Photon Statistics and Classical Fields*

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(Received 27 December 1965; revised manuscript received 31 March 1966)

In view of the misunderstanding which surrounds the use of semiclassical methods in the treatment of optical coherence problems, a simple analysis is given which illustrates the generality of these methods. In particular, it is pointed out that, under commonly occurring circumstances, these methods apply at arbitrarily low light levels, and give correct results for the photoelectric counting of photons. The conclusions are illustrated by treating the interference between laser light and thermal light to demonstrate that this nontrivial case, recently treated by quantum field theory, is correctly described by semiclassical methods.

I. INTRODUCTION

SINCE the publication of the first papers on the quantum theory of optical coherence,[1,2] a substantial amount of discussion[3–10] has been devoted to the relationship between the quantum theory and the older classical and semiclassical theories. Some of these discussions have brought out the fact that, in many situations commonly encountered in practice, the electromagnetic field can be treated in classical terms if the interaction with the measurement apparatus is treated quantum mechanically. This result has nothing to do with the correspondence principle, but is connected with the nature of the electromagnetic field and holds at arbitrarily low light intensities. Of course there exist situations in which the state of the field is not describable classically, except in a formal sense based on the validity of a certain diagonalization theorem [given by Eq. (5) below[4]]. We cite a Fock state of the field as an example.

The distinction between the two situations is frequently misunderstood. It is a common (but mistaken) belief that semiclassical methods always fail when the light intensity is sufficiently low, and that there are statistical features of the field revealed by photoelectric measurements which cannot be accounted for by a semiclassical treatment. As an example we quote from the opening paragraph of a recent publication,[11] in which a field which can be treated in completely classical terms (as we show below)—namely the field resulting from the superposition of light beams from an "ideal" laser and from a thermal source—is treated quantum mechanically: ". . .Although much of the theoretical work uses classical methods with little inhibition, *citing the usual argument* (*high field intensities equal classical limit*),[12] some authors have attempted to formulate the problem on its proper ground, namely on a microscopic quantum mechanical level, with due respect for the statistics underlying the electromagnetic field. . ."

In the following we will discuss the question of the validity of the semiclassical method, and indicate why the method leads to results identical with those obtained from the quantized field theory in appropriate cases, even at arbitrarily low intensities. As an example we then show that the results obtained in Ref. 11 may be derived more simply and directly by semiclassical methods.

II. RELATION BETWEEN QUANTUM AND SEMICLASSICAL TREATMENTS

In quantum field theory the state of an isolated field is described by a density operator[13] $\hat{\rho}$, which is often conveniently represented in the particular basis formed by the eigenstates $|\{v_\lambda\}\rangle \equiv \prod_\lambda |v_\lambda\rangle$ of the annihilation operators $\hat{a}_\lambda$,

$$\hat{a}_\lambda |v_\lambda\rangle = v_\lambda |v_\lambda\rangle, \tag{1}$$

where λ labels the modes and v_λ is any complex number. From (1) it follows that, if we define a configuration space annihilation operator $\hat{\mathbf{A}}(x)$ (we use x for $\mathbf{r}$, t) by an expansion of the type[8]

$$\hat{\mathbf{A}}(x) = (1/L^{3/2}) \sum_\lambda \hat{a}_\lambda \boldsymbol{\varepsilon}_\lambda e^{ik_\lambda x}, \tag{2}$$

and a complex vector wave amplitude $\mathbf{V}(x)$ by

$$\mathbf{V}(x) = (1/L^{3/2}) \sum_\lambda v_\lambda \boldsymbol{\varepsilon}_\lambda e^{ik_\lambda x}, \tag{3}$$

then

$$\hat{\mathbf{A}}(x) |\{v_\lambda\}\rangle = \mathbf{V}(x) |\{v_\lambda\}\rangle. \tag{4}$$

The states $|\{v_\lambda\}\rangle$ have been examined in some detail by Glauber,[1,2] who has called them coherent or classical states and shown that they correspond to a classical wave field of *determinate* complex wave amplitude $\mathbf{V}(x)$. However, the term "classical state" has sometimes tended to mislead. For there exists a classical description for a much wider class of states of the field, namely, those corresponding not to a well-defined function $\mathbf{V}(x)$, but to a statistical ensemble of such functions. The state $|\{v_\lambda\}\rangle$ therefore represents a limiting situation, in which

* This work was supported by the U. S. Air Force Office of Scientific Research, Office of Aerospace Research.

[1] R. J. Glauber, Phys. Rev. **130**, 2529 (1963).
[2] R. J. Glauber, Phys. Rev. **131**, 2766, (1963).
[3] E. T. Jaynes and F. W. Cummings, Proc. IEEE **51**, 89 (1963).
[4] E. C. G. Sudarshan, Phys. Rev. Letters **10**, 277 (1963).
[5] D. Holliday and M. Sage, Phys. Rev. **138**, B485 (1965).
[6] L. Mandel, Phys. Letters **7**, 117 (1963).
[7] L. Mandel, E. C. G. Sudarshan, and E. Wolf, Proc. Phys. Soc. (London) **84**, 435 (1964).
[8] L. Mandel and E. Wolf, Rev. Mod. Phys. **37**, 231 (1965).
[9] I. R. Senitzky, Phys. Rev. Letters **15**, 233 (1965).
[10] A. E. Glassgold and D. Holliday, Phys. Rev. Letters **15**, 741 (1965).
[11] H. Morawitz, Phys. Rev. **139**, A1072 (1965).
[12] The italics are ours.
[13] We use the circumflex throughout to represent operators.

the field is not merely describable classically but is also determinate.

It has been shown[4] that the density operator $\hat{\rho}$ for any free electromagnetic field may always be expressed in the form

$$\hat{\rho}=\int \phi(\{v_\lambda\})|\{v_\lambda\}\rangle\langle\{v_\lambda\}|d^2\{v_\lambda\}, \tag{5}$$

provided the expansion and the real functional $\phi(\{v_\lambda\})$ are interpreted in a certain generalized sense which has recently been made precise.[14–16] However, for a wide class of commonly encountered fields, $\phi(\{v_\lambda\})$ is a real, non-negative, nonsingular functional (except possibly for a δ-function type singularity), which is normalized to unity and may be looked on as a probability functional in the representation (5). The expansion (5) then becomes a simple integral, which has also been referred to as the P representation.[1,2] Obvious examples of such fields are the field due to a thermal source, possibly modified by any linear filter, and a completely coherent field (often referred to as the field due to an "ideal" laser), for which

$$\phi(\{v_\lambda\})=\prod_\lambda(1/\pi w_\lambda)\exp(-|v_\lambda|^2/w_\lambda), \tag{6}$$

and

$$\phi(\{v_\lambda\})=\delta(v_\mu-v_\mu')\prod_{\lambda\neq\mu}\delta(v_\lambda), \tag{7}$$

respectively. In such cases, for the purposes of all commonly encountered measurements, the field may also be described in completely classical terms by an ensemble of classical wave amplitudes $\mathbf{V}(x)$ as in Eq. (3), in which the ensemble distribution is given by the probability functional $\phi(\{v_\lambda\})$. In other words, the field may be represented by a random process $\mathbf{V}(x)$ to which we refer as the fluctuating classical wave amplitude. Any state represented by a non-negative, nonsingular $\phi(\{v_\lambda\})$ may therefore be regarded as a classical state, from the point of view of the usual measurements.[17]

In order to see the correspondence, we first recall that practically all measurements in quantum optics are based on the use of photoelectric detectors, and correspond to expectation values of various normally ordered products of creation and annihilation operators.[1,2] Now the expectation value of any normally ordered functional $L[\{\hat{\mathbf{A}}(x)\},\{\hat{\mathbf{A}}^\dagger(x)\}]$ of the operators $\hat{\mathbf{A}}(x)$ and $\hat{\mathbf{A}}^\dagger(x)$, evaluated according to the usual rules of quantum mechanics, is identical with the expectation value of the random process $L[\{\mathbf{V}(x)\},\{\mathbf{V}^*(x)\}]$, evaluated with respect to the classical ensemble distribution $\phi(\{v_\lambda\})$.[4] For, from (5)

$$\langle L[\{\hat{\mathbf{A}}(x)\},\{\hat{\mathbf{A}}^\dagger(x)\}]\rangle =\mathrm{Tr}\int \phi(\{v_\lambda\})L[\{\hat{\mathbf{A}}(x)\},\{\hat{\mathbf{A}}^\dagger(x)\}]|\{v_\lambda\}\rangle\langle\{v_\lambda\}|d^2\{v_\lambda\},$$

and in view of Eq. (4), together with its Hermitian conjugate,

$$\begin{aligned}\langle L[\{\hat{\mathbf{A}}(x)\},\{\hat{\mathbf{A}}^\dagger(x)\}]\rangle &=\int \phi(\{v_\lambda\})L[\{\mathbf{V}(x)\},\{\mathbf{V}^*(x)\}]d^2\{v_\lambda\}\\ &=\langle L[\{(\mathbf{V}(x)\},\{\mathbf{V}^*(x)\}]\rangle,\end{aligned} \tag{8}$$

where we use the angular brackets to denote either the quantum-mechanical expectation of an operator, or the statistical expectation of a classical random process. This shows that the functional $\phi(\{v_\lambda\})$ may be regarded as either a quantum-mechanical object, namely a diagonal density matrix, or as a classical object, namely the ensemble distribution of a classical wave field.

Actually, as has been pointed out,[4] because of the universal scope of the general representation (5), a mathematical correspondence between the two types of representation of the field persists even when $\phi(\{v_\lambda\})$ is negative and singular. We will briefly return to this point at the end.

As a further illustration that radiation fields for which $\phi(\{v_\lambda\})$ is non-negative and nonsingular may also be described classically, for the purposes of the usual measurements, we will consider the superposition of two fields. According to classical wave theory, when fields of complex random wave amplitudes $\mathbf{V}'(x)$ and $\mathbf{V}''(x)$ are superposed, the resultant random wave amplitude $\mathbf{V}(x)$ is related to $\mathbf{V}'(x)$ and $\mathbf{V}''(x)$ by

$$\mathbf{V}(x)=\mathbf{V}'(x)+\mathbf{V}''(x), \tag{9}$$

and its statistical behavior is governed by the joint probability density $\phi(\{v_\lambda'\},\{v''\})$ for the two fields. According to quantum field theory, the basis states for the combined system are of the type $|\{v_\lambda'\}\rangle|\{v_\lambda''\}\rangle$, and the density operator has the form

$$\hat{\rho}=\int \phi(\{v_\lambda'\},\{v_\lambda''\})|\{v_\lambda'\}\rangle|\{v_\lambda''\}\rangle\langle\{v_\lambda''\}|\langle\{v_\lambda'\}|d^2\{v_\lambda'\}d^2\{v_\lambda''\}. \tag{10}$$

[14] J. R. Klauder, J. McKenna, and D. G. Currie, J. Math. Phys. **6**, 733 (1965).
[15] C. L. Mehta and E. C. G. Sudarshan, Phys. Rev. **138**, B274 (1965).
[16] J. R. Klauder, Phys. Rev. Letters **16**, 534 (1966).
[17] We wish to emphasize that, when we speak of a field which can be described in classical terms, we do not necessarily imply that the description is derivable from completely classical arguments alone. Thus, as is well known, attempts to derive the spectral distribution of black-body radiation by classical arguments encounter serious difficulties. Nevertheless, once the spectral distribution has been found, black-body radiation may be represented in terms of an ensemble of classical wave amplitudes. With this understanding, the term "classical representation" carries no implication about the presence or absence of Planck's constant.

Now the "measurement operator" $\hat{\mathbf{A}}(x)$ operates on the combined system, so that

$$\begin{aligned}\hat{\mathbf{A}}(x)|\{v_\lambda'\}\rangle|\{v_\lambda''\}\rangle &= [\mathbf{V}'(x)|\{v_\lambda'\}\rangle]|\{v_\lambda''\}\rangle + |\{v_\lambda'\}\rangle[\mathbf{V}''(x)|\{v_\lambda''\}\rangle] \\ &= [\mathbf{V}'(x)+\mathbf{V}''(x)]|\{v_\lambda'\}\rangle|\{v_\lambda''\}\rangle.\end{aligned} \tag{11}$$

From (10), (11) and its Hermitian conjugate it follows immediately that the expectation value of any normally ordered operator $L[\{\hat{\mathbf{A}}(x)\},\{\hat{\mathbf{A}}^\dagger(x)\}]$ is

$$\begin{aligned}\langle L[\{\hat{\mathbf{A}}(x)\},\{\hat{\mathbf{A}}^\dagger(x)\}]\rangle &= \mathrm{Tr}\hat{\rho}L[\{\hat{\mathbf{A}}(x)\},\{\hat{\mathbf{A}}^\dagger(x)\}] \\ &= \int \phi(\{v_\lambda'\},\{v_\lambda''\})L[\{\mathbf{V}'(x)+\mathbf{V}''(x)\},\{\mathbf{V}'^*(x)+\mathbf{V}''^*(x)\}]d^2\{v_\lambda'\}d^2\{v_\lambda''\} \\ &= \langle L[\{\mathbf{V}'(x)+\mathbf{V}''(x)\},\{\mathbf{V}'^*(x)+\mathbf{V}''^*(x)\}]\rangle,\end{aligned} \tag{12}$$

which, according to Eq. (9), is just the result to be expected from the classical wave theory.

We emphasize that the classical description of the field is here not to be regarded as an approximation to the quantum-mechanical description. *The ensemble distribution $\phi(\{v_\lambda\})$ governing the fluctuations of the classical wave amplitude carries neither more nor less information than the density operator $\hat{\rho}$.*

Let us now turn to the question of the fluctuation of counts registered by an illuminated photodetector. This problem is often mistakenly claimed to lie outside the scope of any treatment that does not start off with a quantized field, particularly when the light intensities are low. It has been shown by explicit quantum mechanical calculation[18,19] that, when a radiation field with density operator given by (5) (with $\phi(\{v_\lambda\})$ nonsingular) interacts with a photodetector, the probability $p(n;t,t+T)$ that n counts are registered in the time interval $(t,t+T)$ is given by

$$p(n;t,t+T) = \int \phi(\{v_\lambda\})\frac{U^n(\{v_\lambda\})}{n!}\exp[-U(\{v_\lambda\})]d^2\{v_\lambda\}, \tag{13}$$

when the number of detector atoms greatly exceeds n, and certain other commonly encountered conditions are satisfied. Under typical conditions, when the field has a narrow spectral range and propagates normally to the sensitive area of the photodetector, it can be shown with the help of some reasonable assumption about the nature of the photoelectric mechanism that $U(\{v_\lambda\})$ has the simple form

$$U(\{v_\lambda\}) = \beta\int_t^{t+T} \mathbf{V}^*(\mathbf{x},t')\cdot\mathbf{V}(\mathbf{x},t')dt', \tag{14}$$

where β is a constant characteristic of the detector, involving the dipole matrix elements of the atomic states. The counts registered by the detector are frequently identified with the absorption of photons of the field, since a distribution very similar to (13) can be written down for the number of photons localized in a space-time region of sufficiently large dimensions.[20,21]

Nevertheless, when $\phi(\{v_\lambda\})$ is non-negative and non-singular, the formula (13) can also be obtained from semiclassical considerations, and indeed was so obtained in the first place.[22,23,7] From a consideration of the interaction between a classical wave field described by the ensemble distribution $\phi(\{v_\lambda\})$ and the atoms of the photodetector, we readily find for the differential probability of photoelectric detection at time t within δT, under similar conditions to those applying to (13) and (14),[7]

$$p(1;t,t+\delta T) = \beta\langle\mathbf{V}^*(\mathbf{x},t)\cdot\mathbf{V}(\mathbf{x},t)\rangle\delta T. \tag{15}$$

The term $\langle\mathbf{V}^*(\mathbf{x},t)\cdot\mathbf{V}(\mathbf{x},t)\rangle$ will be recognized as the expectation value of the classical intensity. The formula is easily generalized to apply to a finite time interval T, by subdivision of T into a large number of very short intervals δT and by the application of combinatorial statistics. Thus we find[23]

$$p(n;t,t+T) = \lim_{\delta T\to 0}\sum_{r_1=0}^{T/\delta T}\cdots\sum_{r_n=0}^{T/\delta T}\frac{1}{n!}\beta^n\Big\langle \mathbf{V}^*(\mathbf{x},t+r_1\delta T)\cdot\mathbf{V}(\mathbf{x},t+r_1\delta T)\cdots\mathbf{V}^*(\mathbf{x},t+r_n\delta T)\cdot\mathbf{V}(\mathbf{x},t+r_n\delta T) \\ (\delta T)^n\frac{\prod_{s=1}^{T/\delta T}[1-\beta\mathbf{V}^*(\mathbf{x},t+s\delta T)\cdot\mathbf{V}(\mathbf{x},t+s\delta T)\delta T]}{\prod_{q=1}^{n}[1-\beta\mathbf{V}^*(\mathbf{x},t+r_q\delta T)\cdot\mathbf{V}(\mathbf{x},t+r_q\delta T)\delta T]}\Big\rangle, \tag{16}$$

[18] P. L. Kelley and W. H. Kleiner, Phys. Rev. **136**, A316 (1964).
[19] R. J. Glauber, *Quantum Optics and Electronics*, edited by C. de Witt, A. Blandin, and C. Cohen-Tannoudji (Gordon and Breach, Science Publishers, Inc., New York, 1965), p. 65.
[20] F. Ghielmetti, Phys. Letters **12**, 210 (1964).
[21] L. Mandel, Phys. Rev. **144**, 1071 (1966).
[22] L. Mandel, Proc. Phys. Soc. (London) **72**, 1037 (1958).
[23] See for example, L. Mandel, *Progress in Optics*, edited by E. Wolf (John Wiley & Sons, Inc., New York, 1963), Vol. II, p. 181.

and, when the limit $\delta T \to 0$ is evaluated,[24] we arrive at

$$p(n;t,t+T)=\langle (U^n(\{v_\lambda\})/n!)\exp[-U(\{v_\lambda\})]\rangle. \quad (17)$$

This formula is seen to be identical with Eq. (13) when we recall that $\phi(\{v_\lambda\})$ is the ensemble distribution to be used in the calculation of expectation values. Thus, as long as $\phi(\{v_\lambda\})$ is non-negative and nonsingular, the results given by the semiclassical method are identical with those given by quantum field theory. The semiclassical method is therefore able to account for the photoelectric counting of photons, even at arbitrarily low light levels and for arbitrarily small n.

As an illustration we may evaluate the distribution (17) explicitly for a thermal radiation field described by Eq. (6), under the condition that the counting interval T is very short compared with the reciprocal frequency spread of the light. We then find that,[8,22] for polarized light,

$$p(n\,t,t+T)=1/[1+\langle n\rangle][1+1/\langle n\rangle]^n, \quad (18)$$

which is a well-known occupation number distribution for photons in a thermal field.

III. AN ILLUSTRATIVE EXAMPLE

In order to illustrate some of the foregoing remarks, we will examine the problem of the superposition of a light beam from an "ideal" single-mode laser and a beam from a thermal source, which was recently treated by the method of quantum field theory.[11,25] We shall see that identical results are obtainable more simply and directly by a semiclassical treatment. The two fields in question are both classical fields in the sense here defined and are described by Eqs. (7) and (6), respectively. If $\mathbf{V}_1(t)$ and $\mathbf{V}_2(t)$ are the complex wave amplitudes corresponding to the laser and the thermal beams (both assumed to be polarized) at the detector, the resultant wave amplitude $\mathbf{V}(t)$ is given by

$$\mathbf{V}(t)=\mathbf{V}_1(t)+\mathbf{V}_2(t). \quad (19)$$

Recalling that $\mathbf{V}_1(t)$ and $\mathbf{V}_2(t)$ are statistically independent random processes with zero means, we find from (14), (19), (7), and (6), that

$$\langle U(\{v_\lambda\})\rangle=\beta(\langle I_1\rangle+\langle I_2\rangle)T, \quad (20)$$

and, if we make use of the moment theorem for the Gaussian random process,[23,26]

$$\langle I_2(t)I_2(t+\tau)\rangle=\langle I_2\rangle^2[1+|\gamma_{22}(\tau)|^2], \quad (21)$$

we arrive at the result

$$\langle U^2(\{v_\lambda\})\rangle=\beta^2(\langle I_1\rangle+\langle I_2\rangle)^2T^2+\beta^2\langle I_2\rangle^2\int_0^T\int_0^T|\gamma_{22}(t''-t')|^2dt'dt''$$
$$+2\beta^2\langle I_1\rangle\langle I_2\rangle\,\mathrm{Re}\int_0^T\int_0^T\gamma_{11}(t''-t')\gamma_{22}^*(t''-t')dt'dt''. \quad (22)$$

Here $\langle I_1\rangle$ and $\langle I_2\rangle$ and $\gamma_{11}(\tau)$ and $\gamma_{22}(\tau)$ are the mean intensities and the normalized autocorrelation functions[8] associated with $\mathbf{V}_1(t)$ and $\mathbf{V}_2(t)$. Now from (17),[7]

$$\langle n\rangle=\langle U(\{v_\lambda\})\rangle \quad (23)$$

and

$$\langle(\Delta n)^2\rangle=\langle n\rangle+\langle(\Delta U(\{v_\lambda\}))^2\rangle, \quad (24)$$

and with the help of (20) and (22) this becomes

$$\langle(\Delta n)^2\rangle=\langle n\rangle+\frac{\langle n_2\rangle^2}{T^2}\int_0^T\int_0^T|\gamma_{22}(t''-t')|^2dt'dt''$$
$$+\frac{2\langle n_1\rangle\langle n_2\rangle}{T^2}\mathrm{Re}\int_0^T\int_0^T\gamma_{11}(t''-t')\gamma_{22}^*(t''-t')dt'dt'', \quad (25)$$

which is the formula obtained by Morawitz[11,25] [his Eqs. (22) and (23)] from quantum field theory.

The example once again shows explicitly what we have stated at the beginning. When the functional $\phi(\{v_\lambda\})$ representing the density operator of the field is non-negative and nonsingular, the electromagnetic field may be described in a classical way for the purpose of describing the usual measurements, and the results of semiclassical calculations will be identical with those obtained by quantum field theory.

Finally, let us briefly consider the situation when $\phi(\{v_\lambda\})$ is not necessarily restricted to be non-negative definite and nonsingular, but represents an arbitrary field. In that case we can no longer construct a physically realizable ensemble of complex classical Fourier amplitudes $\{v_\lambda\}$ whose probability distribution is $\phi(\{v_\lambda\})$. Moreover, the mathematical significance of the various

[24] Actually, δT should be long compared with a typical period of the light, so that, strictly speaking, we are not entitled to proceed to the mathematical limit $\delta T \to 0$. However, since typical periods of a light beam are far beyond the limit of resolution of available detectors, small values of δT correspond to a very good approximation to the limit $\delta T \to 0$, provided the light intensity is not excessively high.

[25] For another discussion of this problem, see also G. Lachs, Phys. Rev. 138, B1012 (1965).

[26] See C. L. Mehta, *Lectures in Theoretical Physics*, edited by W. E. Brittin (University of Colorado Press, Boulder, Colorado, 1965), Vol. 7(c), p. 345.

expressions involving $\phi(\{v_\lambda\})$ obviously has to be interpreted with some care when $\phi(\{v_\lambda\})$ is singular. The precise sense in which the expressions are then to be understood has recently been found.[14–16] However, with this understanding the various relations we have obtained [such as Eqs. (8), (12), (17), (23), (24)] all remain valid. In other words, even when $\phi(\{v_\lambda\})$ is negative and singular, we may continue to use the formalims of the semiclassical method of calculation *as though* $\phi(\{v_\lambda\})$ *were a probability*, and obtain the correct result. Once this is understood, it becomes clear that the semiclassical method is of very great generality.

ACKNOWLEDGMENT

We would like to acknowledge some helpful critical comments from Dr. J. H. Eberly.

PHYSICAL REVIEW VOLUME 149, NUMBER 4 30 SEPTEMBER 1966

Matter Traversed by Low-Energy Cosmic-Ray Nuclei in Space

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(Received 10 March 1966; revised manuscript received 4 May 1966)

The amount of matter traversed in space by the primary cosmic-ray nuclei of energy 50–150, 200–500, 500–1000, and >1500 MeV/nucleon are obtained as 5.5±1.4, 9.1±0.9, 5.3±0.5, and 3.1±0.4 g cm^{-2} of hydrogen, respectively, using recent measurements of spallation cross sections with proton beams from accelerators and Li/M ratios of cosmic-ray nuclei (where M denotes nuclei with $6\leqslant Z\leqslant 9$). These results and other evidence seem to suggest the hypothesis of two distinct sources (or propagation paths) of cosmic-ray nuclei, one responsible for nuclei of energy roughly 50–150 MeV/nucleon and the other of energies greater than this.

The amount of matter traversed by primary cosmic-ray nuclei can be deduced from a determination of the ratio[1] L/M or L/S assuming that the L nuclei are absent in the source and that they are produced by fragmentations of heavier nuclei in collisions with hydrogen in space during their propagation. Such data available at kinetic energy $E>1.5$ BeV/nucleon show that the matter traversed is 2–3 g cm^{-2} of hydrogen.[2,3] At lower energies, statistically significant results on the energy dependence of L/M have been obtained only recently[4–8]; these are summarized in Fig. 1(a). From this figure it is seen that L/M has a maximum value of ~0.5 at 200–500 MeV/nucleon which drops to ~0.3 at 50–150 MeV/nucleon and to 0.25 at $E>1.5$ BeV/nucleon. It has not been known so far whether this effect is due to the variation of cross sections with energy or to the variation of the amount of matter traversed with energy. The object of this work is to determine the amount of matter traversed by the low-energy cosmic-ray nuclei as a function of energy, using the recently available cross-section data and the experimentally measured ratios of the cosmic-ray nuclei.

The evaluation of the production rate of L nuclei is difficult at present because we need cross sections for the production of a large number of isotopes, both radioactive and stable, of Li, Be, and B for a number of targets, each bombarded by protons of various energies. These data are not available at present. To circumvent this difficulty we make use of the experimental Li/M ratios [Fig. 1(b)] and the recent data on the cross sections for the production of Li isotopes.[9,10]

CROSS-SECTION DATA

The cross sections used by us are given in Table I. The values of σ for $C^{12}(p,x)Li^6$, $C^{12}(p,x)Li^7$, $O^{16}(p,x)Li^6$, and $O^{16}(p,x)Li^7$ have been obtained from recent measurements[9,10]; these are shown in Fig. 2. σ for $N^{14}(p,x)Li^6$ and $N^{14}(p,x)Li^7$ are estimated from the measured cross section[11] of 3 μb at 150 MeV for $N^{14}(p,x)Li^9$ and by

[1] Heavy nuclei of the primary cosmic rays are generally classified into the following groups: L: $3\leqslant Z\leqslant 5$; M: $6\leqslant Z\leqslant 9$; H_1: $10\leqslant Z\leqslant 14$; H_2: $15\leqslant Z\leqslant 19$; H_3: $20\leqslant Z\leqslant 28$; and S: $Z\geqslant 6$. Al is chosen as representative of (H_1+H_2) groups and Fe of H_3 groups.

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