Radiative Equilibrium of a Free-Electron Gas in a Uniform Magnetic Field

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In this paper the subject of radiative equilibrium of a free-electron gas in a uniform magnetic field is studied on the basis of Landau's quantized particle motion of the electrons and the "Golden Rule" of time-dependent perturbation theory. The analysis presented here is somewhat analogous to the quantum theory of blackbody radiation. It is shown that the radiative equilibrium between electrons and their radiation field arises as a consequence of a balance between the two competing processes: photon emission (spontaneous plus induced or stimulated emission) and photon absorption. A coupled set of equations, one describing the time evolution of the photon distribution function and the other describing the time evolution of the electron distribution function, is thus derived. A simple closed-form solution of the equation for the time rate of change of the photon distribution function is presented for situations where absorption exceeeds the stimulated emission. This solution yields equations for the steady-state photon number density and the radiative relaxation time. It is shown that the steady-state photon number density is the familiar distribution function corresponding to Bose-Einstein statistics for "complete thermodynamic equilibrium." The conditions for the existence of overstabilities near the cyclotron frequency and its harmonics are also discussed. The equation for the time rate of change of the electron distribution function reduces, in the classical limit, to a Fokker-Planck equation in which there appear the usual "diffusion" and "dynamical friction" terms. In the classical limit, it is shown that this coupled set of equations (one describing the time evolution of the electromagnetic energy density and the other describing the time evolution of the electron distribution function) is self-consistent, by proving that the average rate of loss of z momentum of the electrons as predicted by one equation is equal to the average rate of gain of z momentum of their radiation field as predicted by the other equation. The z axis is chosen to be along the uniform magnetic field.

I. INTRODUCTION

It is well known that a particle of charge q and mass μ when placed in a uniform magnetic field $\mathbf{B}=B\hat{\imath}_z$ performs a circular cyclotron motion in a plane perpendicular to the uniform magnetic field with a characteristic fundamental frequency $\omega_b=(qB/\mu c)$. Such a circular motion of the charged particle is equivalent to that of an electrical harmonic oscillator. Thus, a free-electron gas placed in a uniform magnetic field is equivalent to a statistical system of a large number of electrical harmonic oscillators possessing the same fundamental frequency ω_b. According to quantum theory, these electrical harmonic oscillators will emit and absorb electromagnetic radiations of frequencies $\omega_k\approx l\omega_b$, where the harmonic number l may take any one of the values $0, \pm1, \pm2, \cdots, \pm\infty$. The emission and absorption at the zeroth harmonic of the electron cyclotron frequency ω_b (that is, when $l=0$) is essentially due to Compton recoil. The positive values (that is, $l=+1, +2, \cdots, +\infty$) and the negative values (that is, $l=-1, -2, \cdots, -\infty$) of l correspond to emission and absorption of circularly polarized plane electromagnetic waves (or photons) whose sense of rotation is the same as and opposite to that of the gyrating electrons, respectively. It is our aim in this paper to study the approach to radiative equilibrium of such a coupled system of electrons and their radiation field in the light of elementary nonrelativistic quantum mechanics.

This subject of radiative equilibrium of a free-electron gas in a uniform magnetic field is, of course, somewhat analogous to the quantum theory of blackbody radiation. It may be recalled that, according to Planck and Einstein,[1,2] the blackbody radiation spectrum is simply a manifestation of the radiative equilibrium between the material oscillators on the walls of the cavity and their radiation field (that is, the equivalent radiation-field oscillators). It is our aim in this paper to examine the radiative equilibrium between the material oscillators (that is, the gyrating electrons) within a box of volume L^3 (say) and their radiation field inside the box. It is therefore clear that there is a very close physical similarity between a blackbody and a free-electron gas placed in a uniform magnetic field. In this paper we will use this physical similarity as a guide for our mathematical discussions.

We will begin by assuming Landau's quantized particle motion of the electrons in a uniform magnetic field and calculate the transition probabilities for emission (spontaneous plus the induced or stimulated emission) and absorption of a photon of momentum $\hbar\mathbf{k}$, energy $\hbar\omega_k$, and polarization vector $\boldsymbol{\varepsilon}_{ks}$ by an electron in some initial quantum state by making use of the Golden Rule of time-dependent perturbation theory. From these, we will derive a coupled set of equations for the time rate of change of the photon and electron distribution functions. This pair of equations provides a simple physical picture of the way radiative equilibrium is established between the electrons and their radiation field. We will then solve the equation for the time rate of change of the photon distribution function and then show that this solution reduces to the familiar

[1] F. K. Richtmyer and E. H. Kennard, *Introduction to Modern Physics* (McGraw-Hill Book Company, Inc., New York, 1947).
[2] L. I. Schiff, *Quantum Mechanics* (McGraw-Hill Book Company, Inc., New York, 1955).

distribution function corresponding to Bose-Einstein statistics for "complete thermodynamic equilibrium"; and it yields the familiar two-stream instability for the zeroth harmonic of the electron cyclotron frequency and the familiar overstabilities near the cyclotron frequency and its higher harmonics. We will also calculate the radiative relaxation time of the system under study. We will then examine the classical limit of this coupled set of equations and show that this pair of equations is self-consistent and that the classical limit of the equation for the time rate of change of the electron distribution function is a Fokker-Planck equation.

II. GENERAL THEORY

We consider the motion of a free electron of charge q and mass μ in a uniform magnetic field $\mathbf{B}=B\hat{\imath}_z$. [See Fig. 1]. Let

$$\begin{aligned}\mathbf{R}&=x\hat{\imath}_x+y\hat{\imath}_y+z\hat{\imath}_z=\mathbf{r}+z\hat{\imath}_z,\\ \mathbf{V}&=v_x\hat{\imath}_x+v_y\hat{\imath}_y+v_z\hat{\imath}_z=\mathbf{v}+v_z\hat{\imath}_z,\\ \mathbf{P}&=p_x\hat{\imath}_x+p_y\hat{\imath}_y+p_z\hat{\imath}_z=\mathbf{p}+p_z\hat{\imath}_z\end{aligned} \tag{1}$$

be the position, velocity, and canonical momentum vectors of the electron. One can show that the energy-level spectrum of the electron is given by[3,4]

$$E_{n,p_z}=(n+\tfrac{1}{2})\hbar\omega_b+p_z^2/2\mu, \tag{2}$$

where $n=0, 1, 2,\cdots,\infty$ and $\omega_b=(qB/\mu c)$ is the electron cyclotron frequency. The nonzero matrix elements of the perpendicular velocity and position operators v_x, v_y, x, and y are given by[4]

$$\begin{aligned}v_{x_{n,n-1}}&=-iv_{y_{n,n-1}}=(\hbar\omega_b/2\mu)^{1/2}n^{1/2}e^{-i\phi_n},\\ v_{x_{n-1,n}}&=iv_{y_{n-1,n}}=(\hbar\omega_b/2\mu)^{1/2}n^{1/2}e^{i\phi_n},\end{aligned} \tag{3a}$$

$$\begin{aligned}x_{n,n-1}&=-iy_{n,n-1}=-i(\hbar/2\mu\omega_b)^{1/2}n^{1/2}e^{-i\phi_n},\\ x_{n-1,n}&=iy_{n-1,n}=i(\hbar/2\mu\omega_b)^{1/2}n^{1/2}e^{i\phi_n},\end{aligned} \tag{3b}$$

where ϕ_n is an arbitrary phase factor and the matrix elements of $e^{\pm ik_z z}$ are given by

$$\langle p_z'|e^{\pm ik_z z}|p_z\rangle=\delta_{p_z',p_z\pm\hbar k_z}. \tag{4}$$

The photon-electron interaction Hamiltonian that is responsible for transitions in which only one light quantum is involved is given by[4,5]

$$\mathcal{H}_{\rm int}=-\frac{q}{c}\left(\mathbf{v}-i\frac{\hbar}{\mu}\frac{\partial}{\partial z}\hat{\imath}_z\right)\cdot\mathbf{A}, \tag{5}$$

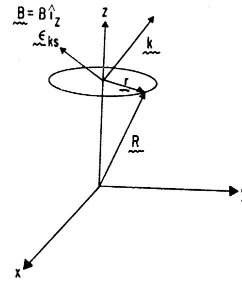

FIG. 1. Motion of an electron in Cartesian coordinates.

where the vector potential $\mathbf{A}(\mathbf{R})$ for the radiation field in a box of volume L^3 can be written in terms of the usual creation and annihilation operators as

$$\mathbf{A}(\mathbf{R})=\sum_{\mathbf{k}}\left(\frac{2\pi\hbar c^2}{L^3\omega_k}\right)^{1/2}\sum_{s=1,2}\boldsymbol{\varepsilon}_{ks}(a_{ks}e^{i\mathbf{k}\cdot\mathbf{R}}+a_{ks}^{\dagger}e^{-i\mathbf{k}\cdot\mathbf{R}}), \tag{6}$$

where $\mathbf{k}\cdot\boldsymbol{\varepsilon}_{ks}=0$ and $|\boldsymbol{\varepsilon}_{ks}|^2=1$. Each term in Eq. (6) represents light quanta all having a momentum $\hbar\mathbf{k}$, energy $\hbar\omega_k$, and polarization vector $\boldsymbol{\varepsilon}_{ks}$. According to time-dependent perturbation theory, the transition probability $j(f;i)$ from an initial state $|i\rangle$ of energy E_i to a final state $|f\rangle$ of energy E_f is given by[5]

$$j(f;i)=(2\pi/\hbar)|\langle f|\mathcal{H}_{\rm int}|i\rangle|^2\delta(E_f-E_i). \tag{7}$$

It is our aim in this paper to examine the way in which radiative equilibrium is established as a consequence of the photon-electron interaction [see Eq. (5)]. The fundamental emission (spontaneous plus induced or stimulated emission) and absorption processes that will drive the coupled electron-photon system towards radiative equilibrium are illustrated in Fig. 2. The two fundamental processes of emission $j_E(2a)$ and absorption $j_A(2a)$ illustrated in Fig. 2(a) will determine the time rate of change of the photon distribution function, while the time rate of change of the electron distribution function will be determined by the four fundamental processes of emission j_E(2a), j_E(2b) and absorption j_A(2a), j_A(2b) illustrated in Fig. 2(a) and Fig. 2(b).

From Eqs. (1), (2), and (4)–(7) we get the following transition probabilities for absorption j_A and emission j_E of a photon of momentum $\hbar\mathbf{k}$, energy $\hbar\omega_k\approx\hbar l\omega_b$ (where the harmonic number l may take any one of the the values 0, ±1, ±2, $\cdots$, $\pm\infty$), and polarization

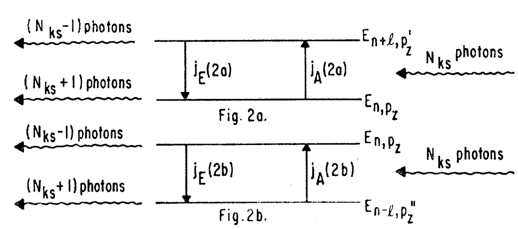

FIG. 2. Emission and absorption of a photon by a "Landau electron."

[3] L. D. Landau and E. M. Lifshitz, *Quantum Mechanics* (Addison-Wesley Publishing Company, Inc., Reading, Massachusetts, 1958).

[4] V. Arunasalam, Princeton Plasma Physics Laboratory Report No. MATT-439, 1966 (unpublished).

[5] W. Heitler, *The Quantum Theory of Radiation* (Clarendon Press, Oxford, England, 1954).

vector $\boldsymbol{\varepsilon}_{ks}$ by an electron

$$j_A(2a)=N_{ks}M^+(n+l,n)\delta_{v_z',v_z+\hbar k_z/\mu}\times\delta[\omega_k-l\omega_b-k_z(v_z+\hbar k_z/2\mu)],$$

$$j_E(2a)=(N_{ks}+1)M^-(n,n+l)\delta_{v_z',v_z+\hbar k_z/\mu}\times\delta[\omega_k-l\omega_b-k_z(v_z+\hbar k_z/2\mu)], \quad (8a)$$

$$j_A(2b)=N_{ks}M^+(n,n-l)\delta_{v_z'',v_z-\hbar k_z/\mu}\times\delta[\omega_k-l\omega_b-k_z(v_z-\hbar k_z/2\mu)],$$

$$j_E(2b)=(N_{ks}+1)M^-(n-l,n)\delta_{v_z'',v_z-\hbar k_z/\mu}\times\delta[\omega_k-l\omega_b-k_z(v_z-\hbar k_z/2\mu)], \quad (8b)$$

where

$$M^\pm(m,n)=\left(\frac{4\pi^2q^2}{L^3\hbar\omega_k}\right)\left|\langle m|\left(\mathbf{v}\pm\frac{\hbar k_z}{\mu}\hat{\imath}_z\right)\cdot\boldsymbol{\varepsilon}_{ks}e^{\pm i\mathbf{k}\cdot\mathbf{r}}|n\rangle\right|^2 \quad (9)$$

and N_{ks} represents the number of light quanta all having a momentum $\hbar\mathbf{k}$, energy $\hbar\omega_k$, and polarization vector $\boldsymbol{\varepsilon}_{ks}$. To evaluate the matrix elements appearing in Eq. (9) we will use the simplest form of the multipole expansion

$$e^{\pm i\mathbf{k}\cdot\mathbf{r}}=1\pm i(k_xx+k_yy)-(1/2!)(k_xx+k_yy)^2+\cdots$$
$$=1\pm i(k_+r_-+k_-r_+)-(1/2!)(k_+r_-+k_-r_+)^2+\cdots, \quad (10)$$

where

$$k_\pm=(1/\sqrt{2})(k_x\pm ik_y), \quad (11a)$$

and

$$r_\pm=(1/\sqrt{2})(x\pm iy) \quad (11b)$$

and the nonzero matrix elements of r_+ and r_- are given by

$$r_{+n-1,\,n}=(r_{-n,\,n-1})^*=i(\hbar/\mu\omega_b)^{1/2}n^{1/2}e^{i\phi_n}, \quad (12)$$

where the asterisk means the complex conjugate.

It is clear from the symmetry of the problem under study that the uniform magnetic field $\mathbf{B}=B\hat{\imath}_z$ defines the $\hat{\imath}_z$ axis uniquely while the $\hat{\imath}_x$ and $\hat{\imath}_y$ axes are not uniquely defined. That is, the $\hat{\imath}_x$ and $\hat{\imath}_y$ axes can be taken as any arbitrary pair of two mutually orthogonal axes in a plane perpendicular to $\mathbf{B}=B\hat{\imath}_z$. We now average Eqs. (8a) and (8b) over all such arbitrary pairs of axes $(\hat{\imath}_x,\hat{\imath}_y)$ since it is not very meaningful to distinguish between any two of such arbitrary pairs of axes $(\hat{\imath}_x,\hat{\imath}_y)$ and we will designate such an average by an angular bracket around the appropriate quantity. It is relatively easy to show that

$$\langle(\hat{\imath}_i\cdot\boldsymbol{\varepsilon}_{ks})(\hat{\imath}_j\cdot\boldsymbol{\varepsilon}_{ks})\rangle=\langle(\hat{\imath}_i\cdot\boldsymbol{\varepsilon}_{ks})^2\rangle\delta_{ij}, \quad (13a)$$

where i, $j=x$, y, and z. Further, one can show that

$$\langle(\hat{\imath}_x\cdot\boldsymbol{\varepsilon}_{ks})^2\rangle=\langle(\hat{\imath}_y\cdot\boldsymbol{\varepsilon}_{ks})^2\rangle=(\tfrac{1}{2})[1-(\hat{\imath}_z\cdot\boldsymbol{\varepsilon}_{ks})^2] \quad (13b)$$

since $\boldsymbol{\varepsilon}_{ks}\cdot\boldsymbol{\varepsilon}_{ks}=1$. From Eqs. (9), (13a), and (13b) we obtain

$$M(m,n)=M(n,m)=\langle M^\pm(m,n)\rangle=\langle M^\pm(n,m)\rangle$$
$$=(4\pi^2q^2/L^3\hbar\omega_k)\{\tfrac{1}{2}[1-(\hat{\imath}_z\cdot\boldsymbol{\varepsilon}_{ks})^2]$$
$$\times[|\langle m|v_xe^{\pm i\mathbf{k}\cdot\mathbf{r}}|n\rangle|^2+|\langle m|v_ye^{\pm i\mathbf{k}\cdot\mathbf{r}}|n\rangle|^2]$$
$$+(\hat{\imath}_z\cdot\boldsymbol{\varepsilon}_{ks})^2(\pm\hbar k_z/\mu)^2|\langle m|e^{\pm i\mathbf{k}\cdot\mathbf{r}}|n\rangle|^2\}, \quad (14)$$

since the perpendicular velocity and position operators $\mathbf{v}$ and $\mathbf{r}$ are Hermitian.

Let $N_0F(E_n,v_z)$ represent the number of electrons per unit volume which are in the quantum state $|E_n,v_z\rangle$. That is, $F(E_n,v_z)$ represents the probability that an electron will have a transverse energy $E_n=(n+\frac{1}{2})\hbar\omega_b$ in the $(\hat{\imath}_x,\hat{\imath}_y)$ plane and a longitudinal velocity v_z along the $\hat{\imath}_z$ axis. We will assume that this probability function $F(E_n,v_z)$ is normalized so that

$$\int dv_z\sum_{n=0}^{\infty}F(E_n,v_z)=1. \quad (15)$$

We now consider the way in which radiative equilibrium is established. We will compute $(\partial N_{ks}/\partial t)$ and $[\partial F(E_n,v_z)/\partial t]$, the time rate of change of the photon and electron distribution functions, in consequence of the photon-electron interaction [see Eq. (5)]. Using Eqs. (8a), (8b), (14), and (15) we will now calculate $(\partial N_{ks}/\partial t)_l$ and $[\partial F(E_n,v_z)/\partial t]_l$, corresponding to the emission (spontaneous plus induced or stimulated emission) and absorption of photons of energy $\hbar\omega_k\approx\hbar l\omega_b$, where the harmonic number $l=0, \pm1, \pm2, \cdots, \pm\infty$. The absorption and emission at the zeroth harmonic of the electron cyclotron frequency (that is, when $l=0$) are essentially due to Compton recoil. The positive values (that is, $l=+1, +2, \cdots, +\infty$) and the negative values (that is, $l=-1, -2, \cdots, -\infty$) of l correspond to absorption and emission of circularly polarized photons whose sense of rotation is the same as and opposite to that of the gyrating electrons, respectively.

We recall that in the box of volume L^3 we have N_{ks} photons all having a momentum $\hbar\mathbf{k}$, energy $\hbar\omega_k$, and polarization vector $\boldsymbol{\varepsilon}_{ks}$. Hence the photon number density is N_{ks}/L^3. By applying the principle of detailed balance for the transition probabilties per unit volume of emission and absorption, we obtain from Eqs. (8a), (14), and (15)

$$\left(\frac{\partial}{\partial t}(N_{ks}/L^3)\right)_l=\int dv_z\sum_{n=0}^{\infty}\times\{N_0F(E_{n+l},v_z')\langle j_E(2a)\rangle-N_0F(E_n,v_z)\langle j_A(2a)\rangle\}.$$

That is,

$$\left(\frac{\partial N_{ks}}{\partial t}\right)_l=\int dv_z\sum_{n=0}^{\infty}\{L^3N_0M(n,n+l)\times\delta[\omega_k-l\omega_b-k_z(v_z+\hbar k_z/2\mu)]\times[(N_{ks}+1)F(E_{n+l},v_z+\hbar k_z/\mu)-N_{ks}F(E_n,v_z)]\}, \quad (16)$$

where $\langle j\rangle$ is the average value of j averaged over all possible choices of axes $\hat{\imath}_x$ and $\hat{\imath}_y$. From Eqs. (8a), (8b), (14), and (15) we find that the time rate of change of the electron distribution function may be written

$$\left(\frac{\partial F(E_n,v_z)}{\partial t}\right)_l=\sum_{\mathbf{k}}\sum_{s=1,2}\left[F(E_{n+l},v_z')\langle j_E(2a)\rangle-F(E_n,v_z)\langle j_A(2a)\rangle+F(E_{n-l},v_z'')\langle j_A(2b)\rangle-F(E_n,v_z)\langle j_E(2b)\rangle\right]$$

$$=\sum_{\mathbf{k}}\sum_{s=1,2}\{[M(n,n+l)]\delta[\omega_k-l\omega_b-k_z(v_z+hk_z/2\mu)][(N_{ks}+1)F(E_{n+l},v_z+hk_z/\mu)-N_{ks}F(E_n,v_z)]$$

$$+[M(n,n-l)]\delta[\omega_k-l\omega_b-k_z(v_z-hk_z/2\mu)][N_{ks}F(E_{n-l},v_z-hk_z/\mu)-(N_{ks}+1)F(E_n,v_z)]\}. \quad (17)$$

Equation (17) may be called the "master equation" in accordance with the terminology used in the conventional transport theory.[6] In Eq. (17), the summation over $\mathbf{k}$ may be written

$$\sum_{\mathbf{k}}[\quad]\rightarrow\int d\mathbf{k}\text{ (density of points in }\mathbf{k}\text{ space)}[\quad]$$

$$=\left(\frac{L}{2\pi}\right)^3\int d\Omega_k\int_0^\infty k^2dk[\quad], \quad (18a)$$

where $d\Omega_k$ is the element of solid angle. If the medium in which the electrons are located has a static dielectric constant D_0, then $D_0\omega_k^2=c^2k^2$. Thus

$$k^2dk=\pm(D_0k\omega_k/c^2)d\omega_k=\pm(D_0^{1/2}/c)^3\omega_k^2d\omega_k. \quad (19)$$

Using Eq. (19), Eq. (18a) becomes

$$\sum[\quad]\rightarrow\left(\frac{LD_0^{1/2}}{2\pi c}\right)^3\int d\Omega_k\int_{-\infty}^{\infty}d\omega_k\omega_k^2[\quad]. \quad (18b)$$

The coupled set of Eqs. (16) and (17) for the photon and electron distribution functions provides a simple physical picture of the way equilibrium is established between the electrons (that is, the particle oscillators) and their radiation field (that is, the field oscillators). It may be noted that the first and third terms of Eq. (17) represent the gain of electrons in the state $|n,v_z\rangle$ as a result of emission (spontaneous plus induced or stimulated emission) and absorption of photons of energy $h\omega_k\approx hl\omega_b$ (where the harmonic number $l=0$, ±1, ±2, $\cdots$, $\pm\infty$), while the second and last terms of Eq. (17) represent the loss of electrons from this state $|n,v_z\rangle$ due to photon absorption and emission. It is therefore clear that the radiative equilibrium arises as a consequence of a balance between the two competing processes: photon emission (spontaneous plus induced or stimulated emission) and photon absorption. This is hardly surprising since this is how Albert Einstein gave the first systematic deduction of Planck's radiation formula.

Let us now examine the terms of Eq. (16) in some detail. In Eq. (16) we write

$$(N_{ks}+1)F(E_{n+l},v_z+hk_z/\mu)-N_{ks}F(E_n,v_z)$$
$$=N_{ks}[F(E_{n+l},v_z+hk_z/\mu)-F(E_n,v_z)]$$
$$+F(E_{n+l},v_z+hk_z/\mu). \quad (20)$$

It is clear that the first and second terms in square brackets of Eq. (20) represent the contributions from induced or stimulated emission and absorption, respectively, while the third term on the right-hand side of Eq. (20) represents the contribution from the spontaneous emission of photons. Usually,

$$I_l=\int dv_z\delta[\omega_k-l\omega_b-k_z(v_z+hk_z/2\mu)]$$
$$\times[F(E_{n+l},v_z+hk_z/\mu)-F(E_n,v_z)]<0, \quad (21)$$

for $l=0$, ±1, ±2, $\cdots$, $\pm\infty$, and consequently the absorption exceeds the induced or stimulated emission. But if, under certain circumstances, the probability function $F(E_n,v_z)$ exhibits an inverted population of states such that $I_l>0$, then the induced or stimulated emission will exceed the absorption and consequently the system of electrons will exhibit maser action for photons of frequency $\omega_k\approx l\omega_b$ (where the harmonic number $l=0$, ±1, ±2, $\cdots$, $\pm\infty$). It will be seen later that, for the zeroth harmonic of the electron cyclotron frequency (that is, for $l=0$), the condition $I_0>0$ yields a two-stream instability for large electron drift velocities, and for $l\neq0$ the condition $I_l>0$ yields the familiar overstabilities[7] near the cyclotron frequency and its harmonics for some values of $T_{\parallel}$ and $T_{\perp}$. Here $T_{\parallel}$ and $T_{\perp}$ are the kinetic temperatures of the electrons in directions parallel and perpendicular, respectively, to the uniform magnetic field $\mathbf{B}=B\hat{\imath}_z$. It is clear that when conditions are such that $I_l>0$, $(\partial N_{ks}/\partial t)_l>0$ and consequently, according to Eqs. (16) and (17), there is no possibility of radiative equilibrium. What may happen is that the number of photons N_{ks} will keep increasing until limited by nonlinear effects in the set of coupled equations (16) and (17), combined with the effects of the higher order terms which have been neglected in Eqs. (5) and (7) [that is, the effects of the higher order terms which occur in the improved form of Eq. (7) due to Heitler and Ma[5] and those arising from the $\mathbf{A}^2$ term which have been neglected[4] in Eq. (5)].

If conditions are such that $I_l<0$, then Eq. (16) has an equilibrium solution. If at time $t=0$ the number of photons $N_{ks}=0$, then the solution of Eq. (16) for a uniform homogeneous system is

$$N_{ks}=N_{ks}^{(0)}[1-e^{-\gamma_l(\omega_k)t}], \quad (22)$$

[6] C. Kittel, *Elementary Statistical Physics* (John Wiley & Sons, Inc., New York, 1958).

[7] T. H. Stix, *The Theory of Plasma Waves* (McGraw-Hill Book Company, Inc., New York, 1962).

where

$$N_{ks}^{(0)}=\frac{\int dv_z \sum_{n=0}^{\infty} M(n,n+l)\delta[\omega_k-l\omega_b-k_z(v_z+\hbar k_z/2\mu)]F(E_{n+l},v_z+\hbar k_z/\mu)}{\int dv_z \sum_{n=0}^{\infty} M(n,n+l)\delta[\omega_k-l\omega_b-k_z(v_z+\hbar k_z/2\mu)][F(E_n,v_z)-F(E_{n+l},v_z+\hbar k_z/\mu)]} \tag{23}$$

is the number of photons at steady state (that is, when $0=\partial N_{ks}/\partial t$), and

$$\gamma_l(\omega_k)=\int dv_z \sum_{n=0}^{\infty} [L^3N_0M(n,n+l)] \times\delta[\omega_k-l\omega_b-k_z(v_z+\hbar k_z/2\mu)] \times[F(E_n,v_z)-F(E_{n+l},v_z+\hbar k_z/\mu)]. \tag{24}$$

From Eq. (24) we obtain the radiative equilibrium time or the radiative relaxation time $t_l(\omega_k)$ for radiations of frequency $\omega_k\approx l\omega_b$ (where the harmonic number $l=0, \pm1, \pm2, \cdots, \pm\infty$),

$$t_l(\omega_k)=1/\gamma_l(\omega_k). \tag{25}$$

It is therefore clear from Eq. (22) that the number of photons N_{ks} of momentum $\hbar\mathbf{k}$, energy $\hbar\omega_k\approx\hbar l\omega_b$, and polarization vector $\boldsymbol{\varepsilon}_{ks}$ (in the box of volume L^3 under consideration) will increase with time towards a steady-state value of $N_{ks}^{(0)}$ as given by Eq. (23).

It is instructive and physically interesting to examine Eqs. (23) and (24) when the system of electrons is very near thermodynamic equilibrium. We may do this by assuming that $F(E_n,v_z)-F^{(0)}(E_n,v_z)=\Delta F(E_n,v_z) \ll F^{(0)}(E_n,v_z)$, where $F^{(0)}(E_n,v_z)$ represents the probability function at thermodynamic equilibrium. For simplicity we further assume that the system of electrons under consideration is nondegenerate and thus $F^{(0)}(E_n,v_z)$ is given by the canonical ensemble distribution function.[6,8] We write

$$F^{(0)}(E_n,v_z)=F_\perp^{(0)}(E_n)F_{\parallel}^{(0)}(v_z), \tag{26}$$

where

$$F_\perp^{(0)}(E_n)=2\sinh(\hbar\omega_b/2\kappa T_\perp) \times\exp[-(n+\tfrac{1}{2})\hbar\omega_b/\kappa T_\perp] \tag{27a}$$

and

$$F_{\parallel}^{(0)}(v_z)=(1/\sqrt{\pi})(1/v_{\parallel})\exp[-(v_z-v_d)^2/v_{\parallel}^2], \tag{27b}$$

where

$$v_{\parallel}=(2\kappa T_{\parallel}/\mu)^{1/2}. \tag{28}$$

Here $\perp$ and $\parallel$ refer to directions perpendicular and parallel, respectively, to the uniform magnetic field $\mathbf{B}=B\hat{\imath}_z$; T is the kinetic temperature of the electrons; and κ is the Boltzmann constant. By replacing F by $F^{(0)}$ in the right-hand side of Eqs. (23) and (24), we obtain

$$N_{ks}^{(0)}\approx\left(\frac{1}{\exp(l\hbar\omega_b/\kappa T_\perp)\exp[\hbar(\omega_k-l\omega_b-k_zv_d)/\kappa T_{\parallel}]-1}\right) \tag{23'}$$

and

$$\gamma_l(\omega_k)\approx\left(\frac{L^3N_02\sinh(\hbar\omega_b/2\kappa T_\perp)}{\pi^{1/2}v_{\parallel}|k_z|}\right)\{1-\exp(-l\hbar\omega_b/\kappa T_\perp)\exp[-\hbar(\omega_k-l\omega_b-k_zv_d)/\kappa T_{\parallel}]\} \times\exp-\left[\frac{\omega_k-l\omega_b-k_z(v_d+\hbar k_z/2\mu)}{k_zv_{\parallel}}\right]^2\{\sum_{n=0}^{\infty} M(n,n+l)\exp[-(n+\tfrac{1}{2})\hbar\omega_b/\kappa T_\perp]\}. \tag{24'}$$

Equations (22), (23′), and (24′) represent the linearized solution of Eq. (16). For "complete thermodynamic equilibrium," $F(E_n,v_z)=F^{(0)}(E_n,v_z)$ and $T_\perp=T_{\parallel}=T$ (say) and $v_d=0$, then Eqs. (23) and (23′) yield

$$N_{ks}^{(\mathrm{BE})}=\left(\frac{1}{e^{\hbar\omega_k/\kappa T}-1}\right). \tag{29}$$

This is the familiar distribution function corresponding to Bose-Einstein statistics.[6,8,9] This result is hardly surprising since the problem under study (that is, the subject of radiative equilibrium of a free-electron gas in a uniform magnetic field) is, of course, somewhat analogous to the quantum theory of blackbody radia-

[8] K. Huang, *Statistical Mechanics* (John Wiley & Sons, Inc., New York, 1963).

[9] L. D. Landau and E. M. Lifshitz, *Statistical Physics* (Addison-Wesley Publishing Company, Inc., Reading, Massachusetts, 1958).

tion. It may be recalled that, according to Planck and Einstein, the blackbody radiation spectrum is simply a manifestation of the radiative equilibrium between the material oscillators on the walls of the cavity and their radiation field (that is, the equivalent radiation-field oscillators). But Eq. (23′) is simply a statement of the radiative equilibrium between the material oscillators (that is, the gyrating electrons) within the box of volume L^3 under consideration and their radiation field. Thus, there is a very close physical similarity between a blackbody and a free-electron gas placed in a uniform magnetic field.

Let us now take a look at the condition $I_l>0$ [see Eq. (21)]. It is relatively easy to see that $I_l>0$ implies that

$$\frac{l\hbar\omega_b}{\kappa T_\perp}+\frac{\hbar(\omega_k-l\omega_b-k_zv_d)}{\kappa T_{\parallel}}<0.$$

That is,

$$\omega_k-k_zv_d<l\omega_b(1-T_{\parallel}/T_\perp). \tag{30}$$

It is clear that Eq. (30) represents the necessary condition for a two-stream instability for $l=0$, and for $l\neq 0$ this equation represents the necessary condition for overstabilities near the cyclotron frequency and its harmonics.[7] Under these conditions Eqs. (23′) and (24′) are invalid and Eqs. (16) and (17) do not possess any equilibrium solution.

Let us now examine specific cases corresponding to the values of the harmonic number l equal to 0, ± 1, and ± 2 in some detail.

III. DETAILED CALCULATIONS FOR A FEW VALUES OF THE HARMONIC NUMBER l

A. Harmonic Number $l=0$

For the zeroth harmonic of the electron cyclotron frequency, we obtain from Eqs. (3a), (10), (12), and (14)

$$M(n,n)=\left(\frac{\pi^2q^2\hbar}{L^3\omega_k\mu^2}\right)\{\tfrac{1}{2}[4(n+\tfrac{1}{2})^2+1]k_\perp^2[1-(\hat{\imath}_z\cdot\boldsymbol{\varepsilon}_{ks})^2]+4k_z^2(\hat{\imath}_z\cdot\boldsymbol{\varepsilon}_{ks})^2\}, \tag{31}$$

where

$$k_\perp^2=k_x^2+k_y^2=k^2-k_z^2=k^2[1-(\hat{\imath}_z\cdot\hat{k})^2]. \tag{32}$$

Here $\hat{k}=\mathbf{k}/k$ is the unit vector along the photon momentum $\hbar\mathbf{k}$. In Eq. (31) we have retained only the leading terms in the multipole expansion. Using Eq. (31) in Eq. (24′) we obtain

$$\gamma_0(\omega_k)=\left(\frac{\pi^{1/2}\omega_p^2\hbar}{4\omega_k\mu v_{\parallel}|k_z|}\right)\{\tfrac{1}{2}[4\langle(n+\tfrac{1}{2})^2\rangle_{\text{av}}+1]\times k_\perp^2[1-(\hat{\imath}_z\cdot\boldsymbol{\varepsilon}_{ks})^2]+4k_z^2(\hat{\imath}_z\cdot\boldsymbol{\varepsilon}_{ks})^2\}\times\{1-\exp[-\hbar(\omega_k-k_zv_d)/\kappa T_{\parallel}]\}\times\exp-\left[\frac{\omega_k-k_z(v_d+\hbar k_z/2\mu)}{k_zv_{\parallel}}\right]^2, \tag{33}$$

where

$$\langle(n+\tfrac{1}{2})^2\rangle_{\text{av}}=\sum_{n=0}^{\infty}(n+\tfrac{1}{2})^2F_\perp^{(0)}(E_n)=\left(\frac{1}{4}+\frac{1}{\exp(\hbar\omega_b/\kappa T_\perp)-1}+\frac{\exp(\hbar\omega_b/\kappa T_\perp)+1}{[\exp(\hbar\omega_b/\kappa T_\perp)-1]^2}\right), \tag{34}$$

and

$$\omega_p^2=(4\pi N_0q^2/\mu) \tag{35}$$

is the plasma frequency of the free-electron gas under study. From Eq. (34) one finds that for $\hbar\omega_b\ll\kappa T_\perp$, $\langle(n+\frac{1}{2})^2\rangle_{\text{av}}\approx(\kappa T_\perp/\hbar\omega_b)^2$. Thus, we find that the classical limit of Eq. (33) becomes

$$\lim_{\hbar\to 0}\gamma_0(\omega_k)=\left(\frac{\pi^{1/2}T_\perp\kappa T_\perp k_\perp^2\omega_p^2}{2T_{\parallel}\mu\omega_b^2\omega_k}\right)\left(\frac{\omega_k-k_zv_d}{v_{\parallel}|k_z|}\right)\times\exp-\left(\frac{\omega_k-k_zv_d}{k_zv_{\parallel}}\right)^2[1-(\hat{\imath}_z\cdot\boldsymbol{\varepsilon}_{ks})^2]. \tag{36}$$

This is the radiative relaxation frequency (that is, the reciprocal of the radiative relaxation time) of a classical plasma in a uniform magnetic field for the case corresponding to $l=0$.

If $\mathcal{E}_{ks}$ represents the energy in the electromagnetic wave of wave vector $\mathbf{k}$ and polarization vector $\boldsymbol{\varepsilon}_{ks}$, then

$$\mathcal{E}_{ks}=N_{ks}\hbar\omega_k. \tag{37}$$

Then the classical limit of Eq. (23′) becomes

$$\lim_{\hbar\to 0}\mathcal{E}_{ks}^{(0)}=\lim_{\hbar\to 0}N_{ks}^{(0)}\hbar\omega_k=(\kappa T_{\parallel})\left(\frac{\omega_k}{\omega_k-k_zv_d}\right). \tag{38}$$

Equation (38) gives the steady-state emission of electromagnetic waves of wave vector $\mathbf{k}$ and polarization vector $\boldsymbol{\varepsilon}_{ks}$ near the zeroth harmonic of the electron cyclotron frequency by a classical plasma placed in a uniform magnetic field. As seen from Eq. (21), Eq. (38) is valid only if $k_zv_d<\omega_k$. It is clear that $\lim_{\hbar\to 0}\mathcal{E}_{ks}^{(0)}\to$ large values as $k_zv_d\to\omega_k$ and the system of electrons tends towards the conditions appropriate to that of two-stream instability or inverse Landau damping. If the medium in which the electrons are located has a large value for its static dielectric constant D_0, then one may be able to achieve the condition $k_zv_d>\omega_k=(ck/D_0^{1/2})$ in practice.

Let us now examine the classical limit of the coupled set of Eqs. (16) and (17). We may do this with the aid of the following relations:

$$E_n=(n+\tfrac{1}{2})\hbar\omega_b \to E_\perp=\tfrac{1}{2}\mu v^2,$$

$$F(E_n,v_z)\to F(E_\perp,v_z),$$

$$\sum_{n=0}^{\infty}[\quad]\to\int_0^\infty dE_\perp[\quad],$$

$$F(E_{n\pm l},v_z\pm\hbar k_z/\mu)\to\left[1\pm\left(\hbar l\omega_b\frac{\partial}{\partial E_\perp}+\frac{\hbar k_z}{\mu}\frac{\partial}{\partial v_z}\right)+\tfrac{1}{2}\left(\hbar l\omega_b\frac{\partial}{\partial E_\perp}+\frac{\hbar k_z}{\mu}\frac{\partial}{\partial v_z}\right)\times\left(\hbar l\omega_b\frac{\partial}{\partial E_\perp}+\frac{\hbar k_z}{\mu}\frac{\partial}{\partial v_z}\right)+\cdots\right]F(E_\perp,v_z),$$

$$\delta[\omega_k-l\omega_b-k_z(v_z\pm\hbar k_z/2\mu)]\to\left[1\pm\left(\frac{\hbar k_z}{2\mu}\frac{\partial}{\partial v_z}\right)+\frac{1}{2}\left(\frac{\hbar k_z}{2\mu}\frac{\partial}{\partial v_z}\right)\left(\frac{\hbar k_z}{2\mu}\frac{\partial}{\partial v_z}\right)+\cdots\right]\delta[\omega_k-l\omega_b-k_zv_z], \tag{39}$$

where $E_\perp$ is the kinetic energy of the electrons in a plane perpendicular to the uniform magnetic field $\mathbf{B}=B\hat{\imath}_z$.

On making use of Eqs. (31), (35), (37), (39), and (18b) one finds, after a certain amount of algebra, that the classical limits of the coupled set of equations (16) and (17) for $l=0$ become

$$\left(\frac{\partial\mathcal{E}_{ks}}{\partial t}\right)_0=\int dv_z\int dE_\perp\left(\frac{\pi\omega_p{}^2k_\perp{}^2[1-(\hat{\imath}_z\cdot\boldsymbol{\varepsilon}_{ks})^2]}{2\omega_b{}^2\mu}\right)E_\perp{}^2\left[\left(\frac{\omega_b}{\omega_k}\mathcal{E}_{ks}\right)[\delta(\omega_k-k_zv_z)Q_0F(E_\perp,v_z)]+[\delta(\omega_k-k_zv_z)F(E_\perp,v_z)]\right], \tag{40}$$

and

$$\left(\frac{\partial F(E_\perp,v_z)}{\partial t}\right)_0=\sum_{\mathbf{k}}\sum_{s=1,2}\left(\frac{2\pi^2q^2k_\perp{}^2[1-(\hat{\imath}_z\cdot\boldsymbol{\varepsilon}_{ks})^2]}{L^3\omega_k\mu^2\omega_b}\right)E_\perp{}^2\left[\left(\frac{\omega_b}{\omega_k}\mathcal{E}_{ks}\right)Q_0[\delta(\omega_k-k_zv_z)Q_0F(E_\perp,v_z)]+Q_0[\delta(\omega_k-k_zv_z)F(E_\perp,v_z)]\right]$$

$$=\left(\frac{q^2D_0{}^{3/2}}{4\pi\mu^2\omega_bc^3}\right)\int d\Omega_k\sum_{s=1,2}\int_{-\infty}^{\infty}d\omega_k\omega_kk_\perp{}^2[1-(\hat{\imath}_z\cdot\boldsymbol{\varepsilon}_{ks})^2]E_\perp{}^2\left[\left(\frac{\omega_b}{\omega_k}\mathcal{E}_{ks}\right)Q_0\times[\delta(\omega_k-k_zv_z)Q_0F(E_\perp,v_z)]+Q_0[\delta(\omega_k-k_zv_z)F(E_\perp,v_z)]\right], \tag{41}$$

respectively, where the linear differential operator Q_0 is

$$Q_0=\left(\frac{k_z}{\mu\omega_b}\frac{\partial}{\partial v_z}\right). \tag{42}$$

The first term of Eq. (40) represents the rate of damping (that is, the so-called Landau damping, the rate of absorption minus the rate of stimulated emission) of the electromagnetic waves, while the second term of this equation gives the appropriate classical rate of spontaneous emission of the electromagnetic waves. It is seen that Eq. (41) is a Fokker-Planck equation whose first and second terms represent a "diffusion" and "dynamical friction," respectively, in the usual sense.

It is interesting and physically instructive to examine the self-consistency of the coupled set of Eqs. (40) and (41). We may do this by examining this coupled set of equations for conservation laws of z momentum. That is, we now wish to show from Eqs. (40) and (41) that the average rate of loss of the z momentum of the L^3N_0 electrons in the box of volume L^3 under consideration is equal to the average rate of gain of the z momentum of the radiation field inside the box.

The average rate of gain of particle z momentum

$$=\int dE_\perp\int dv_zL^3N_0\mu v_z\left(\frac{\partial F(E_\perp,v_z)}{\partial t}\right)_0$$

$$=-\int dE_\perp\int dv_z\left(\frac{L^3\omega_p{}^2D_0{}^{3/2}}{16\pi^2\mu\omega_b{}^2c^3}\right)\times\int d\Omega_k\sum_{s=1,2}[1-(\hat{\imath}_z\cdot\boldsymbol{\varepsilon}_{ks})^2]\int_{-\infty}^{\infty}d\omega_k\omega_kk_\perp{}^2k_zE_\perp{}^2\chi_0, \tag{43}$$

where

$$\chi_0=\{[(\omega_b/\omega_k)\mathcal{E}_{ks}][\delta(\omega_k-k_zv_z)Q_0F(E_\perp,v_z)]+[\delta(\omega_k-k_zv_z)F(E_\perp,v_z)]\}. \tag{44}$$

In obtaining Eq. (43) from Eq. (41) we have carried out an integration by parts and the second term resulting from the integration by parts vanishes at $v_z=\pm\infty$. Since the photon z momentum $=\hbar k_z=\hbar\omega_k(k_z/\omega_k)$, we obtain:

The average rate of gain of the z momentum of the radiation field

$$=\sum_{\mathbf{k}}\sum_{s=1,2}\left(\frac{k_z}{\omega_k}\right)\left(\frac{\partial\mathcal{E}_{ks}}{\partial t}\right)_0$$
$$=\int dE_\perp\int dv_z\left(\frac{L^3\omega_p{}^2D_0{}^{3/2}}{16\pi^2\mu\omega_b{}^2c^3}\right)$$
$$\times\int d\Omega_k\sum_{s=1,2}[1-(\hat{\imath}_z\cdot\boldsymbol{\varepsilon}_{ks})^2]\int_{-\infty}^{\infty}d\omega_k\omega_k k_\perp{}^2k_zE_\perp{}^2\chi_0. \tag{45}$$

In obtaining Eq. (45) from Eq. (40) we have made use of Eq. (18b). Thus, from Eqs. (43) and (45) it is clear that the average rate of loss of the particle z momentum as predicted by Eq. (41) is exactly equal to the average rate of gain of the z momentum of the radiation field as predicted by Eq. (40). This result is hardly surprising and is simply a manifestation of the z-momentum conservation relations implied by the Kronecker δ's of Eqs. (8a) and (8b). The Dirac δ functions of Eqs. (8a) and (8b) indicate the conservation of total energy.

B. Harmonic Numbers $l=+1$ and $l=-1$

When the harmonic number l take the values $+1$ and -1, we obtain from Eqs. (3a), (10), (12), and (14),

$$M(n,n\pm1)=\begin{Bmatrix}n+1\\ n\end{Bmatrix}\left(\frac{2\pi^2q^2\omega_b}{L^3\mu\omega_k}\right)$$
$$\times\left[[1-(\hat{\imath}_z\cdot\boldsymbol{\varepsilon}_{ks})^2]+\left(\frac{\hbar k_z}{\mu\omega_b}\right)^2k_\perp{}^2(\hat{\imath}_z\cdot\boldsymbol{\varepsilon}_{ks})^2\right], \tag{46}$$

where the term

$$\begin{Bmatrix}n+1\\ n\end{Bmatrix}$$

means that $(n+1)$ belongs to $M(n,n+1)$ and n belongs to $M(n,n-1)$. In Eq. (46) we have retained only the leading terms in the multipole expansion. Using Eq. (46) in Eq. (24′) we obtain

$$\gamma_{\pm1}(\omega_k)=\left(\frac{\pi^{1/2}\omega_p{}^2\omega_b}{2\omega_kv_{\shortparallel}|k_z|}\right)\Bigg[[1-(\hat{\imath}_z\cdot\boldsymbol{\varepsilon}_{ks})^2]$$
$$+\left(\frac{\hbar k_z}{\mu\omega_b}\right)^2k_\perp{}^2(\hat{\imath}_z\cdot\boldsymbol{\varepsilon}_{ks})^2\Bigg][\langle n+\tfrac{1}{2}\rangle_{\text{av}}\pm\tfrac{1}{2}]$$
$$\times\{1-\exp(\mp\hbar\omega_b/\kappa T_\perp)\exp[-\hbar(\omega_k\mp\omega_b-k_zv_d)/\kappa T_{\shortparallel}]\}$$
$$\times\exp-\left[\frac{\omega_k\mp\omega_b-k_z(v_d+\hbar k_z/2\mu)}{k_zv_{\shortparallel}}\right]^2, \tag{47}$$

where

$$\langle(n+\tfrac{1}{2})\rangle_{\text{av}}=\left(\tfrac{1}{2}+\frac{1}{\exp(\hbar\omega_b/\kappa T_\perp)-1}\right). \tag{48}$$

From Eq. (48) one finds that for

$$\hbar\omega_b\ll\kappa T_\perp,\ \langle(n+\tfrac{1}{2})\rangle_{\text{av}}\approx(\kappa T_\perp/\hbar\omega_b).$$

Thus, we find that the classical limit of Eq. (47) becomes

$$\lim_{\hbar\to0}\gamma_{\pm1}(\omega_k)=\left(\frac{\pi^{1/2}\omega_p{}^2\omega_b}{2\omega_kv_{\shortparallel}|k_z|}[1-(\hat{\imath}_z\cdot\boldsymbol{\varepsilon}_{ks})^2]\right)$$
$$\times\left(\frac{T_\perp(\omega_k\mp\omega_b-k_zv_d)}{T_{\shortparallel}\omega_b}\pm1\right)$$
$$\times\exp-\left(\frac{\omega_k\mp\omega_b-k_zv_d}{k_zv_{\shortparallel}}\right)^2. \tag{49}$$

$\gamma_{+1}(\omega_k)$ and $\gamma_{-1}(\omega_k)$ are the radiative relaxation frequencies of a classical plasma in a uniform magnetic field for radiations of frequencies $\omega_k\approx\omega_b$ and $\omega_k\approx-\omega_b$, respectively [that is, $\gamma_{+1}(\omega_k)$ and $\gamma_{-1}(\omega_k)$ are the radiative relaxation frequencies appropriate to circularly polarized plane electromagnetic waves whose sense of rotation is the same as and opposite to that of the gyrating electrons, respectively]. Using Eq. (37), the classical limit of Eq. (23′) may be written

$$(\lim_{\hbar\to0}\mathcal{E}_{ks})_{l=\pm1}=(\kappa T_{\shortparallel})\left(\frac{\omega_k}{\omega_k-k_zv_d\mp\omega_b(1-T_{\shortparallel}/T_\perp)}\right). \tag{50}$$

Equation (50) gives the steady-state emission of circularly polarized plane electromagnetic waves of wave vector $\mathbf{k}$ near the electron cyclotron frequency by a classical plasma placed in a uniform magnetic field. ($l=+1$ and $l=-1$ correspond to radiations whose sense of rotation is the same as and opposite to that of the gyrating electrons, respectively.) For the case of $l=+1$, it is clear from Eq. (21) that Eq. (50) is valid only if $\omega_k-k_zv_d>\omega_b(1-T_{\shortparallel}/T_\perp)$; and when $\omega_k-k_zv_d<\omega_b(1-T_{\shortparallel}/T_\perp)$ the system under consideration is unstable in this linearized theory and such a state is usually referred to as the condition of cyclotron overstability. Similarly for the case of $l=-1$, Eq. (50) is valid only if $\omega_k-k_zv_d>-\omega_b(1-T_{\shortparallel}/T_\perp)$, and when $\omega_k-k_zv_d<-\omega_b(1-T_{\shortparallel}/T_\perp)$ the system under study is unstable.

On making use of Eqs. (46), (35), (37), and (39) one finds, after a certain amount of algebra, that the classical limits of the coupled set of equations (16) and (17) for $l=+1$ and $l=-1$ become

$$\left(\frac{\partial\mathcal{E}_{ks}}{\partial t}\right)_{\pm1}=(\pm)\int dv_z\int dE_\perp\left[\frac{\pi}{2}\omega_p{}^2[1-(\hat{\imath}_z\cdot\boldsymbol{\varepsilon}_{ks})^2]\right]$$
$$\times E_\perp\Bigg[\left(\frac{\omega_b}{\omega_k}\mathcal{E}_{ks}\right)[\delta(\omega_k\mp\omega_b-k_zv_z)Q_{\pm1}F(E_\perp,v_z)]$$
$$\pm[\delta(\omega_k\mp\omega_b-k_zv_z)F(E_\perp,v_z)]\Bigg], \tag{51}$$

and

$$\left(\frac{\partial F(E_\perp,v_z)}{\partial t}\right)_{\pm 1}=\sum_{\mathbf{k}}\sum_{s=1,2}\left(\frac{2\pi^2q^2\omega_b[1-(\hat{\imath}_z\cdot\boldsymbol{\varepsilon}_{ks})^2]}{L^3\mu\omega_k}\right)$$

$$\times\left[\left(\frac{\omega_b}{\omega_k}\mathcal{E}_{ks}\right)\{(1+E_\perp Q_{\pm 1})\right.$$

$$\times[\delta(\omega_k\mp\omega_b-k_zv_z)Q_{\pm 1}F(E_\perp,v_z)]\}$$

$$\left.\pm\{(1+E_\perp Q_{\pm 1})[\delta(\omega_k\mp\omega_b-k_zv_z)F(E_\perp,v_z)]\}\right], \tag{52}$$

respectively, where the linear differential operators Q_{+1} and Q_{-1} are

$$Q_{\pm 1}=\left(\frac{\partial}{\partial E_\perp}\pm\frac{k_z}{\mu\omega_b}\frac{\partial}{\partial v_z}\right). \tag{53}$$

In deriving Eq. (52) from Eq. (17) we have made use of the relation

$$\frac{\partial}{\partial E_\perp}[\delta(\omega_k-l\omega_b-k_zv_z)]=0 \tag{54}$$

for any value of the harmonic number l since the above δ function is not an explicit function of the perpendicular energy $E_\perp$. To evaluate the sum $\sum_{\mathbf{k}}$ in Eq. (52) one has to simply use the prescription of Eq. (18b). The first term of Eq. (51) represents the rate of damping (the so-called cyclotron damping; that is, the rate of absorption minus the rate of stimulated or induced emission) of the electromagnetic waves, while the second term of this equation gives the appropriate classical rate of spontaneous emission of the electromagnetic waves. Since $E_\perp=\mu v^2/2$, $(\partial/\partial E_\perp)=(1/\mu v)(\partial/\partial v)$. It is therefore apparent that Eq. (52), which is the classical limit of the "master equation," Eq. (17), is a Fokker-Planck equation whose first and second terms represent a "diffusion" and "dynamical friction," respectively. It may be pointed out that Pines and Schrieffer[10] have previously obtained somewhat structurally similar equations for electron-plasmon and electron-phonon systems. But it must be borne in mind that in this paper we are dealing only with transverse photons (that is, transverse bosons) and not with longitudinal and scalar bosons such as the plasmons or phonons.

Let us now show by examining the conservation laws of the z momentum that the coupled set of equations (51) and (52) is self-consistent. That is, we will now prove that the average rate of loss of z momentum of the electrons as predicted by Eq. (52) is equal to the average rate of gain of the z momentum of their radiation field as predicted by Eq. (51).

The average rate of gain of particle z momentum

$$=\int dv_z\int dE_\perp L^3N_0\mu v_z\left(\frac{\partial F(E_\perp,v_z)}{\partial t}\right)_{\pm 1}$$

$$=(\mp)\int dv_z\int dE_\perp\sum_{\mathbf{k}}\sum_{s=1,2}\left(\frac{\pi}{2}\omega_p^2[1-(\hat{\imath}_z\cdot\boldsymbol{\varepsilon}_{ks})^2]\right)$$

$$\times\left(\frac{k_z}{\omega_k}\right)E_\perp\chi_{\pm 1}, \tag{55}$$

where

$$\chi_{\pm 1}=\{[(\omega_b/\omega_k)\mathcal{E}_{ks}][\delta(\omega_k\mp\omega_b-k_zv_z)Q_{\pm 1}F(E_\perp,v_z)]$$
$$\pm[\delta(\omega_k\mp\omega_b-k_zv_z)F(E_\perp,v_z)]\}. \tag{56}$$

In obtaining Eq. (55) from Eq. (52) we have carried out two integrations by parts, one over $dE_\perp$ and the other over dv_z, and the constant terms resulting from the parts integrations vanish at $E_\perp=0$ and $E_\perp=+\infty$ and at $v_z=\pm\infty$.

The average rate of gain of the z momentum of the radiation field

$$=\sum_{\mathbf{k}}\sum_{s=1,2}\left(\frac{k_z}{\omega_k}\right)\left(\frac{\partial\mathcal{E}_{ks}}{\partial t}\right)_{\pm 1}$$

$$=(\pm)\sum_{\mathbf{k}}\sum_{s=1,2}\int dv_z\int dE_\perp\left(\frac{\pi}{2}\omega_p^2[1-(\hat{\imath}_z\cdot\boldsymbol{\varepsilon}_{ks})^2]\right)$$

$$\times\left(\frac{k_z}{\omega_k}\right)E_\perp\chi_{\pm 1}, \tag{57}$$

where we have made use of Eq. (51). Thus, it is readily seen from Eqs. (55) and (57) that the total z momentum of the electrons and their radiation field in the box of volume L^3 under consideration is really conserved.

C. Harmonic Numbers $l=+2$ and $l=-2$

Similarly, when the harmonic number l take the values $+2$ and -2, we obtain from Eqs. (3a), (10), (12), and (14).

$$M(n,n\pm 2)=\left\{\begin{matrix}(n+2)(n+1)\\ n(n-1)\end{matrix}\right\}\left(\frac{\pi^2q^2hk_\perp^2}{L^3\omega_k\mu^2}\right)$$

$$\times\left[[1-(\hat{\imath}_z\cdot\boldsymbol{\varepsilon}_{ks})^2]+\left(\frac{hk_z}{2\mu\omega_b}\right)^2k_\perp^2(\hat{\imath}_z\cdot\boldsymbol{\varepsilon}_{ks})^2\right], \tag{58}$$

where we have retained only the leading terms in the multipole expansion. Using Eq. (58) in Eq. (24′) we

[10] D. Pines and J. R. Schrieffer, Phys. Rev. 125, 804 (1962).

obtain

$$\gamma_{\pm 2}(\omega_k)=\left(\frac{\pi^{1/2}\omega_p{}^2\hbar k_\perp{}^2}{4\mu\omega_k v_{\shortparallel}|k_z|}\right)\left[[1-(\hat{\imath}_z\cdot\boldsymbol{\varepsilon}_{ks})^2]+\left(\frac{\hbar k_z}{2\mu\omega_b}\right)^2 k_\perp{}^2(\hat{\imath}_z\cdot\boldsymbol{\varepsilon}_{ks})^2\right][\langle(n+\tfrac{1}{2})^2\rangle_{\rm av}\pm 2\langle(n+\tfrac{1}{2})\rangle_{\rm av}+\tfrac{3}{4}]$$
$$\times\{1-\exp(\mp 2\hbar\omega_b/\kappa T_\perp)\exp[-\hbar(\omega_k\mp 2\omega_b-k_zv_d)/\kappa T_{\shortparallel}]\}\exp\left\{-\left[\frac{\omega_k\mp 2\omega_b-k_z(v_d+\hbar k_z/2\mu)}{k_zv_{\shortparallel}}\right]^2\right\}, \quad (59)$$

where $\langle(n+\frac{1}{2})^2\rangle_{\rm av}$ and $\langle(n+\frac{1}{2})\rangle_{\rm av}$ are given by Eqs. (34) and (48), respectively.

The classical limit of Eq. (59) is

$$\lim_{h\to 0}\gamma_{\pm 2}(\omega_k)=\left[\frac{\pi^{1/2}\kappa T_\perp\omega_p{}^2k_\perp{}^2[1-(\hat{\imath}_z\cdot\boldsymbol{\varepsilon}_{ks})^2]}{2\mu\omega_kv_{\shortparallel}|k_z|\omega_b}\right]\times\left[\frac{T_\perp(\omega_k\mp 2\omega_b-k_zv_d}{T_{\shortparallel}2\omega_b}\pm 1\right]\times\exp\left[-\left(\frac{\omega_k\mp 2\omega_b-k_zv_d}{k_zv_{\shortparallel}}\right)^2\right]. \quad (60)$$

Using Eq. (37), the classical limit of Eq. (23′), may be written

$$(\lim_{h\to 0}\mathcal{E}_{ks})_{l=\pm 2}=(\kappa T_{\shortparallel})\left(\frac{\omega_k}{\omega_k-k_zv_d\mp 2\omega_b(1-T_{\shortparallel}/T_\perp)}\right). \quad (61)$$

For the case $l=+2$, Eq. (61) is valid only if $\omega_k-k_zv_d>2\omega_b(1-T_{\shortparallel}/T_\perp)$, and it predicts an overstability when $\omega_k-k_zv_d<2\omega_b(1-T_{\shortparallel}/T_\perp)$. Similarly, for the case $l=-2$, this equation is valid only if $\omega_k-k_zv_d>-2\omega_b(1-T_{\shortparallel}/T_\perp)$, and an overstability occurs when $\omega_k-k_zv_d<-2\omega_b\times(1-T_{\shortparallel}/T_\perp)$. On making use of Eqs. (58), (35), (37), and (39) one finds, after a certain amount of algebra, that the classical limits of the coupled set of Eqs. (16) and (17) for $l=+2$ and $l=-2$ become

$$\left(\frac{\partial\mathcal{E}_{ks}}{\partial t}\right)_{\pm 2}=(\pm)\int dv_z\int dE_\perp\left(\frac{\pi\omega_p{}^2k_\perp{}^2[1-(\hat{\imath}_z\cdot\boldsymbol{\varepsilon}_{ks})^2]}{4\mu\omega_b{}^2}\right)E_\perp{}^2\times\left[\left(\frac{2\omega_b}{\omega_k}\mathcal{E}_{ks}\right)[\delta(\omega_k\mp 2\omega_b-k_zv_z)Q_{\pm 2}F(E_\perp,v_z)]\pm[\delta(\omega_k\mp 2\omega_b-k_zv_z)F(E_\perp,v_z)]\right] \quad (62)$$

and

$$\left(\frac{\partial F(E_\perp,v_z)}{\partial t}\right)_{\pm 2}=\sum_{\mathbf{k}}\sum_{s=1,2}\left(\frac{2\pi^2q^2k_\perp{}^2[1-(\hat{\imath}_z\cdot\boldsymbol{\varepsilon}_{ks})^2]}{L^3\omega_k\omega_b\mu^2}\right)\times\left[\left(\frac{2\omega_b}{\omega_k}\mathcal{E}_{ks}\right)\{(2E_\perp+E_\perp{}^2Q_{\pm 2})\times[\delta(\omega_k\mp 2\omega_b-k_zv_z)Q_{\pm 2}F(E_\perp,v_z)]\}\pm\{(2E_\perp+E_\perp{}^2Q_{\pm 2})[\delta(\omega_k\mp 2\omega_b-k_zv_z)F(E_\perp,v_z)]\}\right], \quad (63)$$

respectively, where the linear differential operators Q_{+2} and Q_{-2} are

$$Q_{\pm 2}=\left(\frac{\partial}{\partial E_\perp}\pm\frac{k_z}{2\mu\omega_b}\frac{\partial}{\partial v_z}\right) \quad (64)$$

and in deriving Eq. (63) from the "master equation" (17) we have made use of Eq. (54). To evaluate the sum $\sum_{\mathbf{k}}$ in Eq. (63) we have simply to use the prescription of Eq. (18b). It is again apparent that Eq. (63) is a Fokker-Planck equation.

The average rate of gain of particle z momentum

$$=\int dv_z\int dE_\perp L^3N_0\mu v_z\left(\frac{\partial F(E_\perp,v_z)}{\partial t}\right)_{\pm 2}$$
$$=(\mp)\int dv_z\int dE_\perp\sum_{\mathbf{k}}\sum_{s=1,2}\left(\frac{\pi\omega_p{}^2k_\perp{}^2[1-(\hat{\imath}_z\cdot\boldsymbol{\varepsilon}_{ks})^2]}{4\mu\omega_b{}^2}\right)\times\left(\frac{k_z}{\omega_k}\right)E_\perp{}^2\chi_{\pm 2}, \quad (65)$$

where

$$\chi_{\pm 2}=\{[(2\omega_b/\omega_k)\mathcal{E}_{ks}][\delta(\omega_k\mp 2\omega_b-k_zv_z)Q_{\pm 2}F(E_\perp,v_z)]\pm[\delta(\omega_k\mp 2\omega_b-k_zv_z)F(E_\perp,v_z)]\}. \quad (66)$$

Here again in obtaining Eq. (65) from Eq. (63) we have carried out two integrations by parts, one over $dE_\perp$ and the other over dv_z, and the constant terms resulting from the parts integrations vanish at $E_\perp=0$, $E_\perp+\infty$, and at $v_z=\pm\infty$. From Eq. (62) we obtain the average rate of gain of the z momentum of radiation field to be

$$\sum_{\mathbf{k}}\sum_{s=1,2}\left(\frac{k_z}{\omega_k}\right)\left(\frac{\partial\mathcal{E}_{ks}}{\partial t}\right)_{\pm 2}=(\pm)\sum_{\mathbf{k}}\sum_{s=1,2}\int dv_z\int dE_\perp\left(\frac{\pi\omega_p{}^2k_\perp{}^2[1-(\hat{\imath}_z\cdot\boldsymbol{\varepsilon}_{ks})^2]}{4\mu\omega_b{}^2}\right)\times\left(\frac{k_z}{\omega_k}\right)E_\perp{}^2\chi_{\pm 2}. \quad (67)$$

Hence, we find that the average rate of loss of z momentum of the electrons as given by Eq. (63) is equal to the average rate of gain of the z momentum of their radiation field as given by Eq. (62). Thus, we believe that the coupled set of Eqs. (62) and (63) is self-consistent.

IV. GENERAL REMARKS

We remarked earlier that when conditions are such that I_l of Eq. (21) is greater than zero, there is no possibility of radiative equilibrium in the above simple theory and the number of photons N_{ks} will keep increasing until limited by nonlinear effects in the set of coupled Eqs. (16) and (17), combined with the effects of the higher order terms which have been neglected in Eqs. (5) and (7) [that is, the effects of the higher order terms which occur in the improved form of Eq. (7) due to Heitler and Ma and those arising from the $\mathbf{A}^2$ term which have been neglected in Eq. (5)]. Under such conditions one has to replace Eq. (5) by[4,5]

$$T_{\text{int}}=\mathcal{H}_{\text{int}}+(q^2/2\mu c^2)\mathbf{A}^2, \tag{68}$$

and according to Heitler and Ma[5] one has to replace Eq. (7) by

$$j(f;i)=(2\pi/\hbar)|\langle f|T|i\rangle|^2\delta(E_f-E_i), \tag{69}$$

where

$$\langle f|T|i\rangle=\langle f|T_{\text{int}}|i\rangle+\sum_{f'\neq i}\frac{\langle f|T_{\text{int}}|f'\rangle\langle f'|T|i\rangle}{E_i-E_{f'}}, \tag{70}$$

where $|f'\rangle$ is an intermediate state of energy $E_{f'}$. The series solution of the integral equation (70) may be written

$$\langle f|T|i\rangle=\langle f|T_{\text{int}}|i\rangle+\sum_{f'\neq i}\frac{\langle f|T_{\text{int}}|f'\rangle\langle f'|T_{\text{int}}|i\rangle}{E_i-E_{f'}}$$
$$+\sum_{f'\neq i}\sum_{f''\neq i}\frac{\langle f|T_{\text{int}}|f'\rangle\langle f'|T_{\text{int}}|f''\rangle\langle f''|T_{\text{int}}|i\rangle}{(E_i-E_{f'})(E_i-E_{f''})}+\cdots, \tag{71}$$

where $|f''\rangle$ is another intermediate state of energy $E_{f''}$. In this way one can, in principle, formulate a quasilinear particle orbit theory of a free-electron gas in a uniform magnetic field.[11,12,4]

The classical Vlasov equation for a free-electron gas (that is, a noninteracting system of electrons) placed in a uniform magnetic field $\mathbf{B}=B\hat{\imath}_z$ may be written

$$\frac{\partial f}{\partial t}+\mathbf{v}\cdot\frac{\partial f}{\partial \mathbf{r}}+v_z\frac{\partial f}{\partial z}=\left[-\frac{q}{\mu c}(\mathbf{v}\times\mathbf{B})\cdot\frac{\partial f}{\partial \mathbf{v}}\right]\equiv\left(\frac{\partial f}{\partial t}\right)_{\mathbf{B}}, \tag{72}$$

where the conventional phase-space distribution function f is a function of the variables $\mathbf{r}=x\hat{\imath}_x+y\hat{\imath}_y$, z, $\mathbf{v}=v_x\hat{\imath}_x+v_y\hat{\imath}_y$, v_z, and t; and $(\partial f/\partial t)_{\mathbf{B}}=[-(q/\mu c)\times(\mathbf{v}\times\mathbf{B})\cdot(\partial f/\partial\mathbf{v})]$ is the rate of change of f due to the magnetic force acting on the electrons. But in the above formalism we defined a probability function F as a function of the variables $\mathbf{r}$, z, $E_\perp=(\mu/2)(\mathbf{v}\cdot\mathbf{v})=(\mu/2)\times(v_x^2+v_y^2)$, v_z and t. Then one can write the kinetic equation appropriate to a free-electron gas placed in a uniform magnetic field $\mathbf{B}=B\hat{\imath}_z$ as

$$\frac{d}{dt}F(\mathbf{r},z,E_\perp,v_z,t)=\frac{\partial F}{\partial t}+\mathbf{v}\cdot\frac{\partial F}{\partial \mathbf{r}}+v_z\frac{\partial F}{\partial z}$$
$$=\sum_{l=-\infty}^{+\infty}\left[\lim_{\hbar\to 0}\left(\frac{\partial F}{\partial t}\right)_l\right], \tag{73}$$

since, for the system under study,

$$\frac{dE_\perp}{dt}=\frac{d}{dt}[\tfrac{1}{2}\mu\mathbf{v}\cdot\mathbf{v}]=\mu\mathbf{v}\cdot\frac{d\mathbf{v}}{dt}=\frac{q}{c}\mathbf{v}\cdot(\mathbf{v}\times\mathbf{B})=0$$

and

$$\mu(dv_z/dt)=0\,.$$

Here $(\partial F/\partial t)_l$ is given by the coupled set of Eqs. (16) and (17). It is therefore apparent from Eqs. (72) and (73) that the transformation $f(\mathbf{r},z,\mathbf{v},v_z,t)\rightleftharpoons F(\mathbf{r},z,E_\perp,v_z,t)$ results in the replacement

$$\left(\frac{\partial f}{\partial t}\right)_{\mathbf{B}}\rightleftharpoons\sum_{l=-\infty}^{+\infty}\left[\lim_{\hbar\to 0}\left(\frac{\partial F}{\partial t}\right)_l\right].$$

This has the following simple physical meaning. $(\partial f/\partial t)_{\mathbf{B}}$ is the rate of change of f due to the magnetic force acting on the electrons. That is, $(\partial f/\partial t)_{\mathbf{B}}$ is the rate of change of f due to the magnetic acceleration of the electrons. But

$$\sum_{l=-\infty}^{+\infty}\left[\lim_{\hbar\to 0}\left(\frac{\partial F}{\partial t}\right)_l\right]$$

is the rate of change of F due to the emission and absorption of electromagnetic radiations by the magnetically accelerated electrons (that is, by the electrons which are being constantly accelerated by the uniform magnetic field). Hence, Eqs. (72) and (73) imply the following simple physical fact: Classically speaking, all accelerated electrons must emit and absorb electromagnetic radiations and, conversely, only accelerated electrons (or, more generally, only accelerated charged particles) can emit and absorb electromagnetic radiations. This is simply a statement of the familiar classical electrodynamics. It is, therefore, clear that under the transformation of $f(\mathbf{r},z,\mathbf{v},v_z,t)$ to $F(\mathbf{r},z,E_\perp,v_z,t)$, the kinetic equation appropriate to a free-electron gas in a uniform magnetic field goes over from the familiar Vlasov equation (72) to a somewhat unfamiliar Fokker-Planck equation (73).

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Stimulated Brillouin Scattering in Liquids*

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Stimulated Brillouin scattering in various liquids is examined in a transverse resonator and in a backward-wave oscillator. Comparison with elementary resonator theory indicates that the data are in satisfactory agreement for many liquids. The theoretical picture for the unstable backward-wave configuration is complicated by phonon transit-time effects, but even here the results establish guidelines for the theory. It is found necessary to discard the results for some liquids because of self-focusing and other nonlinear processes which interfere with the Brillouin effect.

I. INTRODUCTION

SOME experiments on stimulated Brillouin scattering in liquids are devised to test the validity of existing theories on the subject. The use of a resonator transverse to a laser beam provides the most direct confirmation of the quantum theory of Yariv[1] and Pine[2] or the equivalent classical theory of Chiao.[3] Threshold behavior of the more conventional backward-wave configuration is examined but does not yield conclusive proof of either the theory of Kroll[4] or Bloembergen.[5] It does, however, indicate the parameters of importance for any such theory.

Based on measurements of observational thresholds for 180° and 90° scattering and of total output in the transverse resonator, the liquids divide into two classes coincident with the previously distinguished[6] non-self-focusing (NSF) and self-focusing (SF) categories. The NSF liquids behave in stimulated Brillouin scattering as would be expected from their photoelastic properties. Self-focusing liquids however correlate poorly with hypersonic parameters as many parasitic effects are in evidence.

Spectral output, radiation patterns, and temporal behavior identify the Brillouin effect. Other competitive nonlinear effects are disclosed by these measurements and by study of concomitant Raman radiation.

II. EXPERIMENTAL RESULTS

The apparatus of the transverse-resonator experiment is depicted in Fig. 1. It is similar in nature to that used by Dennis and Tannenwald[7] to study the Raman effect and by Dennis[8] to obtain Brillouin radiation. Takuma and Jennings[9] used an off-axis cavity at small angles to study the Brillouin effect. These latter two experiments were performed with a limited range of liquids and in the Dennis case with an ambiguous

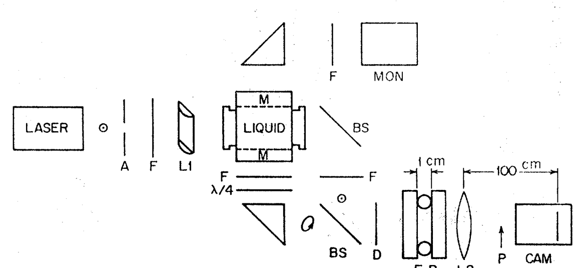

FIG. 1. Experimental schematic for stimulated scattering in a transverse resonator. A, aperture; F, filters; L, lenses; BS, beam splitters; D, diffuser; λ/4, depolarizer; F-P, interferometer; P, polarizer; M, mirrors.

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