Separation of the Relativistic Perihelion Precession from the Precession due to the Gravitational Quadrupole Moment

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The gravitational quadrupole moment is considered insofar as its effect on the perihelion precession of Mercury and pericenter precession of satellites is concerned. It is found possible to separate the quadrupole precession from the precession due to the general theory of relativity. As a corollary, it is found that an exchange of angular momentum between planet and sun takes place due to the quadrupole moment. This exchange may help to explain the phenomenon of sunspots and solar flares.

I. INTRODUCTION

THE general theory of relativity (GTR) is unique in that it arose from a single experimental fact, the principle of equivalence, and the insight of its author into the relationship between coordinate frames and experimental physics. At the time of its promulgation one of the major achievements of the theory was the calculation of the perihelion of the planet Mercury. Subsequent measurements of the bending of light in the gravitational field of the sun provided additional confirmation of the theory. Recently, Schiff[1] and others have claimed that it is possible to obtain an expression for the bending of light, consistent with experimental observation, without reference to the general theory. Furthermore, Dicke[2] has claimed that 25% of the perihelion precession may be due to a solar quadrupole moment.

It is the purpose of this note to indicate that it is possible to design an experiment to distinguish between the precession due to the quadrupole moment and that predicted by the Schwarzschild solution to the field equations of the general theory of relativity.

In the usual nonrelativistic formulation, the gravitational two-body problem is reduced to that of a point particle moving in a Newtonian potential,

$$V(\mathbf{r})=-G\int\frac{d^3r'\rho(r',\theta,\phi')}{|\mathbf{r}-\mathbf{r}'|},\tag{1}$$

where $\rho(r',\theta',\phi')$ is the density function for the mass which produces the gravitational field, and G is the gravitational constant. If the density function is spherically symmetric, $V(\mathbf{r})$ reduces to the usual term for a point mass GM/r. However, if ρ has an angular dependence, the form of $V(\mathbf{r})$ depends upon the coefficients of the expansion of ρ in spherical harmonics,

$$\rho(r,\theta,\phi)=\sum_{l=0}^{\infty}\sum_{m=-l}^{l}C_l{}^mP_l{}^m(\theta)e^{im\phi}.\tag{2}$$

Then, with $1/|\mathbf{r}-\mathbf{r}'|$ expanded in terms of Legendre polynomials $P_l(\cos\Theta)$, where Θ is the angle between $\mathbf{r}$ and $\mathbf{r}'$, the potential becomes

$$V(r,\theta,\phi)=\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{P_l{}^m(\theta)e^{im\phi}}{(l+\frac{1}{2})r^{l+1}}\int_0^R\rho_l{}^m(r')r'^{(l+2)}dr',\tag{3}$$

where R is the radius of the mass distribution and $\rho_l{}^m$ is the radial coefficient.

For the purposes of this discussion we shall restrict ourselves to a consideration of those distributions with axial symmetry, although any actual experiment must take into account the effect of nonaxially-symmetric terms. We choose the origin to be the center of mass and the z axis as the symmetry axis; then the lowest order correction to the $1/r$ potential is the quadrupole term $Q(3\cos^2\theta-1)/r^3$, where

$$Q=\frac{4\pi}{3}\int\rho_2(r')r'^4dr'.\tag{4}$$

The reduced Hamiltonian in spherical coordinates has the form:

$$H=\tfrac{1}{2}m\dot{r}^2+\tfrac{1}{2}mr^2\dot{\theta}^2+\tfrac{1}{2}mr^2\sin^2\theta\dot{\phi}^2-\frac{GMm}{r}+\frac{GQm}{r^3}(3\cos^2\theta-1),\tag{5}$$

where m is assumed small compared to $M=\int_0^R\rho_0(r')dr'$. This equation may be considerably simplified by the usual procedure of restricting the motion to the plane $\theta=\pi/2$ and choosing $\dot{\theta}_0=0$. The Hamiltonian then reduces to the familiar form:

$$H=\tfrac{1}{2}m\dot{r}^2+\tfrac{1}{2}mr^2\dot{\phi}^2-\frac{GMm}{r}-\frac{GQm}{r^3}.\tag{6}$$

The bound solutions (to first order) for this Hamiltonian are precessing ellipses with a rate of precession

$$\delta\psi=\frac{12\pi Q}{a^2(1-e^2)^2}.\tag{7}$$

For Mercury, and, in fact, for all planets except Pluto, the restriction to $\theta=\pi/2$ is quite valid, since the angles between the spin axis of the sun and the normals to the

[1] L. I. Schiff, Am. J. Phys. **28**, 340 (1960).
[2] R. H. Dicke, Nature **202**, 432 (1964).

planetary planes, the so-called inclination angles i, are less than or equal to 6°. It is important to note that the relevant angle is not the ecliptic angle which is the angle between the orbital plane of a planet and the orbital plane of earth, but the angle between the normal to the orbital plane and the spin axis of the sun.

II. ANGULAR DEPENDENCE

Recent developments in satellite technology permit the consideration of orbits with various inclinations and for certain orbits a very large relativistic precession is possible. In fact, for an orbit with perigee at 1.25 earth radii, a relativistic precession of 400″ of arc per century is possible.[3] Therefore, one can consider the case where θ is different from $\pi/2$. The motive behind this generalization, which of course complicates the mathematical analysis, is the possibility of separating the relativistic effect from the quadrupole precession. This must be possible since the relativistic effect is spherically symmetric (S wave) while the quadrupole term is D wave.

For nonequatorial orbits, the Hamiltonian (5) is required and one is immediately faced with an odd effect, the apparent nonconservation of angular momentum. Because of the presence of the $\cos^2\theta$ term, only the z component of angular momentum is conserved:

$$d\mathbf{L}/dt=[\mathbf{L},H],$$

$$\begin{aligned} \dot{L}_x&=\frac{6Q\sin\phi\sin\theta\cos\theta}{r^3}, \\ \dot{L}_y&=\frac{-6Q\cos\phi\sin\theta\cos\theta}{r^3}, \\ \dot{L}_z&=0. \end{aligned} \tag{8}$$

Since angular-momentum conservation is firmly rooted in physics, the apparent nonconservation must be due to elimination of those variables referring to the motion of the sun. However, the retention of the solar variables would not alter the fact that the angular momentum of a planet or satellite is not constant, but that the planet is continually exchanging angular momentum with the sun. This constitutes a kind of classical spin-orbit exchange. This exchange causes not only a variation in the spin rate of the sun, but also a variation in the direction of the spin axis, the consequences of which will be discussed later.

At present we fix our attention on the angular dependence of the perihelion precession, which requires a perturbation calculation. The simplest approach involves the classical analog of the interaction representation where first the unperturbed Hamiltonian-Jacobi equation is solved, and, in the new system,

$$H'=Q(3\cos^2\theta-1)/r^3 \tag{9}$$

is the new Hamiltonian. H' must then be expressed in terms of the new coordinates and momenta (β_i,α_i). These are determined by the usual Hamiltonian equations

$$\dot{\alpha}_i=-\partial H/\partial\beta_i,\quad \dot{\beta}_i=\partial H/\partial\alpha_i. \tag{10}$$

The solution of these equations is a solution of the original problem. For the case in point, these equations are not exactly soluble, but must be solved approximately. The solutions to first order are

$$\begin{aligned} \omega=\frac{3Q}{2a_0^2(1-e_0^2)}\Big\{&(4-5\sin^2 i_0)w+\frac{1}{e_0}[(2-3\sin^2 i_0) \\ &+e_0^2(\tfrac{7}{2}-(17/4)\sin^2 i_0)+\tfrac{1}{4}e_0^2\sin^2 i_0\cos 2\omega_0]\sin w \\ &-\frac{1}{e_0}[e_0^2+(\tfrac{1}{2}-(7/4)e_0^2)\sin^2 i_0]\sin(2\omega_0+w) \\ &+(1-\tfrac{3}{2}\sin^2 i_0)\sin 2w-(1-\tfrac{5}{2}\sin^2 i_0) \\ &\times\sin(2\omega_0+2w)+\tfrac{1}{6}e_0(1-\tfrac{3}{2}\sin^2 i_0)\sin 3w \\ &-\frac{1}{3e_0}[e_0^2-(\tfrac{7}{2}+(19/8)e_0^2)\sin^2 i_0]\sin(2\omega_0+3w) \\ &+\tfrac{3}{4}\sin^2 i_0\sin(2\omega_0+4w) \\ &+\tfrac{1}{8}e_0\sin^2 i_0\sin(2\omega_0+5w)\Big\}, \\ i=i_0+\frac{3Q\sin 2i_0}{4a_0^2(1-e_0^2)}\{&e_0\cos(2\omega_0+w)+\cos(2(\omega_0+w)) \\ &+\tfrac{1}{3}e_0\cos(2\omega_0+3w)\}, \end{aligned} \tag{11}$$

where we have included only the two equations directly affecting this discussion.[4] The first yields the motion of the pericenter, the second the motion of the orbital plane. In the above equations the subscript zero indicates the zeroth-order calculation; e is the eccentricity, a the semimajor axis, w the angle in the orbital plane, ω the angle of the major axis in the orbital plane, i the polar angle of the normal to the orbital plane.

Equations (11) clearly indicate the possibility of separating the pericenter precession due to a quadrupole moment from that due to a Schwarzschild potential, since for an orbital angle given by $\sin i=\sqrt{\tfrac{4}{5}}$ the secular part of the precession vanishes. This effect has already been observed for earth satellites, and therefore the possibility of separating the Schwarzschild effect from the quadrupole precession by means of a satellite experiment is open.

However, an additional theoretical analysis is necessary before an experiment of this type becomes meaningful; that is the modification of the Schwarzschild effect due to an asymmetry in the mass distribution. This

[3] I. Goldberg and E. Marx, Nuovo Cimento (to be published).

[4] Theodore Sterne, *An Introduction To Celestial Mechanics* (Interscience Publishers, Inc., 1960), p. 123.

calculation is currently being carried out, but one can argue on physical grounds that the quadrupole effect will modify the relativistic effect only by a superposition.[5] If this is true, an experiment of the type proposed above is possible; and in addition this information coupled with the known perihelion precession of Mercury would set an upper limit on the quadrupole moment of the sun.

One additional effect should be noted for completeness. This is the Lense-Thirring effect,[6] the effect of rotation of the central mass on the pericenter precession. This effect is a pure rotation effect; that is, it is not due to an asymmetry in the mass distribution. The precession is given by

$$\delta\psi_{\text{rot}}=\frac{GMr_0^2}{24c^2\tau a^3(1-e^2)^{3/2}},$$
$$\frac{\delta_{\text{rot}}\psi}{\delta\psi_s}=\frac{8\pi}{5}\frac{r_0^2}{a^{1/2}}\frac{(GM)^{1/2}}{\tau(1-e^2)^{1/2}}, \qquad (12)$$

where r_0 is the radius of the spherical mass distribution and τ is the period of rotation. For Mercury the ratio of the Lens-Thirring effect to the Schwarzschild effect, $\delta\psi_{\text{rot}}/\delta\psi_s$, is $\sim 4\times 10^{-4}$, which is quite small.

III. ADDITIONAL EFFECTS

The exchange of angular momentum between a planet and the sun discussed in Sec. II can lead to interesting effects, so that it is worthwhile to look more closely at the change of angular momentum. In particular, we shall consider the effect of this exchange on the sun. The most simple means of qualitatively determining the behavior of the system is to substitute the zeroth-order solutions of the Kepler problem into Eqs. (8). The form of the zeroth-order solutions we shall use is

$$r=K/(1+\epsilon\cos w),$$
$$\sin\phi=\cos i\cos\theta, \qquad (13)$$

where w is the angle variable in the plant of the orbit expressed by $\cos\theta=\sin i\cos w$. The first equation is the motion in the orbital plane, the second the equation of the plane, with $\phi_0=0$. Equations (8) then become

$$L_x=3Q\sin 2i\cos^2 w(1+\epsilon_0\cos w)^3/K^3,$$
$$L_y=-3Q\sin i\sin 2w(1+\epsilon_0\cos w)^3/K^3, \qquad (14)$$
$$L_z=0.$$

It is clear that because of the conservation of angular momentum to zeroth order these equations can be changed from time derivatives to angle derivatives and integrated. The L_x contains a secular term which indicates a continual rotation of the orbital plane, but it must be noted that perturbation theory breaks down when the perturbations become large. The L_y term is not secular and leads to an oscillatory behavior of the orbital plane. Although the motions of the orbital planes are interesting, it is far more interesting to consider the effect of this perturbation on the sun. Since the total angular momentum is conserved, the planets whose orbital planes are not perpendicular to the spin axis of the sun exert a torque on the sun which causes the spin axis to wobble. The effect of this perturbation on a plasma in equilibrium must cause similar instabilities in the plasma. We believe that these instabilities may propagate through the plasma in a type of wave motion and result in disturbances similar to those which are called sunspots.

It is clear that the above analysis is qualitative and that detailed models must be considered carefully before sunspots can be attributed to a quadrupole interaction. However, since no satisfactory explanation for sunspots is known, any new idea is worthy of careful appraisal. It must also be remarked that the sunspot phenomenon is periodic and that the period cannot be determined from the period of oscillation of the spin axis but is characteristic of the plasma.

We must also point out that this effect is quite small, $\sim 10^{-18}Q$, and therefore any relation to sunspots is pure speculation. However, experience with plasma stability indicates that even very small perturbations may have a large effect on stability.

IV. CONCLUSION

We have indicated a possible method for measuring the "perihelion" precession of a satellite due to the general theory of relativity. Marx and Goldberg have noted that with the proper choice of orbit the relativistic effect on an earth satellite can be quite large compared to atmospheric and other effects. The above work indicates that the quadrupole precession can also be eliminated in a manner consistent with the orbits suggested by Marx and Goldberg.[3]

As a second consequence of this study, the exchange of angular momentum between a planet and the sun might serve to explain sunspots. At this time a program is under way to look at the result of such perturbation on simple plasma models. A second program seeks to determine the change in the relativistic precession if axial symmetry is substituted for spherical symmetry, in the Schwarzschild problem.

[5] A solar quadrupole moment might account for the small discrepancy between observation and the Schwarzschild calculations of the general theory of relativity.

[6] J. Lense and H. Thirring, Physik. Z. **19**, 156 (1918).