This at least prevents the occurrence of singularities at inelastic thresholds as in Sec. II. As a result we now have

$$\mathrm{Re}D-1 = \frac{-1}{\pi}\mathrm{Re}\left[\sum_n A_n\left\{1-\left(1-\frac{c_n}{s}\right)\ln\left(1-\frac{s}{c_n}\right)\right\}\right]. \quad \text{(A3)}$$

For $c_n\to\infty$ the individual terms at fixed s are $O(s/c_n)$ so that our previous choice of parameters still gives a convergent result. Choosing $A_n=A$, we sum the series using the identity

$$\int^s ds\ln\left(1-\frac{s}{c_n}\right)=(s-c_n)\ln\left(1-\frac{s}{c_n}\right)-s \quad \text{(A4)}$$

to obtain

$$\mathrm{Re}D-1=-\frac{A}{\pi}\left[\frac{1}{s}\int^s ds\ln\left|\frac{\sin(\pi(s/s_0)^{1/2})}{\pi(s/s_0)^{1/2}}\right|\right]. \quad \text{(A5)}$$

For $s\to\infty$, $\ln(\ )\sim\ln s$ and therefore (A5) is of order $\ln s$ also. Our conclusions of Sec. II are therefore unaffected.

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Coupled-Equations Method for the Scattering of Identical Particles. II

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The set of coupled, inhomogeneous equations describing the scattering of a fermion by a system of identical fermions, previously derived from the integral equation for scattering, is shown to follow directly from the Schrödinger equation and the symmetry properties of the exact scattering wave function.

INTRODUCTION

In a previous paper,[1] we derived a set of coupled, inhomogeneous differential equations that described the nonrelativistic scattering of a fermion (boson) by a system composed of identical fermions (bosons). The set of equations was obtained from the integral equation satisfied by the total scattering wave function. Our purpose in the present work is to give an alternative and simpler derivation of the main results of I. We use the Schrödinger equation rather than the integral equation to do this. This treatment enables us to avoid completely the use of certain factors arising from the re-arrangement collision nature of the scattering due to the Pauli principle which were encountered in I and which seem to have caused some confusion concerning the validity of the results of I. The present derivation is intended to clarify the situation, since these factors do not enter the discussion. Only elastic and inelastic scattering of single fermions is considered in the present work.

GENERAL COMMENTS

We follow the notation introduced in I, to which the reader is referred for details not given here. Let $\{\varphi_\alpha\}$ be a complete set of antisymmetrized states for the target, and let Ψ^A be the exact, antisymmetrized scattering wave function. Our goal in I was to expand Ψ^A via the complete set $\{\varphi_\alpha\}$ and then determine a set of coupled equations for the coefficients $\psi_\alpha\equiv\langle\varphi_\alpha|\Psi^A\rangle$, with a specific labeling for the φ_α's. We used the expansion

$$\Psi^A=\textstyle\sum_\alpha \varphi_\alpha[N]\psi_\alpha(N), \quad (1)$$

where $[i]=(1\cdots i-1,\ i+1,\ \cdots N)$ and (i) denotes a function of the coordinates of particle i only. By definition, ψ_α yields the proper scattering amplitude f_α, which is obtained from Ψ^A by first projecting Ψ^A onto the state φ_α and then requiring that $\langle\varphi_\alpha|\Psi^A\rangle$ have the asymptotic behavior $\langle\varphi_\alpha|\Psi^A\rangle\sim e^{i\mathbf{k}\cdot\mathbf{r}}\delta_{\alpha0}+f_\alpha e^{ik_\alpha r}/r$, where $\alpha=0$ denotes the ground state and k_α is the wave number corresponding to excitation of the state φ_α. We have not specified a labeling in φ_α since this is unnecessary due to the identity of the particles: the coordinate of any one of them may be in ψ_α, with the remainder in φ_α; we obtain the *same* scattering amplitude from each $\psi_\alpha(i)$. By choosing the labeling $[N]$, we have $\psi_\alpha(N)=\langle\varphi_\alpha[N]|\Psi^A\rangle$, and thus the expansion (1).

Equation (1) is a valid representation for Ψ^A because $\{\varphi_\alpha\}$ is complete, even though termwise (1) is not antisymmetric. This latter point is unimportant as long as the proper boundary conditions are imposed on the ψ_α in order to secure the correct amplitudes. Since Ψ^A is antisymmetric, we must have

$$P_{MN}\sum \varphi_\alpha[N]\psi_\alpha(N)=-\textstyle\sum_\alpha \varphi_\alpha[N]\psi_\alpha(N),$$

where P_{MN} is the two-body transposition operator.

[1] F. S. Levin, Phys. Rev. **140**, B1099 (1965). We refer to this work as I, and equations from it as (I-1), (I-2), etc. In both I and the present work, we assume that the target is infinitely heavy. This assumption is easily relaxed to include the case of recoil.

Because of this, it is clear that Ψ^A could also be written in the form

$$\Psi^A = N^{-1} \sum_\alpha \mathcal{A}\{\varphi_\alpha \psi_\alpha\},$$

where $\mathcal{A}$ is an antisymmetrizer. Although this latter form is explicitly antisymmetric, all N terms of the form $\sum_\alpha \varphi_\alpha[i]\psi_\alpha(i)$ simply add up to Eq. (1), which we prefer to use.

Clearly the ψ_α are of a special nature, since they reduce $N^{-1}\sum \mathcal{A}\{\varphi_\alpha\psi_\alpha\}$ to $\sum \varphi_\alpha[N]\psi_\alpha(N)$. The advantage of this choice of ψ_α over the F_α from the familiar expansion

$$\Psi^A = N^{-1} \sum_\alpha \mathcal{A}\{\varphi_\alpha F_\alpha\}, \quad (2)$$

with the F_α undetermined (and ambiguous), is that the *continuum exchange contributions* are automatically included in Eq. (1). These arise when one projects Ψ^A onto $\varphi_\alpha[N]$ using (2):

$$\psi_\alpha(N) = \langle \varphi_\alpha[N] | \Psi^A \rangle = F_\alpha(N) - (N-1) \sum_\beta \varphi_\beta[N] \times \langle \varphi_\alpha[N] | \varphi_\beta[N-1] F_\beta(N-1) \rangle .$$

Now, not only does F_α give rise to an amplitude, say $g_\alpha(\theta)$ resulting from $F_\alpha \sim \phi_k \delta_{\alpha 0} + g_\alpha(\theta) \exp(ik_\alpha r)/r$ [here $\phi_k = \exp(i\mathbf{k}\cdot\mathbf{r})$], but all *continuum* states φ_β will give an additional contribution which must be added to g_α to get the total amplitude. This latter contribution we denote as the continuum exchange contribution. *It does not occur when Eq. (1) is used.*

Since the expansion (2) is a formal one, and since the F_α's are not unique, it is always possible to choose them so as to avoid the continuum-exchange problem. Several methods for doing this have been discussed in the literature, but they eventually lead to equations for the ψ_α, or their equivalents. One method for doing this has been given by Feshbach.[2] In his procedure, a projection operator $\mathcal{P}$ is constructed such that $\mathcal{P}\Psi^A = \mathcal{A}\{\varphi_0 u_0\}$, where u_0 is to yield the entire elastic amplitude. Ψ^A is then of the form $\Psi^A = \mathcal{P}\Psi^A + Q\Psi^A$, with $\mathcal{P}\cdot Q = 0$; $Q\Psi^A$ is to give no flux asymptotically. If the terms in $Q\Psi$ are dropped in Feshbach's equation,[2] the resulting truncated equation obeyed by Ψ^A is the same as the equation obeyed by a truncated version of Eq. (2), and similarly for the case in which excited states φ_α, $\alpha \neq 0$, are included in $\mathcal{P}$.[3] However, the exact equation derived by Feshbach is for our ψ_0 [or for $\sum_{\alpha=0}^n \varphi_\alpha[N]\psi_\alpha(N)$ in the case of inelastic scattering] as shown explicitly in his paper.[2] A second method, discussed by Castillejo, Percival, and Seaton[4] and Hahn, O'Malley, and Spruch[5] for the case of e^-+H scattering, is to require that $\sum \langle \varphi_\alpha[N] | \varphi_\beta[N-1] F_\beta(N-1) \rangle = 0$. But then $F_\alpha \equiv \psi_\alpha$, and Eq. (2) is transformed[6] into Eq. (1), which again leads[7] to coupled equations for the ψ_α.

In each of the above approaches,[8] the equations derived for the ψ_α are homogeneous, while in our treatment, the equations are inhomogeneous. Despite the unfamiliarity of inhomogeneous equations, we pursue the approach of I, since on the one hand it automatically incorporates the continuum-exchange terms and on the other hand employs only the Schrödinger equation and the projection operators familiar from the case of a distinguishable projectile to derive the results. Another feature of our method is that the approximate amplitudes obtained by truncating the set of coupled inhomogeneous equations have a simple interpretation in terms of direct and exchange contributions. However, as with all such truncation procedures, we are unable to assess the accuracy or validity of the truncation.

SCHRÖDINGER-EQUATION DERIVATION

As we remarked in I, straightforward substitution of Eq. (1) into the Schrödinger equation $(E-H)\Psi^A = 0$ plus imposition of the boundary conditions[9] $\psi_\alpha \sim \phi_k \delta_{\alpha 0} + f_\alpha(\theta) \exp(ik_\alpha r)/r$ merely leads to the set of coupled equations for the case of N distinguishable. Obviously, this approach fails to include the symmetry of the problem. However, the symmetry can be introduced in a simple manner. We do this by exploiting the antisymmetry of Ψ^A and its asymptotic form. The boundary condition on Ψ^A implies that $\Psi^A = \Phi^A + \Psi_{SC}{}^A$, where Φ^A is an antisymmetric incident wave of the form[10]

$$\Phi^A = \sum_{n=1}^{N} (-1)^{N+n} \Phi_k ,$$

with $\Phi_k(n) = \varphi_0[n]\phi_k(n)$, and $\Psi_{SC}{}^A$ is an antisymmetric function having only outgoing waves when any coordinate is asymptotic. It is evident that $\Psi_{SC}{}^A$ must contain the amplitudes for the various two-body scattering processes and hence $\Psi_{SC}{}^A$ is the antisymmetric scattered wave. Now the unknown portion of Φ^A is just $\Psi_{SC}{}^A$. We may determine an equation for $\Psi_{SC}{}^A$ by solving not for Ψ^A but for $\Psi^A - \Phi^A$. Such a procedure is, of course, familiar from the case of the scattering of a

[2] H. Feshbach, Ann. Phys. (N. Y.) **19**, 287 (1962).

[3] The opposite claim was made in I, but this is clearly false. The equality follows, for example, from the equation $|\varphi_0[i]\rangle\langle\varphi_0[i]|\mathcal{P} = |\varphi_0[i]\rangle\langle\varphi_0[i]|$, as shown by Feshbach (Ref. 2).

[4] L. Castillejo, I. Percival, and M. J. Seaton, Proc. Roy. Soc. (London) **A254**, 259 (1960).

[5] Y. Hahn, T. O'Malley, and L. Spruch, Phys. Rev. **128**, 932 (1962); **134**, B397 (1964).

[6] For further discussion of the equations for the ψ_α in the e^-+H case, see F. S. Levin, Phys. Rev. **142**, 33 (1966).

[7] Alternatively, we could substitute Eq. (2) into the Schrödinger equation $(E-H)\Psi^A = 0$, solve for the F_α's and then impose the above condition. This would seem rather difficult to do in practice.

[8] An additional method for avoiding the continuum exchange problem is given by J. S. Bell and E. J. Squires [Phys. Rev. Letters **3**, 96 (1959)]. However, these authors work with eigenstates of a model Hamiltonian rather than with the true states of the target $\{\varphi_\alpha\}$. We shall discuss their approach to the problem elsewhere.

[9] Here, k_α is the reduced wave number corresponding to excitation of state φ_α; $k_0 \equiv k$.

[10] The extra phase factor $(-1)^N$ is included in Φ^A so that the component $\Phi_k(N)$, which appears explicitly in $\psi(N)$, enters with positive sign. A normalization factor has been removed from Φ^A, but this will cause no difficulty in calculating the amplitudes. A discussion of normalization is given in I.

particle by a potential well. Since we shall only be interested in the projection of Ψ^A onto the channels $\varphi_\alpha[N]$, we may detach the plane wave $\Phi_k(N)$ from Φ^A and solve instead for

$$\Phi_k(N)+\Psi_{SC}{}^A=\Psi^A-\sum_{n=1}^{N-1}(-1)^{N+n}\Phi_k(n).$$

As long as there are no open three-body channels, then $\Phi_k(n)$, $n\neq N$, gives no flux when $r_N\to\infty$; if three-body channels are open, then $\Phi_k(n)$ will contribute to the flux, but the contribution can always be treated in a formal manner. Note that this would *not* be a continuum-exchange contribution.

It should be clear that the above method incorporates the symmetry conditions into the problem. But, we can show that, in effect, this is how they were also included in I. This can be seen from Eq. (I-51), or equivalently, from Eq. (10a). If in these equations, we assume that the projection operator P includes all states φ_α, then P is effectively unity, Q is zero, and the equation reduces to[11]

$$(E-H)[\Psi^A+(N-1)\Phi_k(N-1)]=-(N-1)V(N-1)\Phi_k(N-1), \quad (3)$$

which is just an equation for $\Psi^A+(N-1)\Phi_k(N-1)$. The term $(N-1)\Phi_k(N-1)$ now appears rather than

$$\sum_{n=1}^{N-1}(-)^{N+n}\Phi_k(n),$$

since all exchange terms are identical when projected onto P, and there are a total of $(N-1)$ of them. If we write H as $H=H(n)+V(n)$ where $V(n)$ is the interaction of particle n with all other particles, and note that $[E-H(n)]\Phi_k(n)=0$, then (3) is easily shown to be an alternative form for $(E-H)\Psi^A=0$. Thus we see that Eq. (I-51), and others in I, are equations that incorporate the symmetry conditions in the way described above; we rederive (I-51) below as Eq. (10a).

We proceed as follows. Using

$$(E-H)\Phi^A=-\sum_n(-1)^{N+n}V(n)\Phi_k(n) \quad (4)$$

and subtracting Eq. (4) from $(E-H)\Psi^A=0$, we obtain

$$(E-H)[\Psi^A-\Phi^A]=\sum_{n=1}^{N}(-1)^{N+n}V(n)\Phi_k(n). \quad (5)$$

Now as mentioned above, rather than solve for $\Psi_{SC}{}^A=\Psi^A-\Phi^A$, we wish to determine $\Phi_k(N)+\Psi_{SC}{}^A$, which is the function of interest as long as only two-body channels are considered. Thus, we remove the $\Phi_k(N)$ term from (5), giving

$$(E-H)[\Psi^A-\sum_{n\neq N}(-1)^{N+n}\Phi_k(n)]=\sum_{n\neq N}(-1)^{N+n}V(n)\Phi_k(n). \quad (6)$$

This alternative form of the Schrödinger equation is our starting point.

The set of coupled equations is easily obtained from (6). We introduce the projection operators

$$P=\sum_{\alpha=0}^{n_0}P_\alpha[N],$$

and $Q=1-P$, with $P_\alpha[N]=|\varphi_\alpha[N]\rangle\langle\varphi_\alpha[N]|$ and φ_{n_0} an arbitrary bound state. Evidently,

$$P\Phi^A=\sum_{\alpha=0}^{n_0}\varphi_\alpha[N]\psi_\alpha(N).$$

Regardless of the value of n_0, we have $(P+Q)\Psi^A=\Psi^A$. Using this latter relation, we may write (6) as

$$P(E-H)(P+Q)[\Psi^A+(N-1)\Phi_k(N-1)]=-(N-1)PV(N-1)\Phi_k(N-1) \quad (7a)$$

and

$$Q(E-H)(P+Q)[\Psi^A+(N-1)\Phi_k(N-1)]=-(N-1)QV(N-1)\Phi_k(N-1) \quad (7b)$$

which is the desired result.

We first derive the equation for

$$P[\Psi^A+(N-1)\Phi_k(N-1)],$$

assuming, as we have done, that P contains only two-body bound states. To obtain the requisite equation, we must eliminate the terms $Q[\Psi^A+(N-1)\Phi_k(N-1)]$ from (7a). This may be done by solving (7b) for $Q[\Psi^A+(N-1)\Phi_k(N-1)]$, although some care is needed in carrying this out. A rearrangement of (7b) leads to

$$Q(E-H)Q[\Psi^A+(N-1)\Phi_k(N-1)]=QHP[\Psi^A+(N-1)\Phi_k(N-1)]-(N-1)QV(N-1)\Phi_k(N-1). \quad (8)$$

On the right-hand side of (8), we may write $QHP=QV(N)P$. However, we may not use $H=H(N)+V(N)$ on the left-hand side of (8) to differentiate term by term and write out a set of coupled equations. The reason for this is that we may not interchange the orders of summation (or integration) and differentiation in $H(N)Q\Phi_k(N-1)$: $H(N)Q\Phi_k(N-1)\neq QH(N)\Phi_k(N-1)$, even though the states in Q are eigenstates of $H(N)$. The simplest example that shows this is the case of two noninteracting particles, where it is easily proved that for any continuum state $\varphi_\alpha[N]$,

$$[E-H(N)]|\varphi_\alpha[N]\rangle\langle\varphi_\alpha[N]|\Phi_k(N-1)\rangle\neq|\varphi_\alpha[N]\rangle\langle\varphi_\alpha[N]|V(N-1)|\Phi_k(N-1)\rangle$$

[11] In Eq. (3), and also Eq. (10), we have dropped an irrelevant normalization factor of $N^{-1/2}$ that multiplies each $\Phi_k(n)$ in Eq. (I-51).

because of the surface terms that arise when the Schrödinger equation is used to evaluate $\langle\varphi_\alpha[N]| \times V(N-1)|\Phi_k(N-1)\rangle$.[12] A solution to (8) can only be obtained by treating $Q(E-H)^{-1}Q$ as a formal operator and inverting.

A formal solution to (8) can be written down as soon as the boundary conditions are specified. If we exclude the state $\varphi_0[N]$, then the boundary condition on Ψ^A when projected onto the remaining channels $\varphi_\alpha[N]$, $\alpha\neq 0$, is that it have only outgoing waves for $r_n\to\infty$. The requisite solution of (8) would then seem to be

$$Q[\Psi^A+(N-1)\Phi_k(N-1)]=Q(E^+-H)^{-1} \times Q\{V(N)P[\Psi^A+(N-1)\Phi_k(N-1)] -(N-1)V(N-1)\Phi_k(N-1)\}, \quad (9)$$

where $E^+=E+i\epsilon$, $\epsilon\to 0$, which guarantees the outgoing-wave boundary condition. That this is indeed the proper form also follows from the integral equation for Ψ^A, which may be shown to be (see I)

$$\Psi^A=\Phi^A+(E^+-H)^{-1}\sum(-1)^{N+n}V(n)\Phi_k(n).$$

Operating on both sides of this equation with Q, we find

$$Q[\Psi^A+(N-1)\Phi_k(N-1)] =Q(E^+-H)^{-1}\sum_n(-1)^{N+n}V(N)\Phi_k(n),$$

which suffices to show that (9) does in fact have the proper form. The ingoing waves are contained in $P\Psi^A$ only.

On substituting Eq. (9) into Eq. (7a) we find

$$P[E-H(N)-U]P[\Psi^A+(N-1)\Phi_k(N-1)] =-(N-1)P\bar{U}\Phi_k(N-1) \quad (10a)$$

and

$$P\Psi^A=Pu-(N-1)P\Phi_k(N-1) -(N-1)PGP\bar{U}\Phi_k(N-1), \quad (10b)$$

where

$$U=V(N)+V(N)Q(E^+-H)^{-1}QV(N),$$
$$\bar{U}=V(N-1)+V(N)Q(E^+-H)^{-1}QV(N-1),$$

and

$$PGP=P[E^+-H(N)-U]^{-1}P.$$

Apart from three differences, Eqs. (10a) and (10b) are identical to Eqs. (I-51) and (I-52). These differences are: (1) the normalization of $P\Psi^A$, (2) appearance of $Q(E^+-H)^{-1}Q$ rather than $[E^+-H(N)-QV(N)]^{-1}Q$ as the propagator in U and $\bar{U}$, and (3) the interpretation of the projection operators. Point (1) is trivial, as discussed in I. Point (2) is also trivial, since the presence of the operator P in (10a) and (10b) guarantees the equivalence of the two forms of propagator. Point (3), though not trivial, is easily disposed of. The operators P and Q can be regarded as diagonal n_0+2 by n_0+2 matrices, with P containing the elements $P_\alpha[N]$, $\alpha=0\cdots n_0$ in the first n_0+1 places and 0 in the last place, while Q is everywhere zero except for the last diagonal element, which is

$$\sum_{\alpha=n_0+1}^{\infty}P_\alpha[N].$$

If Ψ^A is written as an (n_0+2)-row column vector with the first n_0+1 rows containing $\varphi_\alpha[N]\psi_\alpha(N)$ and the last row containing

$$\sum_{\alpha=n_0+1}^{\infty}\varphi_\alpha[N]\psi_\alpha(N),$$

then the preceding derivation goes through exactly as above. [Φ_k is now also to be considered as a column vector whose only nonzero element is $\varphi_0[N]\phi_k(N)$ in the first row.] For this reason, the set of coupled equations given by (10a) is in fact identical to those of (I-51), which establishes the equivalence we wished to prove. The meaning of Pu, $P\Phi_k$, etc., is now precisely as given in I. Thus Pu, the solution to the homogeneous portion of Eq. (10a) asymptotic to $P\Phi_k(N)$ plus outgoing waves, is also the scattering-wave-function vector for distinguishable particle scattering. We follow this matrix notation in the remainder of this work.

As indicated in I, the amplitude $T_{k_\beta k}$ for scattering leaving the target in state φ_β is found from (10) to be

$$T_{k_\beta k}=\langle Pu_{k_\beta}^{(-)}|PUP|\Phi_k(N)\rangle -(N-1)\langle Pu_{k_\beta}^{(-)}|P\bar{U}P|\Phi_k(N-1)\rangle \equiv T^{\mathrm{dir}}-(N-1)T^{\mathrm{ex}}. \quad (11)$$

Here, $Pu_{k_\beta}^{(-)}$ is an ingoing-wave scattering function analogous to Pu of Eq. (10) containing a plane wave of momentum k_β (the wave number corresponding to excitation of state φ_β) in channel β only. We showed in I that the expressions for T^{dir} and T^{ex} of Eq. (11) are identical to the following more familiar forms:

$$T^{\mathrm{dir}}=\langle\Psi_{k_\beta}^{(-)}(N)|V(N)|\Phi_k(N)\rangle \quad (12a)$$

and

$$T^{\mathrm{ex}}=\langle\Psi_{k_\beta}^{(-)}(N)|V(N-1)|\Phi_k(N-1)\rangle, \quad (12b)$$

where $\Psi_{k_\beta}^{(-)}(N)$ is the ingoing-wave solution of $(E-H)\Psi_{k_\beta}^{(-)}(N)=0$ generated by a plane wave $\Phi_{k_\beta}(N)$ in which N is assumed to be distinguishable. It is clear by inspection that Eq. (6) also yields the same set of amplitudes T^{dir} and T^{ex}, thus providing a further justification for (6) as the starting point of our analysis.

In practice, of course, it is not possible to solve the above equations for the wave functions or the amplitudes and approximations must be employed. The approximation we consider here is that of truncating the set of coupled equations given by Eqs. (7a) and (7b). We assume that the set of states in P are those to be considered, and accordingly, we ignore the states in Q by setting $Q=0$. This leads to the following set of

[12] The author is indebted to C. F. Clement for pointing out the noninterchangability of the processes of differentiation and integration in this case.

equations for the approximate functions $P\Psi^P = P[\Psi^A + (n-1)\Phi_k(N-1)]$:

$$P[E-H]P\Psi^P = -(N-1)PV(N-1)\Phi_k(N-1), \quad (13)$$

which is the analog of the truncation procedure used in ordinary scattering.[13] The states $P\Phi_k(N-1)$ can be ignored since they give no flux.

By defining $\psi_\alpha' \equiv \langle \varphi_\alpha[N] | \Psi^P \rangle$, and taking the scalar product of both sides of Eq. (13) with $\varphi_\alpha[N]$, we find

$$[E-\epsilon_\alpha - T(N)]\psi_\alpha'(N) - \sum_{\beta=0}^{n_0} V_{\alpha\beta}(N)\psi_\beta'(N) = -(N-1)V_{\alpha k}(N), \quad \alpha = 0, \cdots, n_0 \quad (14)$$

with

$$V_{\alpha\beta}(N) = \langle \varphi_\alpha[N] | V(N) | \varphi_\beta[N] \rangle,$$

and

$$V_{\alpha k}(N) = \langle \varphi_\alpha[N] | V(N-1) | \Phi_k(N-1) \rangle.$$

Also, ϵ_α is the energy of state φ_α and $T(N)$ is the kinetic-energy operator for particle N. Comparing with the derivation of I, we see that Eqs. (14) and (I-55) are identical. Thus, although Eq. (I-55) is derived from Eq. (9) by dropping the terms in U and $\tilde{U}$ containing Q, unlike the derivation of (14) from (7), we see that the two methods lead to identical results. Either method yields a set of equations which treat the set of states in P exactly.

Let us consider now the solution to Eq. (13). We easily find that

$$P\Psi^P = Pu^P - (n-1)P\mathcal{G}PV(N-1)\Phi_k(N-1),$$

where Pu^P satisfies $P(E-H)Pu^P = 0$ and is the analog of the function Pu of Eq. (10), and $P\mathcal{G}P \equiv P(E^+ - H)^{-1}P$ is the full outgoing-wave Green's function in the space of the states retained in P. The amplitude $T_{k_\beta k}{}^P$ for exciting state φ_β is, in this approximation,

$$T_{k_\beta k}{}^P = \langle Pu_{k_\beta}{}^P | V(N) | \Phi_k(N) \rangle - (N-1)\langle Pu_{k_\beta}{}^P | V(N-1) | \Phi_k(N-1) \rangle, \quad (15)$$

where $Pu_{k_\beta}{}^P$ is the counterpart of the state $Pu_{k_\beta}{}^{(-)}$ introduced in Eq. (11). Comparing Eqs. (11) and (15), we see that (15) is an approximation to (11) obtained by the replacement $Pu_{k_\beta}{}^{(-)} \rightarrow Pu_{k_\beta}{}^P$. In other words, the accuracy of the solution to the homogeneous equation (the distinguishable projectile solution) *alone* determines the over-all accuracy of the amplitude. The calculation of the exchange amplitude only involves the assumption that $Pu_{k_\beta}{}^{(-)} \approx Pu_{k_\beta}{}^P$, and it is as accurately determined as the direct amplitude.

Hence by using the inhomogeneous equation, we obtain an approximation to the amplitude $T_{k_\beta k}$ in which the functions $u_{k_\beta}{}^{(-)}$ of Eqs. (12a) and (12b) may be calculated as accurately as possible. Thus, while on the one hand we have given up the explicit antisymmetry of an approximate wave function by choosing to work with Eqs. (13) or (14), rather than a truncated version of Eq. (1), on the other hand we obtain an approximate amplitude $T_{k_\beta k}{}^P$ that includes continuum-exchange contributions and whose relation to $T_{k_\beta k}$ of Eq. (11) is manifest. Our method therefore provides an amplitude in which the direct and exchange terms are treated on an equal footing, and further, in which the exact amplitude $T_{k_\beta k}$ is more closely approximated simply by increasing the number of states in P, thereby making Pu_{k_β} a better approximation to $\Psi_{k_\beta}{}^{(-)}$ of Eq. (12). But as noted in the Introduction, we do not have a means for determining the validity of any particular truncation. The establishing of such a criterion would be a major step in understanding the accuracy of the truncation approximation, both for our approach and for any other.

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[13] See, for example, N. F. Mott and H. S. W. Massey, *Theory of Atomic Collisions* (Oxford University Press, London, 1949).