the distribution (11.1). In fact, this characteristic of the laser statistics is exhibited at all times, displayed in Fig. 2. The laser distribution has the wider dispersion. One would expect the laser distribution to be less optimal because of the influence of spontaneous emission.

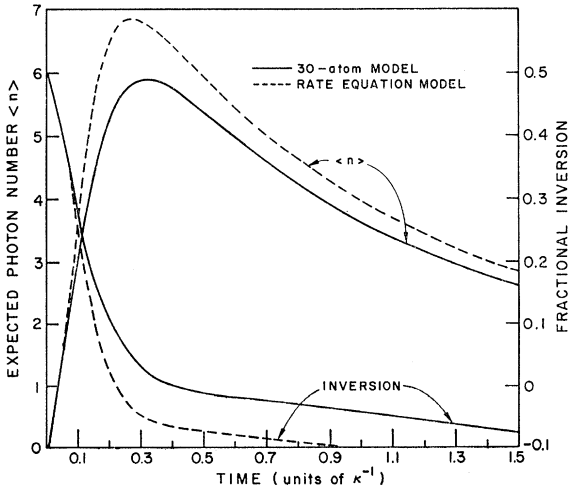

FIG. 2. Comparison of master-equation and rate-equation calculation for emission by 30 atoms in a cavity with loss. The loss-rate constant $\gamma_c = 1$, in units of the absorption lifetime for a single atom. Initially 75% of the 30 atoms present have been excited to the upper state.

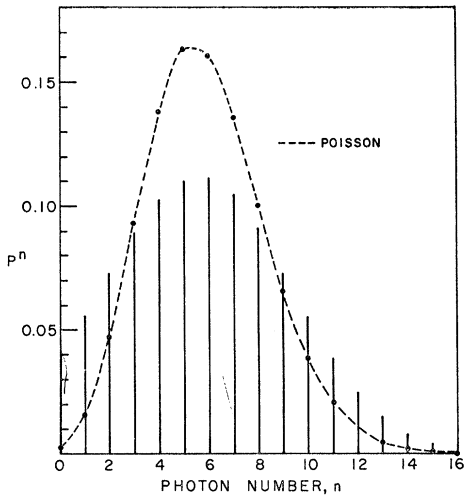

FIG. 3. Statistical distribution for the number of photons in the radiation mode of a Q-spoiled laser. The conditions are the same as in Fig. 2. The photon distribution is calculated at the time of maximum radiation intensity in the cavity and compared with a Poisson distribution with the same mean.

XII. SUMMARY

We have formulated the problem of N atoms radiating into a single cavity radiation mode by deriving a master equation satisfied by the probabilities $P_m{}^n$ for m out of N two-level atoms in the lower state and n photons present in the mode. In deriving the master equation, we have taken into account only multiple single-photon absorption and emission processes. In addition we have assumed that probabilities do not change much in a "collision time," which in turn is introduced as a characteristic time for the dephasing of certain probability amplitudes associated with emission and absorption. When the master equation is used to derive an equation governing the expected photon number $\langle n \rangle$, the resulting equation bears a close resemblance to the familiar rate equation. The master equation has been modified to account for radiation losses from the cavity through the heuristic addition of terms which conserve probability and which lead to expectation values of the proton number that fall off exponentially when no radiation sources are present. The numerical integration of the master equation for the situation of a Q-spoiled laser also indicates a behavior which is qualitatively similar to the rate-equation approach. The photon statistics calculated from the master equation are qualitatively similar to Poisson statistics in that the distribution of the number of photons in the mode is clustered about the mean value.

ACKNOWLEDGMENT

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APPENDIX: FURTHER DISCUSSION OF DEPHASING CONDITIONS

The dephasing of the bilinear quantities $C_{\{m\}'}{}^{n*}C_{\{m\}}{}^{n}$ where $\{m\}'\neq\{m\}$ can be discussed with reference to equations of the form of (5.1) and (5.4). In the present case the appropriate equations are

$$\frac{d}{dt}C_{\{m\}'}{}^{n*}C_{\{m\}}{}^{n}=-\frac{\gamma'}{2}C_{\{m\}'}{}^{n*}C_{\{m\}}{}^{n}+i\alpha\{\sum_{\{m+1\}'}^{\{m\}'}C_{\{m-1\}'}{}^{n+1*}C_{\{m\}}{}^{n}(n+1)^{1/2}+\sum_{\{m-1\}'}^{\{m\}'}C_{\{m-1\}'}{}^{n-1*}C_{\{m\}}{}^{n}(n)^{1/2}$$
$$-\sum_{\{m+1\}}^{\{m\}}C_{\{m\}'}{}^{n*}C_{\{m+1\}}{}^{n+1}(n+1)^{1/2}-\sum_{\{m-1\}}^{\{m\}}C_{\{m\}'}{}^{n*}C_{\{m-1\}}{}^{n-1}(n)^{1/2}\}, \quad \text{(A1)}$$

and

$$\frac{d}{dt}C_{\{m+1\}'}{}^{n+1*}C_{\{m\}}{}^{n}=[i(\omega-\omega_0)-\gamma/2]C_{\{m+1\}'}{}^{n+1*}C_{\{m\}}{}^{n}+i\alpha\{\sum_{\{m\}''}^{\{m+1\}'}C_{\{m\}''}{}^{n*}C_{\{m\}}{}^{n}(n+1)^{1/2}$$
$$-\sum_{\{m+1\}''}^{\{m\}}C_{\{m+1\}'}{}^{n+1}C_{\{m+1\}''}{}^{n+1}(n+1)^{1/2}\}. \quad \text{(A2)}$$

Equations for the other terms on the right-hand side of Eq. (A1) may be obtained by an appropriate shift of indices. We assume the constant γ' in Eq. (A1) to be of the order of magnitude of γ. We examine now those terms on the right-hand side of Eq. (A2) which are independent of phase. If $\{m\}\equiv\{N-1\}$, the second summation reduces to a term proportional to $|C_N{}^n|^2$. Similarly if $\{m\}\equiv\{0\}$, the first summation reduces to a term proportional to $|C_0{}^n|^2$. For $1<m, m'<N-1$, however, the right-hand side does not necessarily contain phase-independent terms. In fact, it is not difficult to see that for given $\{m\}$ and $\{m\}'$ there is at most only one $\{m+1\}'$ which will give a term proportional to $|C_{\{m\}}{}^n|^2$ in the first summation and similarly only one $\{m+1\}'$ which will give a term proportional to $|C_{\{m\}}{}^n|^2$ in the second summation. Put another way, for given $\{m\}$ and $\{m\}'$ there is at most only one way to go via a downward spin flip from $\{m\}'$ to $\{m+1\}'$ and from $\{m+1\}'$ to $\{m\}$ via an upward spin flip. This one way is possible only if the configuration $\{m\}$ differs from $\{m\}'$ by the orientations of two atoms.

We now ignore all terms on the right-hand side of Eq. (A2) which can depend on phase. If we also ignore the derivative on the left-hand side, then in the favorable cases mentioned above,

$$C_{\{m+1\}'}{}^{n+1*}C_{\{m\}}{}^{n}\cong\frac{i\alpha}{-i(\omega-\omega_0)+\gamma/2}|C_{\{m\}}{}^{n}|^2. \quad \text{(A3)}$$

If this result is substituted into Eq. (A1) and the derivative on the left-hand side of the equation is neglected, one obtains

$$C_{\{m\}'}{}^{*}C_{\{m\}}\approx(\kappa/\gamma')|C_{\{m\}}{}^{n}|^2. \quad \text{(A4)}$$

The relation (A4) permits us to make an estimate of the error involved in neglecting the $C_{\{m\}'}{}^{*}C_{\{m\}}$ in the summations in Eq. (5.4), if the system starts out in a highly dephased condition. The constant κ/γ' is the ratio of the inverse absorption lifetime for a single atom to the natural linewidth, and is very small compared to 1. Thus in the summations in Eq. (5.4) mentioned above, there will occur at most two terms of the order of (A4), provided $\{m\}\neq\{N-1\}$ or $\{1\}$. These contributions will be completely negligible. If $\{m\}\equiv\{N-1\}$ or if $\{m\}=\{1\}$ there will be N terms of the form (A4). These contributions may not be negligible. However, as already mentioned, the initial values of $|C_0{}^n|^2$ and $|C_N{}^n|^2$ should be exceedingly small and therefore contribute negligibly to the overall phasing of the configurational amplitudes.