A direct proof that $SU(3)'$ singlets have lower energies than states belonging to other representations of $SU(3)'$ runs as follows. First, we note that the interaction Hamiltonian

$$\mathcal{H}_i = -\mathcal{L}_i = GJ_m{}^{n\alpha}J_n{}^m{}_\alpha\,, \qquad (12)$$

is non-negative. In a semiclassical theory, it vanishes only if $J_m{}^{n\alpha}$ does, i.e., for $SU(3)'$ singlets. In a quantized theory, its expectation value can be written (dropping indices)

$$\langle\mathcal{H}_i\rangle = G\langle J^2\rangle = G[\langle J\rangle^2 + (\Delta J)^2]\,, \qquad (13)$$

where the last equation defines the "uncertainty" ΔJ. For reasonably localized and well separated particles (or aggregates of particles) ΔJ is of the order of the overlap of the wavefunctions, and can be neglected. Hence, once more the condition for lowest energy is $\langle J_m{}^{n\alpha}\rangle = 0$, so that each well separated aggregate of particles must be a $SU(3)'$ singlet.

Note added in proof. Other approaches to the saturation problem have recently been discussed by O. W. Greenberg and D. Zwanziger (unpublished).

ACKNOWLEDGMENTS

I am very much indebted to Professor Harry J. Lipkin and Professor Yuval Ne'eman for helpful criticism.

PHYSICAL REVIEW VOLUME 149, NUMBER 4 30 SEPTEMBER 1966

Dynamical Calculation of Extinct Bound States*

D. Atkinson and M. B. Halpern
Department of Physics, University of California, Berkeley, California

(Received 7 March 1966)

Aspects of the possible dynamical extinction of a bound state (coincident zero in N and D) are discussed within the single and multichannel ND^{-1} framework. A surprising result is that, within any one-channel calculation, the "mass" of the extinct bound state cannot be calculated. This may be reflected in the multichannel formulation by a particular sensitivity of the "mass" to the input.

I. INTRODUCTION

In a recent paper, Chew[1] has conjectured, on the basis of theoretical and empirical arguments, that the second-rank Pomeranchuk trajectory (P'), and perhaps the P trajectory itself, crosses the $J=0$ axis at negative s (energy squared). This could give rise to $\pi\pi$-$\pi\pi$ ($T=J=0$) real parts of phase shifts at infinity of $-\pi(P')$ or $-2\pi(P,P')$ [in the absence of Castillejo-Dalitz-Dyson (CDD) poles]. The crossing of the trajectories in this region corresponds to zeros in the S-wave D function which, if unmodified, would violate unitarity in the cross channel. To preserve unitarity, it is hoped that, at the crossing energy, the N function develops a zero as well. In that such a bound state will no longer be associated with a pole, we shall refer to the state as an extinct bound state (EBS). The state might alternatively be described as a scalar bound state of space-like mass (presumably with velocity greater than that of light) whose renormalized coupling back into the theory has been dynamically damped to zero. (Nevertheless, the EBS can affect phase shifts, etc.)

In this paper, the possibility of the dynamical generation of such an extinct bound state is considered within the single and multi-channel ND^{-1} framework. In particular, we shall focus our attention on the question of whether or not the location of the EBS can be determined from the ND^{-1} equations.

In Sec. II, the general solution of the problem in the one-channel case is found: We exhibit the complete family of input left-hand cuts that will lead to coincident zeros in N and D. We find the rather bizarre circumstance that, assuming an EBS can be formed, it is not possible (within the one-channel framework) to determine its location. This happens because the solutions N and D are not unique (in the presence of an EBS), there being in fact a one-parameter family of solutions, the parametrization of which corresponds to the location of the EBS. The source of the nonuniqueness resides in the fact that, in the presence of an EBS, the associated homogeneous N/D equations always admit one solution, a constant multiple of which can be added to any particular solution. In spite of this nonuniqueness, the ratio N/D is unique, and the phase shift goes to $-\pi$ at infinity (if there is only one EBS).

In Sec. III we consider the two-channel case. It is found that the location of the EBS can sometimes be determined dynamically and sometimes not. When the location can be so determined, the equivalent one-channel problems[2] (including inelasticity) for the di-

* This work was supported in part by the Air Force Office of Scientific Research, Grant No. AF-AFOSR-232-63.

[1] G. F. Chew, Phys. Rev. Letters **16**, 60 (1966); Phys. Rev. **140**, B1427 (1965). Squires and Watson have presented a two-pole left-hand cut model of an extinct bound state [E. J. Squires and P. Watson, Nuovo Cimento **42**, 77 (1966)].

[2] E. J. Squires, Nuovo Cimento **34**, 1751 (1964); M. Bander, P. W. Coulter, and G. L. Shaw, Phys. Rev. Letters **14**, 270 (1965); D. Atkinson, K. Dietz, and D. Morgan, Ann. Phys. (N. Y.) **37**, 77 (1966).

agonal matrix elements break into two cases: (a) There is no CDD pole and there is an EBS of undeterminable position. (b) There is a CDD pole and the location of the EBS can be determined only in terms of the CDD parameters. The phase shifts at infinity for these two cases are $-\pi$ and 0, respectively.

At the end of Sec. III, we will discuss the reflection of the one-channel difficulties in the many-channel case, and make it clear that there is nothing miraculous about going to many channels: where the one-channel equations are singular (which results in inability to locate an EBS), the multi-channel equations may be ill-conditioned (resulting in a particular sensitivity of the location to input).

If one assumes the existence of an $SU(3)$ nonet of EBS's (a Pomeranchuk singlet and an octet, the singlet member of which is the P')—which would be the Regge recurrence of the 2^+ nonet—then arguments similar to those of Ref. 2 indicate that the real part of the phase-shift for $\pi\pi\rightarrow\pi\pi$ scattering tends to $-\pi$ at infinite energy (this is also the case for $K\bar{K}\rightarrow K\bar{K}$).[3] This is in spite of the fact that *both* the P and P' trajectories are assumed to cross $J=0$ in both systems. In fact, under these assumptions, there is one CDD pole in each of the equivalent one-channel amplitudes.

It should be mentioned that, although the P and the P' are presumably EBS's of negative mass squared, all our considerations apply equally to EBS's of any mass.

II. ONE-CHANNEL SYSTEM

Suppose that the scattering amplitude satisfies

$$A=N/D\,, \tag{2.1}$$

where

$$N(s)=\frac{1}{\pi}\int_{-\infty}^{0}\frac{\alpha(s')D(s')ds'}{s'-s}\,,$$

and

$$D(s)=1-\frac{1}{\pi}\int_{\hat{s}}^{\infty}\frac{\rho(s')N(s')ds'}{s'-s}. \tag{2.2}$$

Here $\alpha(s)$ is the left-hand-cut discontinuity, $\rho(s)$ is the relativistic phase-space factor, and $\hat{s}$ is the normal threshold.

We prove the following theorem: The necessary and sufficient condition for N and D to have a set of n coincident zeros (zeros of N and D of equal order), say

$$N(s_0)=N(s_1)=\cdots=N(s_{n-1})=D(s_0)=D(s_1)=\cdots=D(s_{n-1})=0\,, \tag{2.3}$$

is that the input should have the form (for $-\infty<s\leqq 0$)

$$\alpha(s)=-\mathrm{Im}\eta(s)\Big/\frac{1}{\pi}\int_{\hat{s}}^{\infty}\frac{ds'\rho(s')\eta(s')}{s'-s}\,, \tag{2.4}$$

where $\eta(s)$ is a real-analytic function with a left-hand cut, for which the "normalization" integral on the right,

$$\frac{1}{\pi}\int_{0}^{\infty}\eta(s')\rho(s')ds'\equiv\mu\,, \tag{2.5}$$

exists.

We exhibit the proof for the case of just one coincident simple zero, since this is of immediate physical interest.[1] This situation is ensured by requiring $\mu\neq 0$ in (2.5). At the end of this section, we shall state the further requirements on $\eta(s)$ (and the results) for several coincidences and multiple zeros.

Proof

We prove the sufficiency explicitly. The necessity follows essentially by reversing the order. Suppose that a function $\eta(s)$ is given for which μ (in Eq. (2.5) does not vanish. Construct

$$N(s)=\eta(s)\quad(s-s_0)/\mu\,,$$

and

$$D(s)=1-\frac{1}{\pi\mu}\int_{\hat{s}}^{\infty}\rho(s')\eta(s')\frac{s'-s_0}{s'-s}ds'\,. \tag{2.6}$$

These functions satisfy the N/D equations (2.2), with the input (2.4). Moreover, since (2.6) can be rewritten

$$D(s)=-\frac{s-s_0}{\pi\mu}\int_{\hat{s}}^{\infty}\frac{\rho(s')\eta(s')}{s'-s}ds'\,, \tag{2.7}$$

where we have used Eq. (2.5), it is clear that N and D each have a zero at $s=s_0$. Note however, that all the dependence on s_0 cancels out of the ratio N/D so that the amplitude, as well as its imaginary part on the left, is s_0 independent.

Discussion

The structure (2.4) is the most general input capable of generating an EBS. In particular, given some set of input functions η, we can generate a corresponding set of solutions (2.6). In that α is independent of s_0, we see that for each α there is an infinite set of solutions N and D, parametrized by s_0. This set is given by Eq. (2.6). Each pair (N,D) possesses coincident zeros, but the input does not contain the information necessary to locate the coincidence. This may be paraphrased: *It is not possible, in a one-channel N/D framework, to calculate dynamically the location of an* EBS.

Another way of understanding the multiplicity of solutions of the N and D equations, with inputs of the form (2.4), is to note that the homogeneous version of Eq. (2.2), namely

$$N_H(s)=\frac{1}{\pi}\int_{-\infty}^{0}\frac{ds'\alpha(s')}{s'-s}D_H(s')\,,$$

$$D_H(s)=-\frac{1}{\pi}\int_{\hat{s}}^{\infty}\frac{ds'\rho(s')}{s'-s}N_H(s')\,, \tag{2.8}$$

[3] D. Atkinson and M. B. Halpern, Phys. Rev. 149, 1133 (1966).

is soluble. One can show explicitly that the solution is

$$N_H(s)=\eta(s),$$

and

$$D_H(s)=-\frac{1}{\pi}\int_{\bar{s}}^{\infty}\frac{ds'\rho(s')}{s'-s}\eta(s'). \quad (2.9)$$

If one solution of the inhomogeneous system (2.2) has a coincident zero at $s=s_p$, the general solution is then

$$N(s)=(s-s_p)\eta(s)+\lambda\eta(s)=(s-s_p+\lambda)\eta(s),$$

and

$$D(s)=(s-s_p)D_H(s)+\lambda D_H(s)=(s-s_p+\lambda)D_H(s), \quad (2.10)$$

where $D_H(s)$ is given by Eq. (2.9), and λ is an arbitrary constant. Thus the coincident zero of the general solution occurs at $s=s_p-\lambda$, which is entirely arbitrary. The nonuniqueness arises from the ability to add to any inhomogeneous solution an arbitrary multiple of the homogeneous solution.

It is worth reiterating that, although the coincidence location is not determined, the ratio N/D is unique, and agrees with N_H/D_H. Because of this, the phase shift is unique: Indeed, because of the zero of D, the real part of the phase shift tends to $-\pi$ at infinite energy.[4]

Next, we mention other forms of the initial N/D equations. If there is some inelasticity, then Eq. (2.2) is replaced by

$$N(s)=\frac{1}{\pi}\int_{-\infty}^{0}\frac{ds'\alpha(s')D(s')}{s'-s},$$
$$D(s)=1-\frac{1}{\pi}\int_{\bar{s}_1}^{\infty}\frac{ds'R(s')\rho(s')N(s')}{s'-s} \quad (2.11)$$

where $R(s)$ is the inelasticity factor.[5] The general solution goes through as before, but with the substitution

$$\rho(s)\to R(s)\rho(s).$$

In particular, given $\alpha(s)$ and $R(s)$, the location of the EBS cannot be determined.

Again, if the equations are subtracted, or, equivalently, if a CDD pole is inserted into the D equation, then the general problem can be solved as before. However, the amplitude is not now independent of s_0, so that the location of an EBS could be determined from the N/D equations—but only in terms of the subtraction or CDD parameters, which would have to be specified in addition to $\alpha(s)$.

Relation to Fredholm Theory

Evidently we can construct both Fredholm and non-Fredholm EBS solutions, with appropriate functions $\eta(s)$. However, according to Fredholm theory, if a homogeneous Fredholm equation has a solution, then the inhomogeneous equation can only have a solution if the inhomogeneous term is orthogonal to any solution of the adjoint homogeneous equation.[6] Let us make certain that this orthogonality holds, so that there is no conflict with Fredholm theory.

In the case of a single EBS, there is only one independent solution of the adjoint homogeneous equation, which can be shown to be $\rho(s)\eta(s)$. The inhomogeneous integral equation for $N(s)$ is

$$N(s)=F(s)+\frac{1}{\pi}\int_{\bar{s}}^{\infty}ds'\frac{F(s')-F(s)}{s'-s}\rho(s')N(s'), \quad (2.12)$$

where

$$F(s)=\frac{1}{\pi}\int_{-\infty}^{0}ds'\frac{\alpha(s')}{s'-s}. \quad (2.13)$$

Evaluating this at $s=s_0$, we have

$$0=F(s_0)\left[1-\frac{1}{\pi\mu}\int_{\bar{s}}^{\infty}\rho(s')\eta(s')ds'\right]+\frac{1}{\pi}\int_{\bar{s}}^{\infty}ds'F(s')\rho(s')\eta(s').$$

The term in square brackets vanishes, by Eq. (2.5), so the orthogonality between the inhomogeneous term, F, and $\rho\eta$, is proven. (Note that the orthogonality holds even when the equation is not Fredholm.)

Simple examples of N/D systems with coincident zeros may be easily generated by taking simple forms of $\eta(s)$. For example, in the case

$$\eta(s)\propto 1/(s+s_1)-1/(s+s_2) \quad (s_1,\ s_2 \text{ on the left}),$$

one can easily work through all the steps above analytically. (This corresponds to taking a two-pole left cut, but with particular residues.) Note that a one-pole form for η (and correspondingly for α) does not generate an EBS—because μ [Eq. (2.5)] is not finite.

Several EBS and Higher Order Coincident Zeros

The requirement that $\mu\neq 0$ in Eq. (2.5) actually specifies the case of one coincident zero of first order. For two first-order zeros, or the degenerate case of one coincident second-order zero, the necessary and sufficient constraints on $\eta(s)$ are

$$\frac{1}{\pi}\int_{\bar{s}}^{\infty}\eta(s')\rho(s')ds'=0,$$

and (2.14)

$$\frac{1}{\pi}\int_{\bar{s}}^{\infty}\eta(s')s'\rho(s')ds'\equiv\kappa\neq 0.$$

[4] R. L. Warnock, Phys. Rev. **131**, 1320 (1963); D. Atkinson and D. Morgan, Nuovo Cimento **41**, 559 (1966).
[5] G. F. Chew and S. Mandelstam, Phys. Rev. **119**, 467 (1960).
[6] E. C. Titchmarsh, *The Theory of Functions* (Oxford University Press, New York, 1950).

The most general N and D functions are

$$N(s)=1/\kappa[(s-s_1)(s-s_2)\eta(s)+\lambda_1(s-s_1)\eta(s)+\lambda_2(s-s_2)\eta(s)],$$

$$D(s)=-[(s-s_1)(s-s_2)+\lambda_1(s-s_1)+\lambda_2(s-s_2)]\times\frac{1}{\kappa\pi}\int_s^\infty\frac{ds'\rho(s')\eta(s')}{s'-s}, \quad (2.15)$$

where λ_1 and λ_2 are arbitrary. (There are now two solutions of the homogeneous equations.) The ratio N/D and, in particular, the input $\alpha(s)$ are independent of s_1, s_2, and κ. Thus, again, the location of the zeros is not dynamically determinable.

In the case of n EBS's, the first n-1 moments of $\eta\rho$ must be set to zero, and the nth moment must not vanish. In general one concludes that, *in the one-channel case, coincident zeros (independently of their order or multiplicity) cannot be determined dynamically.*

III. MANY-CHANNEL CALCULATION

While the one-channel discussion was complete, no attempt at generality will be made in the many-channel case. We shall be satisfied to show that, although there are situations in which the coincident zero location cannot be dynamically determined in a two-channel problem, there are also instances in which it can be so determined.

The decomposition of a many-channel amplitude A_{ik} is defined by

$$A_{ik}=(ND^{-1})_{ik}=N_{ij}\Delta_{jk}/\det D \quad \text{(summation convention)}, \quad (3.1)$$

where Δ is the adjoint of the matrix D, and $\det D$ is its determinant. The condition for a coincident zero relating to, say, A_{11}, is

$$N_{1j}(s_0)\Delta_{j1}(s_0)=0 \quad \text{and} \quad \det D(s_0)=0. \quad (3.2)$$

Clearly, it is not necessary for any particular $N_{ij}(s_0)$ or $D_{ij}(s_0)$ to vanish. However, a subclass of Eq. (3.2) will certainly be given by

$$N_{ij}(s_0)=0 \text{ for all } i, j, \quad \text{and} \quad \det D(s_0)=0. \quad (3.3)$$

In order to simplify the manipulations, we shall limit our discussion entirely to the subclass (3.3).

The many-channel equations are

$$N_{ik}(s)=\frac{1}{\pi}\int_{-\infty}^0\frac{ds'}{s'-s}\alpha_{ij}(s')D_{jk}(s'), \quad (3.4)$$

$$D_{ik}(s)=\delta_{ik}-\frac{1}{\pi}\int_{\hat{s}}^\infty\frac{ds'}{s'-s}\rho_{ij}(s')N_{jk}(s'), \quad (3.5)$$

where $\rho_{ij}(s)=\delta_{ij}[(s-\hat{s}_j)/s]^{1/2}$, and $\hat{s}_j$ is the jth threshold. Following our approach in the one-channel case, we introduce a matrix $\eta(s)$, with a (matrix) left-hand cut, and construct the N matrix in terms of it:

$$N_{ij}(s)=(s-s_0)\eta_{ij}(s). \quad (3.6)$$

The D matrix is constructed in terms of N by Eq. (3.5). It will be convenient to define

$$\hat{\eta}_{ik}\equiv\frac{1}{\pi}\int_{\hat{s}_1}^\infty ds'\rho_{ij}(s')\eta_{jk}(s') \quad (3.7)$$

so that

$$D_{ik}(s_0)=\delta_{ik}-\hat{\eta}_{ik}. \quad (3.8)$$

If we specialize to the two-channel case, the condition $\det D(s_0)=0$ becomes

$$(1-\hat{\eta}_{11})(1-\hat{\eta}_{22})-\hat{\eta}_{12}\hat{\eta}_{21}=0. \quad (3.9)$$

Time-reversal invariance,

$$A_{ij}(s)=A_{ji}(s), \quad (3.10)$$

is guaranteed if we choose for η the form

$$\eta=\begin{pmatrix}\eta_{11} & \eta_{12}\\ \eta_{12} & \eta_{11}\end{pmatrix}. \quad (3.11)$$

Then Eq. (3.9) reduces to

$$\hat{\eta}_{12}=\pm(1-\hat{\eta}_{11}). \quad (3.12)$$

If $\hat{\eta}_{11}=1$, so that $\hat{\eta}_{12}=0$, then each matrix element $D_{ik}(s_0)$ vanishes separately. An analysis similar to that for the one-channel case shows that all s_0 dependence factors out of the inputs, and thus the coincident zero location cannot be determined from the two-channel equations. This is completely analogous to the one-channel case.

However, if $\hat{\eta}_{11}\neq1$, so that $\hat{\eta}_{12}\neq0$, then $D_{ik}(s_0)\neq0$, and it can be shown easily that the coincident zero of $N_{ik}\Delta_{kj}$ and of $\det D$ does not factor out in a trivial way. In particular, the inputs (the discontinuities across the left-hand cuts) remain s_0-dependent. Thus the location of the EBS can be determined from the two-channel equations.

If a pseudo-one-channel calculation were to be set up in this case, in the sense that the same input $\alpha_{11}(s)$, and an equivalent inelasticity, defined by

$$R(s)=1+\left|\frac{A_{12}}{A_{11}}\right|^2\left(\frac{s-s_2}{s-s_1}\right)^{1/2} \quad (3.13)$$

were used, then, in order to reproduce A_{11} it would be necessary in some cases to introduce a CDD pole. Detailed conditions under which this would, or would not be necessary can be found in Ref. 2. We shall consider only two of the possible two-channel phase-shifts at infinity:

(1) $\mathrm{Re}\delta_{11}(\infty)=-\pi$

The equivalent one channel N and D can be constructed in terms of

$$\mathfrak{D}_{11}(s)=\exp\left[-\frac{s}{\pi}\int_{\hat{s}}^{\infty}\frac{ds'\,\mathrm{Re}\delta_{11}(s')}{s'(s'-s)}\right] \tag{3.14}$$

according to

$$\begin{aligned} D_1(s)&=(s-s_0)\mathfrak{D}_{11}(s)\\ N_1(s)&=(s-s_0)\mathfrak{D}_{11}(s)A_{11}(s). \end{aligned} \tag{3.15}$$

There is no one-channel CDD pole, but there is a coincident zero (at s_0). However, its position is arbitrary: It need not coincide with the location derived from the two-channel calculation. None the less, the ratio N/D agrees with A_{11}.

(2) $\mathrm{Re}\delta_{11}(\infty)=0$

In this case, a particular one-channel decomposition is

$$\begin{aligned} D_1(s)&=\frac{s-s_0}{s-s_c}\mathfrak{D}_{11},\\ N_1(s)&=\frac{s-s_0}{s-s_c}\mathfrak{D}_{11}A_{11} \end{aligned} \tag{3.16}$$

where there is a coincident zero at s_0 and a CDD pole at s_c. The EBS position can be calculated, since the CDD pole prevents the disappearance of the factor $(s-s_0)$ from the ratio N/D. However, the two CDD parameters need to be specified in addition to α_{11} and R. An alternative N/D decomposition exists in this case with no coincident zero, nor a CDD pole, namely

$$\begin{aligned} D_1'(s)&=\mathfrak{D}_{11}(s)\\ N_1'(s)&=\mathfrak{D}_{11}(s)A_{11}(s). \end{aligned} \tag{3.17}$$

Needless to say, the coincident zero cannot be calculated from N'/D', since it has disappeared without trace. Note that, in all the cases (3.15), (3.16), (3.17), the localized EBS of the two-channel problem has been "folded" into the N function of the one-channel problem, in that A_{11} is a factor of N_1. In this sense then the location of the double zero in the one-channel problem (if any) is disconnected from the location of the coincidence in the two-channel calculation.

IV. SUMMARIZING DISCUSSION

It might seem that there is a fundamental difference between the one-channel case, in which the EBS location cannot be calculated, and the two-channel case, in which it can often be calculated. However, we shall see that, when the second channel is very weak, the determination is poor, in the sense that the computed location is very sensitive to any small perturbation of the inputs. In this sense, there is nothing miraculous about the many-channel formulation—the difficulties of the one-channel case are strongly reflected in the many-channel case.

To understand this, we recall that the one-channel N integral equation is singular, i.e., the kernel $K(x,x')$ has eigenvalue unity. In the two-channel equations the kernel is a matrix $K_{ij}(x,x')$, with $K_{11}(x,x')\equiv K(x,x')$. This matrix integral equation can be reduced to a new single integral equation by a standard procedure[7] in which the norm of the new kernel is the sum of the norms of the matrix elements $K_{ij}(x,x')$. If the left-hand discontinuities $\alpha_{12}(x)$, $\alpha_{21}(x)$, and $\alpha_{22}(x)$ are small compared with $\alpha_{11}(x)$, i.e., the norms of K_{12}, K_{21}, K_{22} are small compared with that of K_{11}, then it follows easily that the new kernel has an eigenvalue close to unity. In other words, the two-channel equations are ill-conditioned, in the mathematical sense. (That is, the solution will be very sensitive to small perturbations of the inputs—because the Fredholm resolvent kernel has a nearby pole in the eigenparameter plane.) Evidently, this sensitivity is a strong dependence of the EBS location on the inputs.

In summary then, we have learned that the location of an EBS cannot be determined dynamically within a one-channel calculation (because the equations are singular), and that this is reflected in a possible sensitivity of the location of the EBS to small changes of input in a many-channel calculation (because the many-channel equations are in general ill-conditioned). Imagine for example that a π-π input could be found of the form (2.4), and an attempt were made to fix the location of the resulting EBS by the addition of a weak channel (i.e., some channel which has little to do with the binding of the EBS). Then one could not trust the resulting location unless the inputs for both strong and weak channels were very accurately given. On the other hand, of course, one hopes that the physically relevant multichannel equations will not be badly ill-conditioned, and that the resulting location of the EBS will be relatively accurate.

ACKNOWLEDGMENTS

We would like to thank Professor G. F. Chew for encouragement in this work, and Professor S. Weinberg for a helpful remark.

[7] F. G. Tricomi, *Integral Equations* (Interscience Publishers Inc., New York, 1957).