Thick-Target Bremsstrahlung and Target Considerations for Secondary-Particle Production by Electrons*

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Bremsstrahlung spectra as a function of target thickness up to several radiation lengths, to be used for particle production in high-energy electron machines, are investigated. The shower equations are cast in integral forms which are then solved by iteration. The iterations are performed up to the second-generation photons; and the numerical results show that for most experiments the first-generation photons alone will give sufficiently accurate results. For example, for a target thickness of two radiation lengths and for $k/E_0=0.5$, where k is the photon energy and E_0 is the incident electron energy, the ratio of the second- to the first-generation photon intensity is 8%. This ratio decreases rapidly as one increases k/E_0 and decreases the target thickness. A very simple formula which approximates the first-generation photon spectra as a function of target thickness is derived. This approximate formula is shown to be accurate enough for estimating the secondary-beam production by electrons. As by-products of our investigation the first- and the second-generation electron and positron spectra were obtained as functions of target thickness. These spectra are useful in estimating the electron and positron background. Some aspects of target considerations for the secondary-beam production are given as an illustration of the use of our formulas.

I. INTRODUCTION

It is expected that muon, pion, kaon, and antibaryon beams of usable intensity can be photoproduced by an electron machine such as the Stanford Linear Accelerator. In order to estimate the flux of these secondary beams, it is necessary to have a simple and reliable formula for the bremsstrahlung spectra covering target thicknesses up to several radiation lengths and the photon energy k, in the range $\frac{1}{2}E_0<k<E_0$, where E_0 is the incident-electron energy.

Most of the articles in the literature[1] are mainly concerned with photon energy much smaller than an incident-electron energy E_0, and hence are inapplicable to our problem. Among experimentalists, a computer program by Alvarez[2] seems to be widely used. Alvarez's program essentially treats the emission of the bremstrahlung by the first-generation electrons (degraded in energy using Heitler's straggling formula) and the absorption of the bremsstrahlung by a factor $1-(7/9)\times(t-t')$ due to pair production, where $t-t'$ is the target thickness (in radiation lengths, r.l.) from the point of gamma production t' to the point t where the gammas are to be used. Actually the factor $1-(7/9)\times(t-t')$ in Alvarez's program is a series expansion of $\exp[-(7/9)(t-t')]$ and hence is applicable only for target thicknesses very small compared with a radiation length. There are many other technical notes about this subject by various people. Every experimentalist seems to have his own version of a thick-target bremsstrahlung formula, and each one of them seems to be widely different from the others, none of them being very convincing. Apparently what is needed is a detailed derivation of a formula whose error can be evaluated and range of applicability stated. Also, in case a refinement is required, the treatment given can be a useful reference.

The method we used is similar to that of Bhabha and Heitler.[3] This method is particularly suited to calculating the shower components whose energies are not small compared with the incident-electron energy. The method consists of successive approximations for solving the shower equations: First calculate the energy distribution of the first-generation electrons, $I_e^{(1)}(t,E)$, as a function of target thickness. $I_e^{(1)}(t,E)$ represents the straggling of the incident electrons owing to the emission of bremsstrahlung. Then $I_e^{(1)}(t,E)$ is used to calculate the first-generation photon spectra $I_\gamma^{(1)}(t,k)$ with absorption of the resultant photons due to pair production taken into consideration. $I_\gamma^{(1)}(t,k)$ is then used to calculate the second-generation electron spectra, $I_e^{(2)}(t,E)$, by pair production with straggling of the resultant electrons due to the bremsstrahlung taken into consideration. $I_e^{(2)}(t,E)$ is then used to calculate the second-generation photons, $I_\gamma^{(2)}(t,k)$, and so forth, until the contribution becomes negligible. The energy distribution of the photons as a function of thickness is then given by the sum

$$I_\gamma(t,k)=I_\gamma^{(1)}(t,k)+I_\gamma^{(2)}(t,k)+I_\gamma^{(3)}(t,k)+\cdots.$$

Bhabha and Heitler[3] were mainly concerned with the multiplicity and energy distribution of electron showers, whereas we are interested in obtaining a reasonably accurate and compact formula for $I_\gamma(t,k)$ to be used in photoproduction of particles.

Our main results are contained in Eqs. (24), (25), and (29). Equation (24) gives the first-generation bremsstrahlung energy distribution as a function of target thickness $I_\gamma^{(1)}(t,k)$.

* Work supported by the U. S. Atomic Energy Commission.
[1] B. Rossi, *High Energy Particles* (Prentice-Hall, Inc., Englewood Cliffs, New Jersey, 1956).
[2] R. A. Alvarez, High Energy Physics Laboratory, Stanford University, Internal Reports HEPL–228 and 229, 1961 (unpublished).
[3] H. J. Bhabha and W. Heitler, Proc. Roy. Soc. (London) **A159**, 432 (1937).

We have used the complete screening formula for the bremsstrahlung cross section. Hence our $I_\gamma^{(1)}(t,k)$ is not reliable at the tip of the bremsstrahlung spectra. Our formula is correct only when

$$1-\frac{k}{E_0}\equiv\epsilon=\frac{137m}{2z^{1/3}E_0+137m},$$

where z is the atomic number of the target and m is the mass of the electron. For example, for $E_0=10$ BeV and for a Be target ($z=4$), ϵ must be much larger than 1.5×10^{-3}. For the type of experiment in which a precise shape of the bremsstrahlung tip is required, one can easily insert the exact formula for the bremsstrahlung cross section in Eq. (23) and obtain an adequate $I_\gamma^{(1)}(t,k)$. Equation (25) is a compact expression which gives approximately $I_\gamma^{(1)}(t,k)$. From Fig. 3 and Table I, we see that $[I_\gamma^{(1)}(t,k)]_{\text{approx}}$ is indeed an excellent approximation to $I_\gamma^{(1)}(t,k)$; thus it may be used safely for the purpose of estimating the secondary beam production. Equation (29) gives $I_\gamma^{(2)}(t,k)$. The small numerical values of $I_\gamma^{(2)}(t,k)$ compared with $I_\gamma^{(1)}(t,k)$ assure us that $I_\gamma^{(2)}(t,k)$ is negligible. Numerical values of $E_0I_e^{(1)}(t,E)$ and $E_0I_e^{(2)}(t,E)$ are shown in Table II.

It is a major problem to dispose of electron beams after they have been used to produce secondary beams. The formulas for $I_e^{(1)}(t,E)$ and $I_e^{(2)}(t,E)$ given in this paper may be used to calculate the intensity and energy distribution of the electron beams and thereby assist in the problem of disposing of electrons.

In Sec. III, we state some of the practical problems involved in using a thick target and also illustrate how our formula may be used for the calculation of secondary beam yield.

II. CALCULATIONS

A. Shower Equations at High Energies

We are interested in deriving a reasonably compact formula for the photon spectrum as a function of target thickness produced by a single incident electron with an incident energy $E_0>1$ BeV. Since E_0 is high and we are interested only in the high-energy component of the shower, we need to consider only the energy loss of electrons due to bremsstrahlung and the attenuation of the photon beam due to pair production. It is convenient to measure the thickness of the target in units of radiation length. We also use the complete screening formulas for both the bremsstrahlung and pair production cross sections. As mentioned in the Introduction, this will cause inaccuracy near the very tip of the bremsstrahlung, especially when the target is very thin. However, this difficulty can be remedied easily later.

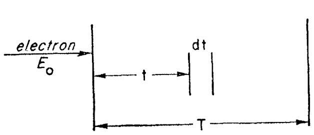

FIG. 1. A thick target.

When an electron with energy E passes through an infinitesimally thin target dt (in r.l.), the number of photons produced in the energy internal dk is[4]

$$dtdk(1/k)[\tfrac{4}{3}(1-k/E)+(k/E)^2]. \qquad (1)$$

When a photon with energy k passes through dt, the number of electrons plus positrons produced in the energy interval dE is

$$dtdE(2/k)[\tfrac{4}{3}(1-k/E)+(k/E)^2]E^2/k^2. \qquad (2)$$

The number of photons lost due to pair production in dt per incident photon is

$$(7/9)dt. \qquad (3)$$

Consider an electron with energy E_0 incident on a target, as shown in Fig. 1. Let the intensity of the electrons in the energy range dE at depth t be $I_e(t,E)dE$ and that of the photons in the energy range dk be $I_\gamma(t,k)dk$. Then after passing through an additional thickness dt, I_e and I_γ are altered by bremsstrahlung and pair creations. From Eqs. (1), (2), and (3) we obtain the shower equations:

$$\frac{\partial I_\gamma(t,k)}{\partial t}=-\frac{7}{9}I_\gamma(t,k)+\int_k^{E_0} I_e(t,E)\times\left[\frac{4}{3}\left(1-\frac{k}{E}\right)+\left(\frac{k}{E}\right)^2\right]\frac{dE}{k}, \qquad (4)$$

and

$$\partial I_e(t,E)/\partial t$$
$$=2\int_E^{E_0} I_\gamma(t,k)\left[\frac{4}{3}\left(1-\frac{k}{E}\right)+\left(\frac{k}{E}\right)^2\right]\frac{E^2}{k^3}dk$$
$$+\int_0^{E_0-E} I_e(t,\,E+k)\left[\frac{4}{3}\left(1-\frac{k}{E+k}\right)+\left(\frac{k}{E+k}\right)^2\right]\frac{dk}{k}$$
$$-I_e(t,E)\int_0^{E}\left[\frac{4}{3}\left(1-\frac{k}{E}\right)+\left(\frac{k}{E}\right)^2\right]\frac{dk}{k}. \qquad (5)$$

The last two integrations in Eq. (5) have infrared divergence; however, the difference of the two integrations is finite.

Our objective is to obtain $I_\gamma(t,k)$ and $I_e(t,E)$ with the boundary conditions

$$I_\gamma(0,k)=0, \qquad (6)$$

$$I_e(0,E)=\delta(E-E_0). \qquad (7)$$

Substituting $I_\gamma(t,k)=F(t,k)e^{-(7/9)t}$ in (4), and solving for $F(t,k)$ using the boundary condition (6), Eq. (4)

[4] H. A. Bethe and J. Ashkin, in *Experimental Nuclear Physics*, edited by E. Segre (John Wiley & Sons, Inc., New York, 1953).

may be integrated into the form

$$I_\gamma(t,k)=\frac{1}{k}\int_0^t e^{-(7/9)(t-t')}dt'\int_k^{E_0} I_e(t',E)\times\left[\frac{4}{3}\left(1-\frac{k}{E}\right)+\left(\frac{k}{E}\right)^2\right]dE. \quad (8)$$

The physical meaning of Eq. (8) is clear. The last integration represents the bremsstrahlung produced at t' and the term $\exp[-(7/9)(t-t')]$ just represents the attenuation of the photons due to the pair production in going through the thickness $(t-t')$.

Equation (5) can also be cast in a similar form. To do that we first let $I_\gamma=0$ and solve Eq. (5) with the boundary condition (7); the result must be, by definition, equal to $I_e^{(1)}(t,E)$, the intensity of the first generation electrons per incident electron. $I_e^{(1)}(t,E)$ is also called the straggling formula because it represents the energy distribution of the incident electron itself after having passed through thickness t. For convenience, let us restore the E_0 dependence of $I_e^{(1)}(t,E)$ and write

$$I_e^{(1)}(t,E)\equiv G(t,E,E_0). \quad (9)$$

Then obviously Eq. (5) with the boundary condition (7) may be cast in the form

$$I_e(t,E)=I_e^{(1)}(t,E)+\int_0^t dt'\int_E^{E_0} dE'G(t-t',E,E')\times\int_{E'}^{E_0} 2I_\gamma(t',k)\times\left[\frac{4}{3}\left(1-\frac{k}{E'}\right)+\left(\frac{k}{E'}\right)^2\right]\frac{E'^2}{k^3}dk. \quad (10)$$

By a direct substitution, one can show that Eq. (10) satisfies Eq. (5). The physical meaning of Eq. (10) is as follows. The net flux of electrons and positrons at thickness t consists of an energy-degraded incident electron beam $I_e^{(1)}(t,E)$ plus photon-induced pairs. The k integration in the second term times $dE'dt'$ represents the number of electrons and positrons in the energy interval dE' photoproduced in dt'. These electron-positron pairs suffer straggling in traveling from t' to t and their contribution to the number of e^+ and e^- with energy E at t is given by the E' and k integrations. Finally, dt' is integrated from 0 to t.

Now Eqs. (8) and (10) are completely equivalent to shower Eqs. (4) and (5) with boundary conditions (6) and (7). Once $I_e^{(1)}$ is obtained, Eqs. (8) and (10) can be solved by iterations;

$$I_\gamma^{(n)}(t,k)=\frac{1}{k}\int_0^t e^{-(7/9)(t-t')}dt'\int_k^{E_0} I_e^{(n)}(t',E)\times\left[\frac{4}{3}\left(1-\frac{k}{E}\right)+\left(\frac{k}{E}\right)^2\right]dE, \quad (11)$$

$$I_e^{(n+1)}(t,k)=\int_0^t dt'\int_E^{E_0} dE'G(t-t',E,E')\int_{E'}^{E_0} 2I_\gamma^{(n)}(t',k)\times\left[\frac{4}{3}\left(1-\frac{k}{E'}\right)+\left(\frac{k}{E'}\right)^2\right]\frac{E'^2}{k^3}dk, \quad (12)$$

$n=1, 2, 3, \cdots.$

The final solutions are

$$I_\gamma(t,k)=\sum_{n=1}^{\infty} I_\gamma^{(n)}(t,k), \quad (13)$$

and

$$I_e(t,E)=\sum_{n=1}^{\infty} I_e^{(n)}(t,E). \quad (14)$$

B. First-Generation Electron Intensity, $I_e^{(1)}(t,E)$

By definition $I_e^{(1)}(t,E)$ is the solution of Eq. (5) with $I_\gamma=0$. A very good approximate solution for $I_e^{(1)}(t,E)$ can be obtained by using a Heitler trick[5]; namely, the bremsstrahlung shape in the last two integrations is approximated by a more convenient form:

$$\frac{4}{3}\left[1-\frac{k}{E}\right]+(k/E)^2=\frac{4}{3}\frac{k/E}{\ln[1/(1-k/E)]}. \quad (15)$$

In Fig. 2 we compare the shape of the above two expressions. It should be noted that for $k/E=0$ the two expressions coincide, and for $k/E<0.5$ the approximate expression is, at most, 10% higher than the exact expression. Since the approximate expression for the photon emission is more accurate for low k, the resultant solution for $I_e^{(1)}(t,E)$ must be more accurate for E closer to E_0—this is exactly what we want.

With this approximation we have, from Eq. (5),

$$\partial I_e^{(1)}(t,E)/\partial t=-\frac{4}{3}\int_0^1\left[I_e^{(1)}(t,E)-\frac{1}{1-v}I_e^{(1)}\left(t,\frac{E}{1-v}\right)\right]\frac{dv}{\ln(1/1-v)}. \quad (16)$$

Multiplying both sides by E^s and integrating E from 0 to ∞ (i.e., taking the Mellin Transform of both sides), we have[6]

$$\partial M_e(t,s)/\partial t=-\frac{4}{3}\int_0^1 [M_e(t,s)-(1-v)^s M_e(t,s)]\frac{dv}{\ln(1/(1-v))}=-\tfrac{4}{3}M_e(t,s)\ln(1+s), \quad (17)$$

[5] W. Heitler, *Quantum Theory of Radiation* (Oxford University Press, Oxford, England, 1954).

[6] L. Eyges, Phys. Rev. **76**, 264 (1949). The analytical method of obtaining the Bethe and Heitler straggling formula was first used in this reference. We carried out these manipulations explicitly here for completeness.

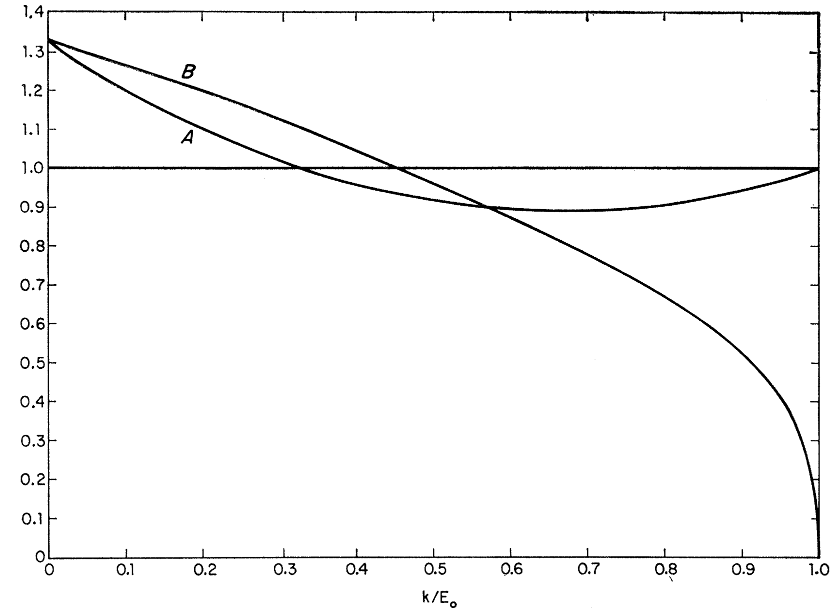

FIG. 2. Bremsstrahlung spectral shape. Curve A for complete screening, curve B was used to calculate straggling formula. See Eq. (15).

where

$$M_e(t,s)=\int_0^\infty E^s I_e^{(1)}(t,E)dE. \tag{18}$$

Applying the boundary condition $I_e^{(1)}(0,E)=\delta(E-E_0)$ to Eq. (18), we have

$$M_e(0,s)=E_0^s. \tag{19}$$

Hence from Eq. (17) we have

$$\begin{aligned}M_e(t,s)&=E_0^s e^{-(4/3)t\ln(1+s)}\\&=E_0^s(1+s)^{-(4/3)t}.\end{aligned} \tag{20}$$

Using the inversion formula for the Mellin transform we obtain

$$I_e^{(1)}(t,E)=\frac{1}{2\pi i}\int_c E^{-(1+s)}E_0^s(1+s)^{-(4/3)t}ds, \tag{21}$$

where the integration path c runs parallel to the imaginary axis with $\mathrm{Re}s>0$. If one completes the contour by an infinite semicircle on the left plane and evaluates the residue at $s=-1$, one obtains

$$\begin{aligned}I_e^{(1)}(t,E)&=\frac{1}{E_0}\frac{[\ln(E_0/E)]^{(4/3)t-1}}{\Gamma(\frac{4}{3}t)},\\&\equiv G(t,E,E_0).\end{aligned} \tag{22}$$

This is the well-known formula first obtained by Bethe and Heitler.[7] [See Heitler,[5] page 378. Instead of $\frac{4}{3}$ in the formula he used a different value. We used $\frac{4}{3}$ in order to force the approximate formula (15) to agree with the exact value at the infrared limit ($k=0$); thus $I_e^{(1)}(t,E)$ has the correct value for E near E_0.]

[7] H. A. Bethe and W. Heitler, Proc. Roy. Soc. (London) **A146**, 83 (1934).

C. First-Generation Photons, $I_\gamma^{(1)}(t,k)$

Substituting Eq. (22) into Eq. (11) we obtain

$$I_\gamma^{(1)}(t,k)=\frac{1}{k}\int_0^t\frac{e^{-(7/9)(t-t')}dt'}{\Gamma(\frac{4}{3}t')}\int_k^{E_0}\left(\ln\frac{E_0}{E}\right)^{(4/3)t'-1}\times\left[\frac{4}{3}\left(1-\frac{k}{E}\right)+\left(\frac{k}{E}\right)^2\right]\frac{dE}{E_0}. \tag{23}$$

Equation (23) as it stands has a singularity in the integrand when $E_0=E$ and $\frac{4}{3}t'<1$. In order to carry out numerical integration by a computer, it is convenient to write it in a different form. Changing the variable of integration $E=E_0e^{-x}$, and integrating by parts, the second integration in Eq. (23) can be written as

$$\frac{4}{3t'}\left(\ln\frac{E_0}{k}\right)^{(4/3)t'}\frac{k}{E_0}-\int_0^{\ln(E_0/k)}\frac{3}{4t'}x^{(4/3)t'}\times\left[-\tfrac{4}{3}e^{-x}+\frac{k^2}{E_0^2}e^x\right]dx.$$

The terms e^{-x} and e^x are then expanded into a power series and the integration carried out term by term. We then obtain

$$kI_\gamma^{(1)}(t,k)=e^{-(7/9)t}\int_0^t\frac{e^{(7/9)t'}}{\Gamma(\frac{4}{3}t'+1)}\left(\ln\frac{1}{u}\right)^{(4/3)t'}\times\left\{u+\sum_{n=0}^\infty\frac{1}{n!(n+\frac{4}{3}t'+1)}[\tfrac{4}{3}(-1)^n-u^2]\times\left(\ln\frac{1}{u}\right)^{n+1}\right\}dt', \tag{24}$$

where $u=k/E_0$.

TABLE I. Numerical values of $F_{\text{thin}} \equiv f(k/E_0)$ from Eq. (1), $F_{\text{exact}} \equiv kI_\gamma^{(1)}(t,k)/t$ from Eq. (24), and $F_{\text{approx}} \equiv k[I_\gamma^{(1)}(t,k)]_{\text{approx}}/t$ from Eq. (25).

k/E_0	F_{thin}	F_{exact}	F_{approx}	F_{exact}	F_{approx}	F_{exact}	F_{approx}	F_{exact}	F_{approx}	F_{exact}	F_{approx}
		$t=0.01$		$t=0.05$		$t=0.1$		$t=0.2$		$t=0.3$	
0.10	1.20997	1.20417	0.99542	1.18110	0.97735	1.15277	0.95529	1.09772	0.91288	1.04475	0.87265
0.20	1.10664	1.10071	0.99464	1.07713	0.97350	1.04815	0.94775	0.99178	0.89840	0.93745	0.85178
0.30	1.02331	1.01740	0.99375	0.99387	0.96916	0.96490	0.93929	0.90840	0.88234	0.85378	0.82890
0.40	0.95998	0.95406	0.99273	0.93046	0.96419	0.90130	0.92966	0.84416	0.86427	0.78868	0.80349
0.50	0.91665	0.91055	0.99153	0.88616	0.95835	0.85591	0.91843	0.79638	0.84353	0.73841	0.77475
0.60	0.89332	0.88668	0.99005	0.86013	0.95126	0.82716	0.90494	0.76225	0.81904	0.69929	0.74142
0.70	0.88999	0.88222	0.98816	0.85123	0.94223	0.81289	0.88793	0.73811	0.78886	0.66670	0.70125
0.80	0.90666	0.89671	0.98550	0.85735	0.92970	0.80929	0.86467	0.71774	0.74883	0.63319	0.64951
0.82	0.91239	0.90181	0.98481	0.86001	0.92648	0.80921	0.85877	0.71311	0.73888	0.62521	0.63693
0.84	0.91893	0.90760	0.98403	0.86302	0.92289	0.80908	0.85223	0.70791	0.72797	0.61636	0.62326
0.86	0.92626	0.91407	0.98316	0.86625	0.91886	0.80873	0.84489	0.70186	0.71587	0.60634	0.60825
0.88	0.93440	0.92119	0.98216	0.86956	0.91422	0.80789	0.83653	0.69456	0.70224	0.59471	0.59155
0.90	0.94333	0.92889	0.98097	0.87272	0.90878	0.80616	0.82679	0.68543	0.68658	0.58084	0.57261
0.92	0.95306	0.93710	0.97951	0.87534	0.90219	0.80289	0.81508	0.67353	0.66806	0.56373	0.55058
0.94	0.96360	0.94566	0.97765	0.87674	0.89377	0.79691	0.80030	0.65720	0.64520	0.54161	0.52393
0.96	0.97493	0.95421	0.97502	0.87538	0.88210	0.78565	0.78010	0.63289	0.61480	0.51085	0.48942
0.98	0.98707	0.96164	0.97056	0.86653	0.86261	0.76136	0.74715	0.59021	0.56736	0.46102	0.43769
0.999	0.99933	0.95451	0.95157	0.79765	0.78482	0.64338	0.62504	0.43255	0.41355	0.30367	0.28809
		$t=0.5$		$t=0.7$		$t=1.0$		$t=1.5$		$t=2.0$	
0.10	1.20997	0.94458	0.79825	0.85154	0.73117	0.72421	0.64258	0.54181	0.52158	0.39397	0.42679
0.20	1.10664	0.83465	0.76612	0.73931	0.68960	0.60981	0.58975	0.42932	0.45615	0.29138	0.35446
0.30	1.02331	0.75015	0.73170	0.65415	0.64610	0.52503	0.53641	0.35073	0.39398	0.22525	0.28992
0.40	0.95998	0.68318	0.69446	0.58584	0.60025	0.45695	0.48237	0.28993	0.33512	0.17727	0.23286
0.50	0.91665	0.62848	0.65360	0.52828	0.55143	0.39899	0.42738	0.24005	0.27958	0.14029	0.18298
0.60	0.89332	0.58142	0.60786	0.47669	0.49868	0.34675	0.37101	0.19731	0.22737	0.11064	0.13990
0.70	0.88999	0.53678	0.55508	0.42626	0.44038	0.29659	0.31250	0.15918	0.17837	0.08604	0.10317
0.80	0.90666	0.48695	0.49091	0.37071	0.37331	0.24448	0.25032	0.12351	0.13225	0.06475	0.07214
0.82	0.91239	0.47525	0.47595	0.35813	0.35829	0.23333	0.23717	0.11645	0.12330	0.06074	0.06653
0.84	0.91893	0.46256	0.45999	0.34476	0.34253	0.22178	0.22368	0.10934	0.11439	0.05676	0.06110
0.86	0.92626	0.44863	0.44280	0.33041	0.32587	0.20973	0.20976	0.10214	0.10552	0.05279	0.05581
0.88	0.93440	0.43307	0.42409	0.31481	0.30810	0.19701	0.19531	0.09480	0.09662	0.04880	0.05065
0.90	0.94333	0.41535	0.40341	0.29757	0.28890	0.18345	0.18017	0.08723	0.08766	0.04474	0.04557
0.92	0.95306	0.39463	0.38006	0.27817	0.26780	0.16872	0.16409	0.07929	0.07851	0.04055	0.04052
0.94	0.96360	0.36950	0.35283	0.25556	0.24397	0.15231	0.14661	0.07077	0.06901	0.03610	0.03540
0.96	0.97493	0.33710	0.31921	0.22785	0.21571	0.13317	0.12685	0.06121	0.05878	0.03117	0.03001
0.98	0.98707	0.28960	0.27224	0.18982	0.17839	0.10847	0.10229	0.04937	0.04672	0.02511	0.02378
0.999	0.99933	0.16770	0.15839	0.10381	0.09802	0.05768	0.05447	0.02607	0.02462	0.01325	0.01252

The infinite series converges rapidly when u is near 1. For example, summing the series up to $n=5$, one obtains an accuracy of better than 0.1% at $u=0.5$.

D. Approximate Expression for $I_\gamma^{(1)}(t,k)$

In many problems in which the photoproduction cross section is given by a simple formula, one may use Eq. (24) directly. However, in some problems in which the photoproduction cross section is expressed in terms of multifold integrations, a simplified expression representing $I_\gamma^{(1)}(t,k)$ is desirable. Let us go back to Eq. (23) and make the following approximations.

(1) Replace $[\frac{4}{3}(1-k/E)+(k/E)^2]$ by 1. An inspection of Fig. 2 shows that this approximation will at most make a 10% overestimate in the high-energy half of the bremsstrahlung spectrum.

(2) Replace $\ln(E_0/E)$ by $(E_0-E)/E_0$. This approximation is also good when E is close to E_0.

(3) Replace $\Gamma(\frac{4}{3}t')$ by $3/(4t')$. This replacement is correct to within 12% as long as $4t'/3\leq 1.25$.

We obtain then an approximate expression for $I_\gamma^{(1)}(t,k)$;

$$[I_\gamma^{(1)}(t,k)]_{\text{approx}}=\frac{1}{k}\frac{(1-k/E_0)^{(4/3)t}-e^{-(7/9)t}}{[(7/9)+\frac{4}{3}\ln(1-k/E_0)]}. \quad (25)$$

The approximations made in deriving this formula are extremely crude and not valid at all when $E/E_0\to 0$ and $t'>1$. However, the numerical comparisons shown in Table I and Fig. 3 indicate that up to $t=2$ r.l. and $0.2<k/E_0<1$, the difference between $I_\gamma^{(1)}(t,k)$ and $[I_\gamma^{(1)}(t,k)]_{\text{approx}}$ is about 0 to 15%. The reason for this miraculous agreement between the numerical values of the two expressions can be understood by closer inspection of the behavior of the integrand of Eq. (23). We notice that the integrand is big only when $\frac{4}{3}t'<1$ and $E\to E_0$. But our approximations are good under these circumstances, and hence, even though they are very bad in other regions of integration, the integrand there hardly contributes anything to the final result.

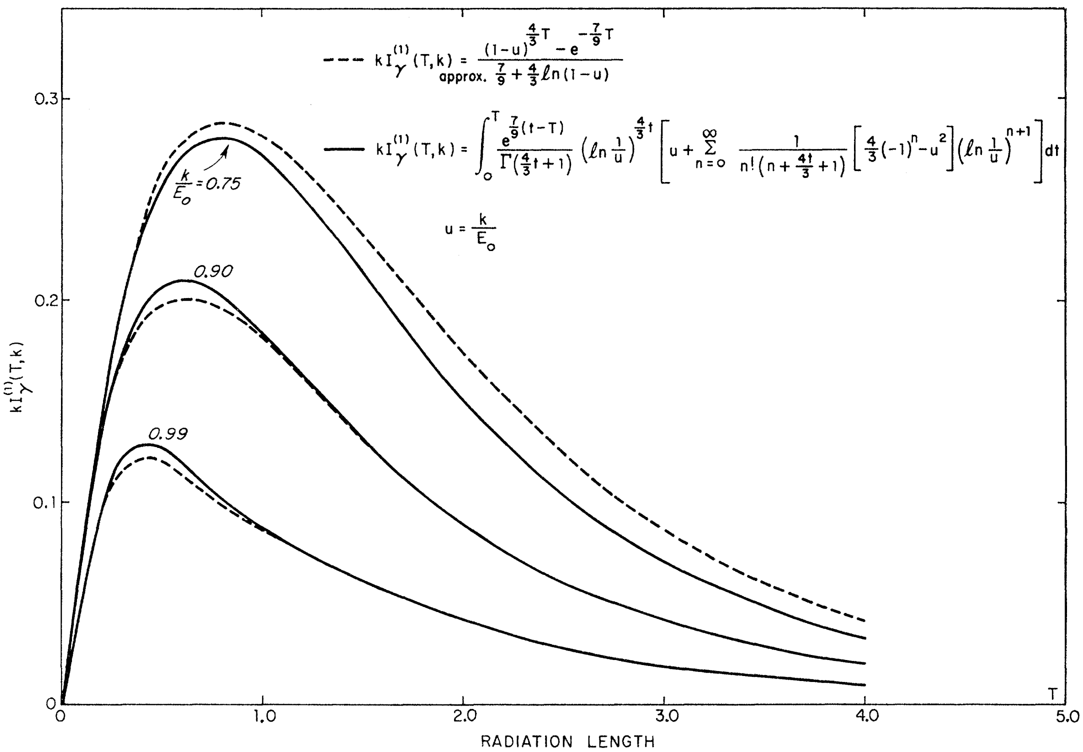

FIG. 3. Exact and approximate first-generation photon spectra as a function of target thickness.

E. The Second-Generation Electron and Bremsstrahlung Spectra: $I_e^{(2)}(t,E)$ and $I_\gamma^{(2)}(t,k)$

In this section we would like to make rough estimates of $I_e^{(2)}(t,E)$ and $I_\gamma^{(2)}(t,k)$ in order to have some feeling for the errors involved in using $I_e^{(1)}(t,E)$ and $I_\gamma^{(1)}(t,k)$ in doing further calculations.

Substituting Eqs. (25) and (22) into Eq. (12), we obtain

$$I_e^{(2)}(t,E)=\int_0^t dt'\int_E^{E_0} dE'\frac{[\ln(E'/E)]^{(4/3)(t-t')-1}}{\Gamma[\frac{4}{3}(t-t')]}\times\int_{E'}^{E_0} 2[I_\gamma^{(1)}(k,t')]_{\text{approx}}\times\left[\frac{4}{3}\left(1-\frac{k}{E'}\right)+\left(\frac{k}{E'}\right)^2\right]\frac{E'^2}{k^3}dk. \quad (26)$$

Since we are interested in E very close to E_0 and $E\leq E'\leq k\leq E_0$, we may approximate $[\frac{4}{3}(1-k/E')+(k/E')^2]\times E'^2/k^2$ by 1. Using the identity

$$\int_E^{E_0} dE'\int_{E'}^{E_0} dk=\int_E^{E_0} dk\int_E^{k} dE',$$

the integration with respect to dE' can be carried out and we obtain

$$I_e^{(2)}(t,E)=2\int_E^{E_0}\frac{dk}{k}\int_0^t dt'\times([\ln(k/E)]^{(4/3)(t-t')}/\Gamma[\tfrac{4}{3}(t-t')+1])\times[I_\gamma^{(1)}(k,t')]_{\text{approx}}. \quad (27)$$

Now the gamma function may be replaced by 1 because when $\frac{4}{3}(t-t')\leq 1.25$, this replacement causes 12% error at most, and when $\frac{4}{3}(t-t')>1.25$ the term $(\ln k/E)^{(4/3)(t-t')}$ decreases rapidly with increasing $t-t'$ when k/E is very close to 1. With these approximations the integration with respect to t' can be carried out and we obtain

$$I_e^{(2)}(t,E)=\frac{2}{E_0}\int_v^1\frac{dx}{x^2}\frac{1}{(7/9)+\frac{4}{3}\ln(1-x)}\times\left[\frac{(1-x)^{(4/3)t}-(1-v/x)^{(4/3)t}}{\frac{4}{3}\ln[(x-x^2)/(x-v)]}+\frac{e^{-(7/9)t}-(1-v/x)^{(4/3)t}}{(7/9)+\frac{4}{3}\ln(1-v/x)}\right], \quad (28)$$

where $v=E/E_0$ and $x=k/E_0$.

In Table II we compare $E_0 I_e^{(1)}(t,E)$ with $E_0 I_e^{(2)}(t,E)$.

TABLE II. Numerical values of $E_0I_e^{(1)}(t,E)$ and $E_0I_e^{(2)}(t,E)$ as functions of t and E/E_0.

E/E_0	$E_0\times I_e^{(1)}$	$E_0\times I_e^{(2)}$	$E_0\times I_e^{(1)}$	$E_0\times I_e^{(2)}$	$E_0\times I_e^{(1)}$	$E_0\times I_e^{(2)}$	$E_0\times I_e^{(1)}$	$E_0\times I_e^{(2)}$	$E_0\times I_e^{(1)}$	$E_0\times I_e^{(2)}$
	$t=0.2$		$t=0.4$		$t=0.6$		$t=0.8$		$t=1.0$	
0.50	0.3863	0.0288	0.7128	0.0849	0.9243	0.1424	1.0106	0.1902	0.9911	0.2263
0.55	0.4306	0.0231	0.7638	0.0666	0.9520	0.1094	1.0006	0.1434	0.9434	0.1666
0.60	0.4832	0.0184	0.8219	0.0518	0.9824	0.0833	0.9902	0.1071	0.8952	0.1220
0.65	0.5475	0.0144	0.8900	0.0397	1.0165	0.0625	0.9790	0.0786	0.8458	0.0877
0.70	0.6289	0.0112	0.9719	0.0299	1.0556	0.0458	0.9668	0.0562	0.7942	0.0614
0.75	0.7362	0.0084	1.0745	0.0217	1.1020	0.0324	0.9530	0.0388	0.7393	0.0414
0.80	0.8870	0.0060	1.2097	0.0151	1.1594	0.0217	0.9370	0.0252	0.6793	0.0262
0.85	1.1191	0.0040	1.4026	0.0097	1.2353	0.0134	0.9175	0.0150	0.6112	0.0151
0.90	1.5379	0.0024	1.7171	0.0053	1.3472	0.0070	0.8913	0.0075	0.5290	0.0073
0.95	2.6072	0.0010	2.4026	0.0020	1.5558	0.0025	0.8496	0.0025	0.4161	0.0023
	$t=1.2$		$t=1.4$		$t=1.6$		$t=1.8$		$t=2.0$	
0.50	0.8983	0.2479	0.7656	0.2580	0.6205	0.2589	0.4819	0.2527	0.3609	0.2416
0.55	0.8220	0.1807	0.6735	0.1850	0.5247	0.1827	0.3918	0.1758	0.2820	0.1656
0.60	0.7480	0.1291	0.5876	0.1300	0.4390	0.1264	0.3144	0.1206	0.2170	0.1121
0.65	0.6753	0.0911	0.5070	0.0902	0.3619	0.0863	0.2476	0.0805	0.1633	0.0738
0.70	0.6030	0.0625	0.4304	0.0608	0.2922	0.0572	0.1901	0.0526	0.1192	0.0476
0.75	0.5300	0.0412	0.3573	0.0393	0.2290	0.0363	0.1407	0.0329	0.0833	0.0294
0.80	0.4551	0.0255	0.2867	0.0239	0.1717	0.0217	0.0986	0.0194	0.0546	0.0171
0.85	0.3762	0.0143	0.2178	0.0131	0.1199	0.0117	0.0633	0.0103	0.0322	0.0090
0.90	0.2901	0.0067	0.1496	0.0060	0.0734	0.0053	0.0345	0.0046	0.0156	0.0040
0.95	0.1884	0.0021	0.0802	0.0018	0.0325	0.0016	0.0126	0.0013	0.0047	0.0011

TABLE III. Intensity ratio of second-generation to first-generation photon spectra as functions of k/E_0 and t.

k/E_0	$t=0.2$	$t=0.4$	$t=0.6$	$t=0.8$	$t=1.0$	$t=1.2$	$t=1.4$	$t=1.6$	$t=1.8$	$t=2.0$
0.50	0.002	0.007	0.015	0.023	0.032	0.041	0.050	0.059	0.069	0.078
0.55	0.002	0.006	0.011	0.017	0.024	0.031	0.038	0.045	0.053	0.060
0.60	0.001	0.004	0.008	0.013	0.018	0.023	0.029	0.034	0.040	0.045
0.65	0.001	0.003	0.006	0.009	0.013	0.017	0.021	0.025	0.029	0.033
0.70	0.001	0.002	0.004	0.007	0.009	0.012	0.015	0.018	0.021	0.024
0.75	0.000	0.001	0.003	0.004	0.006	0.008	0.010	0.012	0.014	0.016
0.80	0.000	0.001	0.002	0.003	0.004	0.005	0.006	0.007	0.009	0.010
0.85	0.000	0.000	0.001	0.001	0.002	0.003	0.003	0.004	0.005	0.005
0.90	0.000	0.000	0.000	0.001	0.001	0.001	0.001	0.002	0.002	0.002

The second-generation photons can be obtained by substituting $I_e^{(2)}(t,E)$ in Eq. (11). Since we are interested only in the order of magnitude, we may again approximate $\frac{4}{3}(1-k/E)+(k/E)^2$ by 1. After integration with respect to t', we obtain the second-generation bremsstrahlung spectra as a function of target thickness:

$$I_\gamma^{(2)}(t,k)=\frac{2}{k}\int_x^1 dv\int_v^1 \frac{dy}{y^2}\frac{1}{D_1}\left[\frac{N_1-N_2}{D_1D_2}-\frac{N_3-N_2}{D_2D_3}+\frac{tN_2}{D_3}-\frac{N_3-N_2}{D_3^2}\right], \quad (29)$$

where

$$x=k/E_0,\quad (v=E/E_0,\ y=k'/E_0),$$
$$D_1=(7/9)+\tfrac{4}{3}\ln(1-y),$$
$$D_2=\tfrac{4}{3}\ln((y-y^2)/(y-v)),$$
$$D_3=(7/9)+\tfrac{4}{3}\ln[1-v/y],$$
$$N_1=(1-y)^{(4/3)t},$$
$$N_2=e^{-(7/9)t},$$
$$N_3=[1-v/y]^{(4/3)t}.$$

In Table III the intensity ratio of the second-generation to the first-generation bremsstrahlung, $I_\gamma^{(2)}(t,k)/[I_\gamma^{(1)}(t,k)]_{\text{approx}}$, from Eqs. (25) and (29) is given. It is seen that this ratio is 0.078 at $t=2$ r.l., $k/E_0=0.5$ and it becomes smaller as t is decreased or k/E_0 is increased. The smallness of these ratios assures us that for those experiments which only utilize bremsstrahlung in the energy range $\frac{1}{2}<k/E_0<1$ and thickness $t<2$ r.l., the second-generation photons can safely be ignored.

III. APPLICATIONS

Let us briefly mention some of the problems[8] involved in using a thick target at high energies with an intense electron beam.

1. Temperature: Consider, for example, a beam of 20-BeV electrons with an intensity of 10^{14} electrons per second. This beam has a power of 320 kW; if its energy is completely absorbed in a target, it will probably destroy the target immediately, even with normal cooling. Of course one can overcome this problem by

[8] Z. Guiragossian (private communication).

using a less intense beam or a thinner target so that only a fraction of the beam energy is deposited in the target.

2. Electron and Positron Backgrounds: With high-energy and high-intensity beams it is not practical to sweep away all the electrons after they are used to produce the bremsstrahlung in a thick target ($t>0.03$). The same target must be used for both the bremsstrahlung and particle production. Thus there will be many electrons with intensity spectrum $I_e(t,E)$ in addition to μ, π, K, p, n, $\bar{p}$, $\bar{n}$, etc. This blast of electron beam can be avoided in two ways. (a) Shun the near forward angle and use a thinner target (in order to suppress the angular spread of the electrons due to multiple scatterings); or (b) Use a very high z absorber after the main target to slow down all the high-energy electrons. High z materials are better because their absorption coefficients per radiation length (roughly proportional to $A^{-4/3}$ where A is atomic weight) for the strongly interacting particles are smaller. To estimate the high-energy component of the electrons and positrons after the absorber, Eqs. (22) and (28) may be used. Because of the heating problem, this scheme will work only when a relatively low-intensity beam is used. The separation of other particles is a very complicated technical problem and beyond the scope of this article.

Let us assume that the problems mentioned above can be solved and ask what is the optimum thickness and material of the target for producing high-energy secondary particles. Since the same target is used for producing the bremsstrahlung and the secondary particles, the yield Y per unit energy and solid angle of the secondary particle per incident electron is given by

$$Y=(NX_0/A)\times\int_0^T dt\, e^{-\eta(T-t)}\int_{k_{\min}}^{E_0} I_\gamma(t,k)(d^2\sigma/d\Omega dq_0)dk\,, \quad (30)$$

where N is 6×10^{23}, Avogadro's number; A is the atomic weight of the target nucleus; X_0 is the unit radiation length in g/cm^2 of the target material and equals

$$\frac{[1+0.12(z/82)^2]A}{4\alpha Nz(z+1)r_e^2\ln(183z^{-1/3})}.$$

η is $(N/A)X_0\sigma_a$, nuclear absorption coefficient per radiation length for the secondary particle in the target; $I_\gamma(t,k)$: Use $[I_\gamma^{(1)}(t,k)]_{\text{approx}}$ given by Eq. (25) for this purpose; $d^2\sigma/d\Omega dq_0$ is the differential cross section for the secondary particle production by a photon with energy k; T is the target thickness in radiation lengths; E_0 is the incident-electron energy; and $k_{\min}$ is the minimum energy of the photon kinematically allowed for the photoproduction process.

σ_a is the absorption cross section of the secondary particles for the target nucleus. For most purposes, we may assume the absorption cross section to be 80% of the total cross section and the total cross sections for π, K, and $\bar{p}$, etc., on the nuclei are given by $\sigma^t(A)=\sigma^t(1)A^{2/3}$, that is, the total cross section for the proton times $A^{2/3}$. The total cross sections at high energies for π, K^-, K^+, and $\bar{p}$ incident on protons can be obtained from experimental data[9];

$$\sigma_{\pi^\pm p}{}^t=25\text{ mb},$$

$$\sigma_{K^-p}{}^t=22\text{ mb},$$

$$\sigma_{K^+p}{}^t=18\text{ mb},$$

$$\sigma_{\bar{p}p}{}^t=\left(45+\frac{80}{p}\right)\text{ mb},$$

where p is the incident $\bar{p}$ momentum in BeV/c in the lab system. ($\sigma_{\bar{p}p}{}^t\approx50$ mb for our purpose.) From these we obtain the values of η for various secondary particles in some target materials. (See Table IV.)

TABLE IV. Nuclear absorption coefficient η per radiation length.

Particle	Target	Z	A	X_0	η
π	Be	4	9	65	3.75×10^{-1}
π	C	6	12	44.6	2.34×10^{-1}
π	Cu	29	63.57	13.1	3.93×10^{-2}
π	Pb	82	207.2	6.5	1.32×10^{-2}
K^-	Be				3.3×10^{-1}
K^+	Be				2.7×10^{-1}
$\bar{p}$	Be				7.5×10^{-1}

Using Eq. (30) we have calculated[10] the yield of π, K^-, k^+, k^0, $\bar{p}$ from the electron machine using various kinds of production mechanisms. The numerical results show that the smaller the z the better the yield since no strongly interacting particles are photoproduced with z^2 dependence; hence the yield per radiation length is larger from small z materials than from high z materials. The nuclear absorption coefficient, η, is larger for the low z materials than for the high z ones, but it is not a very decisive factor in the choice of materials because there is not much sense in using a target thicker than 2 radiation lengths. Thus, Y is rather weakly dependent upon η.

We have used Be targets to calculate the yields for various particles. The results indicate the optimum thickness of the target is around $T=2$ for π and K productions and $T=1.6$ for $\bar{p}$ photoproduction. For muon production, the nuclear absorption coefficient η is zero; thus as far as the yield is concerned, the thicker the target the better. However, more than 90% of the maximum number of muons with energy $>\frac{1}{2}E_0$ are produced within a thickness of four radiation lengths.

[9] T. Kycia *et al.*, quoted by Lindenbaum in the *Proceedings of the Twelfth Annual International Conference on High Energy Physics, Dubna, 1964* (Atomizdat, Moscow, 1965).

[10] See *SLAC User's Handbook* (Stanford University, Stanford, California, 1966).

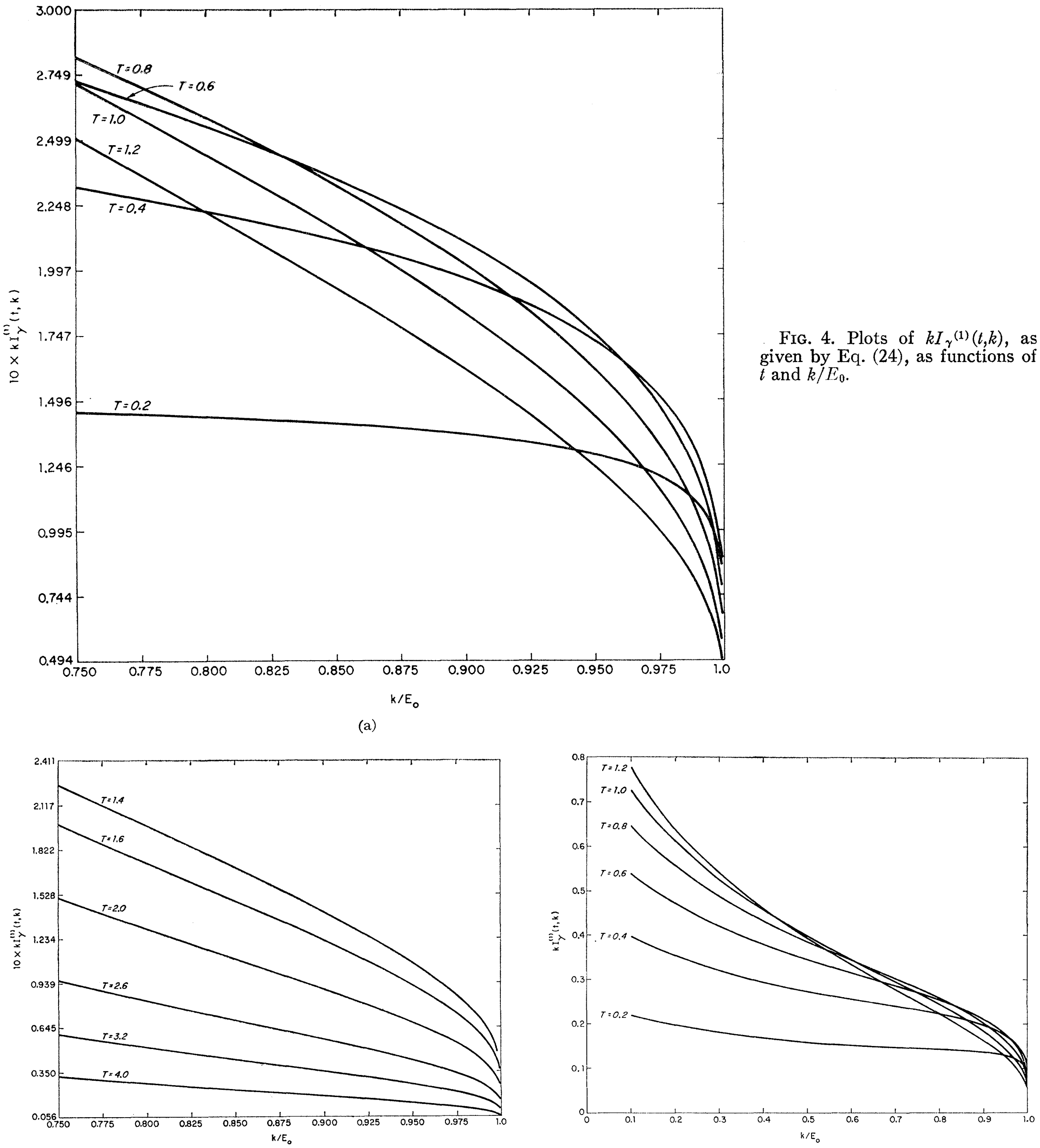

FIG. 4. Plots of $kI_\gamma^{(1)}(t,k)$, as given by Eq. (24), as functions of t and k/E_0.

IV. DISCUSSION

From the numerical examples, the following conclusions may be drawn:

(i) Second-generation bremsstrahlung is negligible for $T<2$ r.l. and $k/E_0>0.5$.

(ii) For calculating yield per equivalent quanta, either $I_\gamma^{(1)}(t,k)$ or $[I_\gamma^{(1)}(t,k)]_{\text{approx}}$, given by Eqs. (24) and (25), respectively, may be used, depending upon the degree of accuracy desired.

(iii) For the photon-difference type of experiments in which the accurate shape of the bremsstrahlung tip is required, one may insert a more accurate bremsstrahlung spectrum[11] in Eq. (23) and calculate $I_\gamma^{(1)}(t,k)$ accordingly.

(iv) To estimate the electron and positron background from a thick target, $I_e^{(1)}(t,E)$ and $I_e^{(2)}(t,E)$ given by Eqs. (22) and (28) may be used. The angular

[11] H. W. Koch and J. W. Motz, Rev. Mod. Phys. **31**, 920 (1959).

spread of these electrons and positrons is mainly caused by multiple scatterings, and not by the production mechanisms.

Some qualitative features of $I_\gamma^{(1)}(t,k)$ may be understood in the following way. Since we are interested in k comparable to the incident-electron energy E_0, the energy of the electrons E from which these γ's are produced must also be very close to E_0. Now the electron spectrum given by $I_e^{(1)}(t,E)$, Eq. (22) changes its shape abruptly at $t=0.75$ r.l. For $t<0.75$, we have $I_e^{(1)}(t,E_0)=\infty$; whereas for $t>0.75$, we have $I_e^{(1)}(t,E_0)=0$. This tells us qualitatively that practically all the high-energy γ's are produced from $t=0$ to $t=0.75$, and after $t=0.75$, the intensity of the γ's is just attenuated by the absorption factor $e^{-7/9(t-0.75)}$, as shown in Fig. 3.

In Figs. 4(a), (b), and (c) the curves for $kI_\gamma^{(1)}(t,k)$, given by Eq. (24) as functions of t and k/E_0 are plotted. The computer programs (in ALGOL) for generating all the numerical values given in this article are available upon request.

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Noncompact Groups in Classifying Hadrons*

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We show that the noncompact groups $U(6,6)$, $GL(6,C)$, and $U(6)\times U(6)\times O(3,1)$ proposed by many authors to underlie a classification of hadrons can be excluded on the basis of existing information on the hadron spectrum. A scheme which at least avoids the obvious difficulties of the above noncompact groups is mentioned.

I. INTRODUCTION

MANY papers[1,2] have been devoted lately to the idea[3] that much as in the case of the hydrogen atom[4] a noncompact group G should be used in classifying hadrons. All baryons are then expected to lie in one infinite-dimensional unitary representation (i.d.u.r.) D_B of G. Of course, the mass of the higher members should increase as a consequence of G-symmetry-breaking effects, but one should be able to foretell unambiguously the spectrum of baryon resonances that follow the lowest lying **56** states. Similarly all mesons are expected to belong to an i.d.u.r. D_M of G. Concerning G, the following candidates have been discussed[1-3] $G=U(6,6)$, $GL(6)$, and $U(6)\times U(6)\times O(3,1)$. We wish to show in this paper (Sec. II) that existing experimental evidence, especially on the baryon spectrum, is sufficient to exclude all these three possibilities. In Sec. III we will propose an alternative possibility based on the group $U(12)\times U(12\times SL(2C)$ which has at least a chance of survival besides some speculatively interesting features.

Let us still recall the differences between the groups mentioned above: $SL(6,C)$ ($U(6,6)$) starts with a given $U(6)$ ($U(6)\times U(6)$) multiplet and keeps adding s-wave mesons to it in the most symmetrical way. Thus at each step the maximum spin increases by one unit and also new $SU(3)$ representations appear. $SL(6,C)$ and $U(6,6)$ thus lead to large spins paired with large unitary ($SU(3)$) spins. $U(6)\times U(6)\times O(3,1)$, on the other hand, starts with a given $U(6)\times U(6)\times O(3)$ representation and keeps exciting its constituent quarks into higher and higher orbital states. Again, maximal spin increases by one unit upon each step, but no higher $SU(3)$ representations are introduced. High spins appear with low unitary spins. We could label $SL(6,C)$ and $U(6,6)$ as "$SU(3)$ and spin explosive" and $U(6)\times U(6)\times O(3,1)$ as "orbital explosive" groups.

Now in order to get a precise idea about the particles appearing in a given i.d.u.r., we shall always write out the few lowest terms in an i.d.u.r. By lowest terms we mean lowest representations of the maximal compact

* This work supported in part by the U. S. Atomic Energy Commission.

[1] Y. Dothan, M. Gell-Mann, and Y. Ne'eman, Phys. Letters **17**, 148 (1965); C. Fronsdal, Trieste Report No. IC/65/68 (to be published).

[2] A. Salam and J. Strathdee (Trieste Report) have developed techniques for constructing vertex functions for $U(6,6)$ and $GL(6,C)$.

[3] R. P. Feynman (unpublished).

[4] See, e.g., A. Salam, in *Proceedings of the Oxford International Conference on Elementary Particles, 1965* (Rutherford High Energy Laboratory, Harwell, England, 1966), p. 241.