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S-Matrix Method for Calculation of Radiative Correction for $\pi^+ \rightarrow \pi^0 e^+ \nu$

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Dashen and Frautschi's method of calculating the p-n mass difference is extended to the electromagnetic correction of πe_3 decay. The $\gamma \rightarrow 3\pi$ contribution is evaluated.

1. INTRODUCTION

A METHOD of calculating the p-n mass difference was proposed by Dashen and Frautschi,[1] and it has been applied in deriving mass differences of the octet and the decuplet baryons.[2] It is the purpose of this paper to extend this method to the evaluation of the electromagnetic correction in the weak processes. In this paper we restrict ourselves to $\pi^+ \rightarrow \pi^0 e^+ \nu$.

When the electromagnetic interaction is switched on, the vector current is no longer conserved, because of the π^+ and π^0 mass difference and the electromagnetic correction to the scattering processes. Evaluating the latter effect by the S-matrix method which is given in Ref. 1, we calculate the deviation from the universal Fermi coupling constant. It is assumed that the unrenormalized pion mass is equal to the neutral one.[3]

The radiative correction to πe_3 can be classified into two types of diagrams, which are illustrated in Fig. 1. In this paper we investigate the contribution from Fig. 1(b), because Fig. 1(a) can be approximated by a perturbation series.

2. CONSERVED-CURRENT LIMIT

In the lowest order in the weak coupling constant, the S-matrix element for the process $\pi^+ \rightarrow \pi^0 e^+ \nu$ is given by

$$\begin{aligned}\langle e^+(q),\nu(q'),\pi^0(k')|S|\pi^+(k)\rangle &= -i(2\pi)^4\delta(k-k'-q-q')f\langle e^+(q),\nu(q')|J_\mu^{(l)}(0)|0\rangle \\ &\quad\times\langle\pi^0(k')|J_\mu(0)|\pi^+(k)\rangle\,,\end{aligned} \quad (2.1)$$

where f is the coupling constant, and $J_\mu^{(l)}(0)$ and $J_\mu(0)$ are the lepton and the hadron current, respectively. In order to calculate the matrix element of Eq. (1), we define the following quantities:

$$\begin{aligned}\langle 0|J_\mu(0)|\pi^0(k');\pi^+(k)\rangle &= (2\pi)^{-3}[4\omega_{\pi^+}(k)\omega_{\pi^0}(k')]^{-1/2} \\ &\quad\times[f_+(s)(k-k')_\mu+f_-(s)(k+k')_\mu]\,,\end{aligned} \quad (2.2)$$

where $s=-(k+k')^2$, $f_\pm(s)$ are the πe_3 form factors, and $\omega_{\pi^+}(k)$ and $\omega_{\pi^0}(k)$ are the π^+ and π^0 total energy, respectively.

Let us calculate $\mathrm{Im}f_\pm(s)$ by assuming elastic unitarity and T invariance: Taking the two-pion intermediate state and averaging over incoming state and outgoing state, we have

$$\begin{aligned}&\{s-2(m_{\pi^+}{}^2+m_{\pi^0}{}^2)\}\,\mathrm{Im}[f_+(s)]-(m_{\pi^+}{}^2-m_{\pi^0}{}^2)\,\mathrm{Im}[f_-(s)] \\ &=\mathrm{Re}\bigg[\bigg\{-\frac{(m_{\pi^+}{}^2-m_{\pi^0}{}^2)^2}{s}a_0^*(s)+4p^2a_1^*(s)\bigg\}f_+(s) \\ &\qquad-(m_{\pi^+}{}^2-m_{\pi^0}{}^2)a_0^*(s)f_-(s)\bigg]\,,\end{aligned} \quad (2.3)$$

$$\begin{aligned}&(m_{\pi^+}{}^2-m_{\pi^0}{}^2)\,\mathrm{Im}[f_+(s)]+s\,\mathrm{Im}[f_-(s)] \\ &=\mathrm{Re}[(m_{\pi^+}{}^2-m_{\pi^0}{}^2)a_0^*(s)f_+(s)+sa_0^*(s)f_-(s)]\,,\end{aligned} \quad (2.4)$$

where $a_l(s)$ stands for the lth partial-wave amplitude for the process $\pi^+\pi^0 \rightarrow \pi^+\pi^0$, which is defined by the equations

$$\begin{aligned}\langle\pi^+(p),\pi^0(p')|S|\pi^+(k),\pi^0(k')\rangle &=\delta^3(\mathbf{p}-\mathbf{k})\delta^3(\mathbf{p}'-\mathbf{k}')+i\delta(p+p'-k-k') \\ &\quad\times[16\omega_{\pi^+}(p)\omega_{\pi^0}(p')\omega_{\pi^+}(k)\omega_{\pi^0}(k')]^{-1/2}M\,, \\ M&=-\frac{1}{\pi}\frac{\sqrt{s}}{p}\sum_{l=0}^{\infty}(2l+1)a_l(s)P_l(\cos\theta)\,,\end{aligned}$$

where $\cos\theta=(\mathbf{p},\mathbf{k})/\mathbf{p}^2$. Here $|\mathbf{k}|$ is the pion momentum in the center-of-mass system, i.e.,

$$\mathbf{p}^2=(s+m_{\pi^+}{}^2-m_{\pi^0}{}^2)^2/4s-m_{\pi^+}{}^2\,. \quad (2.5)$$

Let us now discuss the case when the electromagnetic interaction is switched off. Then the vector current is

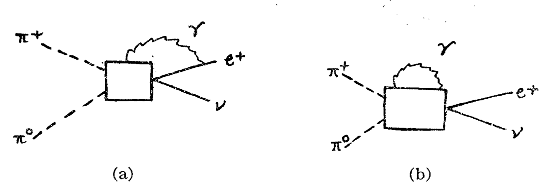

FIG. 1. Radiative correction to πe_3.

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[2] R. F. Dashen and S. C. Frautschi, Phys. Rev. **137**, B1331 (1965); **137**, B1318 (1965); F. J. Ernst, R. L. Warnock, and K. C. Wali, *ibid.* **141**, 1354 (1966).

[3] J. H. Wojtaszek, R. E. Marshak, and Riazuddin, Phys. Rev. **136**, B1053 (1964).

assumed to be conserved;

$$(k+k')_\mu\langle 0|J_\mu(0)|\pi^0(k'),\pi^+(k)\rangle=0. \quad (2.6)$$

That is to say,

$$(m_{\pi^+}{}^2-m_{\pi^0}{}^2)f_+{}^{(0)}(s)+sf_-{}^{(0)}(s)=0, \quad (2.7)$$

where $f_\pm{}^{(0)}(s)$ denote the form factors when the electromagnetic interaction is switched off. In this case $m_{\pi^+}=m_{\pi^0}$, and so

$$f_-{}^{(0)}(s)=0. \quad (2.8)$$

Equation (2.3) turns out to be

$$\mathrm{Im}[f_+{}^{(0)}(s)]=\mathrm{Re}[f_+{}^{(0)}(s)a_1{}^*(s)]. \quad (2.9)$$

We assume the once-subtracted dispersion relation for

$$f_+{}^{(0)}(s)=1+\frac{s}{\pi}\int_{4m_{\pi^0}{}^2}^{\infty}ds'\frac{\mathrm{Im}f_+{}^{(0)}(s')}{s'(s'-s)}. \quad (2.10)$$

Here it is assumed that the unperturbed pion mass is equal to the neutral pion mass and that the form factor is normalized as $f_+{}^{(0)}(0)=1$. Then we get the solution for Eqs. (2.9) and (2.10) by the standard method.

$$f_+{}^{(0)}(s)=\exp\left[\frac{s}{\pi}\int_{4m_{\pi^0}{}^2}^{\infty}ds'\frac{\Delta_1(s')}{s'(s'-s)}\right], \quad (2.11)$$

where

$$\Delta_1(s)=\frac{1}{2i}\ln\left[\frac{1+ia_1(s)}{1-ia_1{}^*(s)}\right]. \quad (2.12)$$

3. ELECTROMAGNETIC CORRECTION

We express the electromagnetic correction to the form factors $\delta f_\pm(s)$ as follows:

$$f_+(s)=f_+{}^{(0)}(s)+\delta f_+(s), \quad (3.1)$$

$$f_-(s)=\delta f_-(s). \quad (3.2)$$

Approximating Eqs. (2.3) and (2.4) to order α, we obtain the following equation for $\mathrm{Im}[\delta f_\pm(s)]$:

$$\mathrm{Im}[\delta f_+(s)]=\mathrm{Re}[a_1{}^*(s)\delta f_+(s)]+X(s), \quad (3.3)$$

$$\mathrm{Im}[\delta f_-(s)]=\mathrm{Re}[a_0{}^*(s)\delta f_-(s)]+Y(s), \quad (3.4)$$

where $X(s)$ and $Y(s)$ are expressed as

$$X(s)=\mathrm{Re}[f_+{}^{(0)}(s)\delta a_1{}^*(s)], \quad (3.5)$$

$$Y(s)=\frac{(m_{\pi^+}{}^2-m_{\pi^0}{}^2)}{s}\{\mathrm{Re}[a_0{}^*(s)f_+{}^{(0)}(s)]+\mathrm{Im}f_+{}^{(0)}(s)\}. \quad (3.6)$$

Here $\delta a_1(s)$ is the electromagnetic correction to P-wave π-π scattering. Unsubtracted dispersion relations are assumed for $\delta f_\pm$, namely,

$$\delta f_\pm(s)=\frac{1}{\pi}\int_{m_{\pi^0}{}^2}^{\infty}ds'\frac{\mathrm{Im}[\delta f_\pm(s')]}{s'-s}. \quad (3.7)$$

As is mentioned in the Appendix, the solution of Eqs. (3.4), (3.5), (3.6), and (3.7) are given by

$$\delta f_+(s)=\frac{f_+{}^{(0)}(s)}{\pi}\int_{m_{\pi^0}{}^2}^{\infty}ds'\,\mathrm{Re}\left[\frac{\delta a_1(s')}{1+ia_1(s')}\right]\frac{1}{s'-s}, \quad (3.8)$$

$$\delta f_-(s)=\frac{Z(s)}{\pi}\int_{m_{\pi^0}{}^2}^{\infty}ds'\frac{Y(s')}{(1+ia_0(s'))Z(s'-i\epsilon)}\frac{1}{s'-s}, \quad (3.9)$$

where $Z(s)$ is given by

$$Z(s)=\exp\left[\frac{s}{\pi}\int_{4m_{\pi^0}{}^2}^{\infty}ds'\frac{\Delta_0(s')}{s'(s'-s)}\right], \quad (3.10)$$

$$\Delta_0(s')=\frac{1}{2i}\ln\left[\frac{1+ia_0(s')}{1-ia_0{}^*(s')}\right]. \quad (3.11)$$

In order to evaluate δa_1, we adopted Dashen and Frautschi's method. Expressing $A_i(s)$ by the equation

$$A_i(s)=\rho a_i(s)=N_i(s)/D_i(s), \quad (i=0,1), \quad (3.12)$$

where $\rho=(\sqrt{s})/|\mathbf{p}|$, we have

$$\delta A_i(s)=\frac{1}{D_i{}^2(s)}\frac{1}{\pi}\left[\int_L ds'\frac{D_i{}^2(s')\,\mathrm{Im}\delta A_i(s')}{s'-s}+\int_R ds'\frac{\mathrm{Im}\{D_i{}^2(s')\delta A_i(s')\}}{s'-s}\right]. \quad (3.13)$$

Here R and L stand for the right-hand cut and the left-hand cut, respectively. According to Dashen and Frautschi, the right-hand-cut contribution is given as

$$\mathrm{Im}\{D_i{}^2(s)\delta A_i(s)\}=-|D_i|^2\delta I_i+\tfrac{1}{2}\,\mathrm{Re}(D_i{}^2\delta\rho), \quad (3.14)$$

where $I_i=\frac{1}{2}\rho|e^{2i\delta_i}|$, δ_i being the π-π phase shift.

As an application of our formula, we evaluate the contribution of $\gamma\to 3\pi$ to πe_3 decay, which is illustrated in Fig. 2. The effective Hamiltonian is assumed to be

$$H_I=\frac{ie\lambda}{m_\pi{}^3}\epsilon_{\lambda\mu\nu\rho}A_\lambda(x)\frac{\partial\varphi_1}{\partial x_\mu}\frac{\partial\varphi_2}{\partial x_\nu}\frac{\partial\varphi_3}{\partial x_\rho}, \quad (3.15)$$

where φ_i stand for π-meson field and λ is a coupling constant. The contribution from f_- to πe_3 decay is

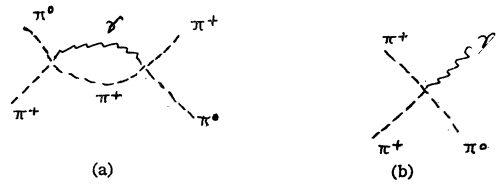

FIG. 2. (a) Left-hand cut. (b) Right-hand cut.

negligible compared with f_+, because δf_- is proportional to $m_{\pi^+}^2-m_{\pi^0}^2$, and besides it is multiplied by the electron mass in the transition probability.

By taking $\lambda=-0.732$,[4] we obtain

$$\delta f_+(0)=-5\times 10^{-4}. \tag{3.16}$$

Here $D_1(s)$ is approximated as

$$D_1(s)=[s_\rho-s-i\gamma_\rho\theta(s-4)]/s_\rho$$

where $\gamma_\rho=5.0$ and $s_\rho=32$ (the neutral pion mass is taken as a unit). The result is quite sensitive to the cutoff; $\propto\Lambda^5$. Here we take Λ slightly larger than the ρ-meson mass; $\Lambda=36$ for the right-hand cut and $\Lambda=-28$ for the left-hand cut.

The contribution from Fig. 1(a) can be approximated by the perturbation and the result is[5]

$$(\tau-\tau_0)/\tau_0=-1.12\times 10^{-2}, \tag{3.17}$$

where τ and τ_0 are the mean lifetime with and without radiative corrections, respectively. Hence, the $\gamma\to 3\pi$ contribution for πe_3 decay is very small and it is in the opposite direction from the perturbation-theory result.

Although we specialized to $\pi^+\to\pi^0e^+\nu$, our treatment is clearly of general applicability. By means of it we may investigate all effects of broken symmetry in weak interactions.

ACKNOWLEDGMENT

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[5] Ngee-Pong Chang, Phys. Rev. **131**, 1272 (1963).

APPENDIX

Let us give Muskhelishvili's method[6] of solving the integral equation which is given in Sec. 3;

$$\mathrm{Im}[\delta f_+(s)]=\mathrm{Re}[a_1^*(s)\delta f_+(s)]+X(s),\quad (s\in L) \tag{A1}$$

$$\delta f_+(s)=\frac{1}{\pi}\int_L ds'\frac{\mathrm{Im}[\delta f_+(s')]}{s'-s}, \tag{A2}$$

where $X(s)$ and a_1 are arbitrary functions and L is an interval on the real axis. We assume that $f_+^{(0)}$ is a solution of the following equation:

$$\mathrm{Im}[f_+^{(0)}(s)]=\mathrm{Re}[a_1^*(s)f_+^{(0)}(s)],\quad (s\in L') \tag{A3}$$

$$f_+^{(0)}(s)=1+\frac{s}{\pi}\int_{L'} ds'\frac{\mathrm{Im}f_+^{(0)}(s')}{s'(s'-s)}, \tag{A4}$$

where L' is an interval on the real axis. The solution of Eqs. (A3) and (A4) is given by Eqs. (2.11) and (2.12). Eliminating a_1^* from Eqs. (A1) and (A2), we have

$$\frac{\delta f_+(s+i\epsilon)}{f_+^{(0)}(s+i\epsilon)}-\frac{\delta f_+(s-i\epsilon)}{f_+^{(0)}(s-i\epsilon)}=2i\frac{X(s)}{\{1+ia_1(s)\}f_+^{(0)}(s-i\epsilon)},\quad (s\in L). \tag{A5}$$

Therefore, $\delta f_+/f_+^{(0)}$ is expressed as

$$\delta f_+(s)/f_+^{(0)}(s)=P(s)+\frac{1}{\pi}\int_L ds'\frac{X(s')}{\{1+ia_1(s)\}f_+^{(0)}(s-i\epsilon)}\frac{1}{s'-s}, \tag{A6}$$

where $P(s)$ is an arbitrary polynomial. We may take $P(s)=0$, because $\delta f_+(s)\to 0$ $(s\to\infty)$ by Eq. (A2).

[6] N. I. Muskhelishvili, *Singular Integral Equations* (P. Noordhoff Ltd., Groningen, The Netherlands, 1953).