14. Also, our considerations are more exhaustive.

We finally mention that a calculation of strong coupling constants has been recently made by Dashen *et al.* in an entirely different investigation of broken symmetry.[15]

Other related questions, particularly regarding the form factors $K_{12\alpha}(k^2)$, will be discussed elsewhere.

[15] R. F. Dashen, Y. Dothan, S. C. Frautschi, and D. H. Sharp, Phys Rev. 143, 1185 (1966).

The author is grateful to Professor E. C. G. Sudarshan for comments and for reading through the manuscript; to Dr. S. Pakvasa for a detailed discussion of Cabibbo's theory and for information about recent data on leptonic decays; to Dr. Y. Hara for a discussion of the PV coupling; to Professor J. Leitner for comments on the experimental data on leptonic decays; and to Dr. N. Mukunda for calling our attention to Riazuddin's work.

PHYSICAL REVIEW VOLUME 149, NUMBER 4 30 SEPTEMBER 1966

Combined Spin and G_2 Symmetry

C. R. Garibotti*
Palmer Laboratory, Princeton University, Princeton, New Jersey
(Received 22 March 1966)

The algebra of the group D_5 is the most direct generalization of that of the direct product of G_2 with the ordinary spin group SU_2. Furthermore it also contains that of R_7 times SU_2. In this paper we analyze the particle classification provided by D_5; it is found that a 16-dimensional representation appears to contain the baryons and gives results for the magnetic moment and mass formula equivalent to those obtained with the nonbroken $G_2\otimes SU_2$ group, but now the Λ particle is directly included. The 45-dimensional regular representation of D_5 provides an appropriate basis for the proposed $G_2\otimes SU_2$ representations of the mesons and boson resonances.

I. INTRODUCTION

THE idea of using the group SU_4 to classify nuclear states has been extended to the elementary-particle case, using SU_3 as the internal symmetry group. This has led to the study of SU_6.[1] There is another group that can be used to describe the particle internal symmetry, G_2. It is more difficult to work out than SU_3, but recent work[2] shows that its results are not less physical. In this paper we want, in analogy with SU_4 and SU_6, to consider a group whose Lie algebra embodies those of the direct product of G_2 with the ordinary spin group SU_2.[3] The simplest group that satisfies this condition is the unimodular orthogonal rotation group in ten dimensions, D_5 (Ref. 4); since G_2 is a subgroup of R_7 we can write the induction chain for the Lie algebras as

$$G_2\otimes SU_2\subset R_7\otimes R_3\subset R_{10}\equiv D_5. \quad (1)$$

Let us note that D_5 embodies $R_7\otimes R_3$, which has been also proposed as an elementary-particle symmetry algebra.[5] The representation of D_5 allows a classification of elementary particles, suggestive enough to be considered. We have studied here its representations and a possible assignment of particles. The 16-dimensional representation appears to contain the baryons, and such a possibility is strengthened by the results obtained for the magnetic moments. The bosons fit in a 45-dimensional representation and the possible splitting of this supermultiplet, under a G_2-noninvariant interaction, in representations of $G_2\otimes SU_2$ with definite A parity, makes **45** a very good candidate.

II. REPRESENTATIONS

D_5 is a semisimple group, whose Lie algebra is described by 45 generators, and its representations are defined by five integers. The states that describe the

TABLE I. Some representations of $G_2\otimes SU_2$.

Dimension	Decomposition in $G_2\otimes SU_2$
10	(7,1)+(1,3)
16	(7,2)+(1,2)
45	(14,1)+(7,3)+(7,1)+(1,3)
54	(27,1)+(7,3)+(1,5)+(1,1)
144	(27,2)+(14,2)+(7,4)+(7,2)+(1,4)+(1,2)
210	(27,3)+(14,3)+(27,1)+2(7,3)+2(7,1)+(1,3)+(1,1)
560	(27,4)+(14,4)+2(7,4)+(1,4)+(64,2) +2(27,2)+2(14,2)+3(7,2)+(1,2)

* Permanent address: Facultad de Ciencias Exactas, Universidad de Buenos Aires, Peru 222, Buenos Aires, Argentina.

[1] F. Gürsey and L. Radicati, Phys. Rev. Letters 13, 173 (1964); B. Sakita, Phys. Rev. 6, B1756 (1964).

[2] R. Behrends and L. Landovitz, Phys. Rev. Letters 11, 296 (1963); R. Behrends, L. Landovitz, and B. Tunkelang, Phys. Rev. 142, 1092 (1966); R. Behrends, *ibid.* 142, 1101 (1966); R. Behrends and A. Sirlin, *ibid.* 142, 1095 (1966).

[3] There are similar structures about the relativistic validity of D_5, as at SU_6.

[4] E. Dynkin, Am. Math. Soc. Transl. 2, 319 (1957).

[5] D. Peaslee, Phys. Rev. 117, 873 (1960).

elementary-particle multiplets will be the basis of an irreducible representation of D_5. The effect of an interaction that breaks the D_5 symmetry but is invariant under the product group $G_2\otimes SU_2$ (as a G_2-spin-dependent interaction could) is to split the supermultiplets into irreducible representations of $G_2\otimes SU_2$. The dimensions of the representations of D_5 are $N=1$, 10, 16, $\overline{16}$, 45, 54, 120, 126, 144, $\overline{144}$, 210, $\overline{210}$, 320, 560, $\cdots$; in Table I we have written the contents of some of these in terms of the $G_2\otimes SU_2$ representations, the numbers (m,n) are the G_2 and SU_2 multiplicity, respectively.

If we introduce a purely spin-dependent interaction, D_5 will not decompose according to $G_2\otimes SU_2$, but as representations of $R_7\otimes SU_2$; thus we will have $\mathbf{16}\rightarrow(8,2)$ and $\mathbf{45}\rightarrow(21,1)+(7,3)+(1,3)$, where the first number of (m',n') is the dimension of the R_7 irreducible representation.

A difference between SU_6 and D_5 is that the D_5 supermultiplets have a high degree of degeneracy, which makes the group more flexible, but also the calculations more difficult.

The products between the representations of most interest to us are:

$$\begin{aligned}&\mathbf{10}\otimes\mathbf{10}=\mathbf{54}\oplus\mathbf{45}\oplus\mathbf{1},\\&\mathbf{10}\otimes\mathbf{16}=\mathbf{144}\oplus\overline{\mathbf{16}},\\&\mathbf{16}\otimes\overline{\mathbf{16}}=\mathbf{210}\oplus\mathbf{45}\oplus\mathbf{1},\\&\mathbf{16}\otimes\mathbf{45}=\mathbf{560}\oplus\overline{\mathbf{144}}\oplus\mathbf{16},\\&\mathbf{45}\otimes\mathbf{45}=\mathbf{975}\oplus\mathbf{770}\oplus\mathbf{210}\oplus\mathbf{54}\oplus\mathbf{45}\oplus\mathbf{1}.\end{aligned}\tag{2}$$

The fact that there is no multiplicity of representations in the products shows *a priori* that we cannot obtain satisfactory results in first order, since only linear combinations of generators will appear in the physical operators and not quadratic invariants.[6]

The simplest possibility to accommodate the particles could be the baryons in the 16- and the bosons in the 10-dimensional representation, but this forces all the baryon resonances to be in **144**, which has only seven particles with spin $\frac{3}{2}$, so this choice does not appear satisfactory. A more reasonable choice is to consider the bosons as states of the **45** representation; in this way **45** under a symmetry-breaking interaction is split into the multiplets of G_2 with definite A parity proposed by Behrends, Landovitz, and Tunkelang,[2] that is, (with $M_{JPA}{}^{N}$):

$$\begin{aligned}&(7,1)\equiv M_{0--}{}^{7}\colon \pi,K;\quad (14,1)\equiv M_{0--}{}^{14}\colon \eta\cdots;\\&(7,3)\equiv M_{1-+}{}^{7}\colon \rho,K^*;\quad (1,3)\equiv M_{1--}{}^{1}\colon \omega.\end{aligned}\tag{3}$$

We must assign negative parity to the **45** states. If we ask that the ϕ belong to $M_{1-+}{}^{14}$ in G_2, that is, a (14,3) subrepresentation, the lowest possible representation where it can be included will be **126**. Then the baryonic resonances must belong to **560** and **144**; the number of states with spin $\frac{3}{2}$ allows us to place the resonances $\mathcal{J}^P=\frac{3}{2}^+$ in the **560** representation and those $\mathcal{J}^P=\frac{3}{2}^-$ in the **144**, i.e., N^{**}, Y_1^{**}, and Ξ^{**} in the (7,4) and Y_0^{**} in the (1,4) multiplet.[7] The interpretation of the higher spin resonances, as for SU_6, is limited, because in this case the spin and space variables are coupled by the spin-orbit forces which are constituents of the structure (or potential) that defines the particles, and since the angular momentum is not included in D_5 a more careful interpretation or extension of the group is required.

III. MAGNETIC MOMENT AND MASS SPLITTING

The next step is to consider whether with simple assumptions for the symmetry-breaking interaction we can derive some physical results for the mass and the magnetic moment. For that we must set up a scheme to interpret the quantum numbers as linear combinations of the commuting operators of D_5. Denoting by C_{ij} $(i,\,j\equiv\pm1,\,\cdots,\,\pm5)$ the generators of D_5 which satisfy the commutation rule[8]:

$$[C_{ik},C_{mn}]=\delta_{k,-m}C_{in}-\delta_{k,-n}C_{im}-\delta_{i,-m}C_{kn}+\delta_{i,-n}C_{km}\tag{4}$$

and using the reduction chain (1), the generators of the commuting G_2 and SU_2 groups can be written:

$$\begin{aligned}G_2\colon\quad &T_3=C_{3-3}-\tfrac{1}{2}(C_{2-2}+C_{1-1}),\qquad \mathcal{J}=C_{1-1}-C_{2-2},\\&E_1=\frac{1}{\sqrt{3}}(C_{34}+C_{3-4}+C_{-1-2}),\quad E_2=C_{3-2},\\&E_3=\frac{1}{\sqrt{3}}(C_{-24}+C_{-2-4}+C_{31}),\quad E_4=C_{1-2},\\&E_5=\frac{1}{\sqrt{3}}(C_{14}+C_{1-4}+C_{-2-3}),\quad E_6=C_{-31};\\SU_2\colon\quad &\mathcal{J}_3=C_{5-5},\quad \mathcal{J}_+=\frac{1}{\sqrt{2}}(C_{54}-C_{5-4}),\quad \mathcal{J}_-=\frac{1}{\sqrt{2}}(C_{-54}-C_{-5-4}).\end{aligned}\tag{5}$$

[6] T. Ginibre, J. Math. Phys. 4, 720 (1963).

[7] It is not clear, considering the $N_{5/2}{}^{**}$–$N_{1/2}{}^{**}$ experimental splitting, whether we must accept a privileged set of 8 baryons in **16**, and not include all the particles with baryon number 1, positive parity and spin $\frac{1}{2}$ or $\frac{3}{2}$ in the single representation **144**.

[8] G. Racah, Institute for Advanced Study, Princeton, 1951 (unpublished).

With this reduction we can label the particle states.

First we will calculate the magnetic moment of the **16** supermultiplet. We ask the magnetic current to transform as a tensor of the product group (14,3) multiplet, and furthermore it must be a bilinear combination of the **16** and $\overline{\mathbf{16}}$ representations; thus it must belong to **210**. The electric current will transform as a tensor of the subrepresentation (14,1) of the regular representation **45**. Thus we choose the magnetic moment as a tensor of **210**, which transforms as the product $Q\cdot\mathcal{J}_3$ in $G_2\otimes SU_2$ (Ref. 9); we get then

$$\mu_B=\mu_p Q\,,$$

where μ_p is the proton magnetic moment and Q is the charge of the baryon. This result is similar to those obtained in G_2 with the additional relation $\mu(\Lambda)=0$, and leads us to believe that the electromagnetic properties of the baryons in D_5 will have similar difficulties to those in G_2.[10] A way of getting a solution to these poor results in G_2 is to mix different representations[10]; in the D_5 case, if we assign the baryons to a higher representation (144) and define the quantum numbers in the framework of D_5 independently of the reduction chain (1), such a mixture could be automatic. Another possibility is first to break the symmetry.[11]

Now we will consider the baryon mass splitting. There are different possibilities for breaking the symmetry. We can choose an interaction Hamiltonian so that it: (a) breaks the D_5 symmetry but not $R_7\otimes SU_2$, (b) breaks the D_5 symmetry but not $G_2\otimes SU_2$, (c) has an R_7-noninvariant part, (d) has a G_2-noninvariant part. We have chosen a G_2-noninvariant symmetry-breaking Hamiltonian, and as usual we have assumed that it transforms like a $T=Y=\mathcal{J}=0$ member of the regular representation, so it will belong to the subrepresentation (14,1) of **45**. The bases of this supermultiplet are antisymmetric second-rank tensors in the 10-dimensional vector space. In first order we obtain

$$M_B=a+bY\,.$$

The unique contribution in second order comes from the 210-dimensional representation and there, because of the condition $T=\mathcal{J}=Y=0$, the single contribution comes from the center of (27,1). This tensor is a fourth-rank one, totally antisymmetric, and can be constructed from the basis of **45**; then we get:

$$M_B=a+bK_3+CK(K+1)\,,\qquad(6)$$

where $K_3=Y/2$ is the third component of the hypercharge rotation subgroup of G_2, defined by K.[1] Unfortunately, there is no splitting between Σ and Λ.

Let us turn now to the meson representation. By the symmetry of the weight diagram we have no first-order breaking; in second order, because $\mathbf{45}\otimes\mathbf{45}=\mathbf{975}\oplus\mathbf{770}\oplus\mathbf{210}\oplus\mathbf{54}\oplus\mathbf{45}\oplus\mathbf{1}$, there are contributions from many centers but, since we are dealing with bosons, it is not necessary to consider the antisymmetric representations. We have calculated only the contribution from a symmetry-breaking term which transforms like a center of **54** (traceless symmetric second-rank tensor); for that it must belong to the multiplet (27,1). For the different subrepresentations of **45**, this term gives the following mass expression:

$$\begin{aligned}\Delta M^2(14,1)&=-\tfrac{1}{3}\alpha[T(T+1)-9K(K+1)]\,,\\ \Delta M^2(7,1)&=\tfrac{4}{3}\alpha T(T+1)\,,\\ \Delta M^2(7,3)&=\tfrac{2}{3}\alpha[5K(K+1)-3T(T+1)]\,,\\ \Delta M^2(1,3)&=-4\alpha\,.\end{aligned}\qquad(7)$$

It is possible to write the mass equation in closed form, but for that it is necessary to obtain the 5 invariant operators of D_5. In Eq. (7) we only show the splitting pattern. These relations do not give good agreement with the experimental masses (all the states with $K=\frac{1}{2}$ in **45** have the same mass correction, so K, K^*, and δ remain degenerate), nor do they corroborate the suggestions for the masses of the (14,1) multiplet in Ref. 2; but we must recall that they represent only a part of and not the full second-order contribution to the meson representation.

We must conclude that D_5 provides a highly reasonable classification for the particles, but at first sight the physical properties seem to have similar difficulties to those of G_2.

ACKNOWLEDGMENTS

It is a pleasure to thank Professor L. Landovitz for suggesting this investigation and for many stimulating discussions, and to thank Professor R. Behrends for very useful suggestions.

[9] Let us recall that we have supposed that we are dealing with the static limit of a relativistic theory.

[10] N. Mukunda, A. J. Macfarlane, and E. C. G. Sudarshan, Phys. Rev. **138**, 3665 (1965); **133**, 3475 (1964).

[11] R. Behrends, Ref. 2.