From the definitions $D=\hbar^2/m$ and the definition (1.2) of λ, it follows that $D\beta=\lambda^2/2\pi$. When we use this relation in (3.4) we obtain the result (1.6) stated in the Introduction.

In conclusion, we note that our procedure could be used to evaluate additional terms in (3.4). We also observe that we could have introduced dimensionless variables from the beginning, and then our expansion would have been in terms of a dimensionless ratio rather than in terms of a^{-1}. Finally, we should point out that our procedure yields the asymptotic expansion of $B_D(T)$ for λ/a small, rather than a convergent power series, since $B_D(T)$ has an essential singularity at $\lambda/a=0$. This singularity will be manifested by exponentially small terms similar to those occurring in $B_E(T)$ and described in footnote 2.

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Cyclotron Excitation of Electromagnetic Waves by a Gyrating Electron Beam*

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When a tenuous electron stream gyrates around the field lines of a static magnetic field in the presence of a stationary plasma, electromagnetic waves are excited which propagate perpendicular to the magnetic field. Previous theoretical studies involved the assumption that these excited waves are almost longitudinal (quasistatic approximation). Such an assumption is not made in this analysis. It is shown that the excited waves have the electron cyclotron frequency or some multiple thereof. The electric intensity vector rotates in a plane perpendicular to the magnetic field; the phase velocities are of the order of, or exceed, the velocity of light, and for certain plasma-beam systems lower harmonics of the electron cyclotron frequency cannot be excited. Expressions are given for the rate of growth of the waves as a function of the plasma-beam parameters and the harmonic number.

I. INTRODUCTION

THERE has been considerable interest recently[1] in processes which produce emission of waves from plasma, perpendicular to a magnetic field, with frequencies which are multiples of electron gyrofrequency. The relevant theory has been developed in terms of a dispersion equation based on quasistatic approximation. This approximation involves the assumption that the electrostatic effects alone control the wave propagation and is valid when the phase velocity of the wave, when compared to the velocity of light in vacuum, is sufficiently small. In a quasistatic approximation the waves are almost longidudinal, i.e., the electrical intensity **E** is almost parallel to the wave vector **k**.

A suggestion that waves moving perpendicular to the magnetic field are electrostatic was made by Canobbio and Croci[2] in their analysis of radiation observed in a Penning ion-gauge (PIG) discharge by Landauer.[3] Subsequent investigations based on quasistatic approximation were made by Dory, Guest, and Harris,[4] Crawford and Tataronis,[5] Ikegami,[6] and others.

This investigation is based on a different approach to the problem. It has not been assumed that the excited waves are quasistatic (i.e., no *a priori* restrictions are imposed on the orientation of **E** with respect to **k**), and it is shown that there is an emission of excited harmonic waves which are not almost longitudinal. The main characteristics of these excited waves are: (a) each is elliptically (or circularly) polarized and the electric

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[1] E. G. Harris, Phys. Rev. Letters **2**, 34 (1959) and J. Nucl. Energy **2**, 138 (1961); K. Kato, J. Phys. Soc. Japan **15**, 1093 (1960); K. Mitani, H. Kubo, and S. Tanaka, in *Proceedings of the Sixth International Conference on Ionization Phenomena in Gases, Paris, 1963* (Serma Publishing Company, Paris, 1964), p. 28; J. L. Hirshfield and S. C. Brown, Phys. Rev. **122**, 719 (1961); A. B. Kitzenko and K. N. Stepanov, Zh. Tekhn. Fiz. **31**, 176 (1961) [English transl.: Soviet Phys.—Tech. Phys. **6**, 127 (1961)]; G. Bekefi, J. D. Coccoli, E. B. Hooper, Jr., and S. J. Buchsbaum, Phys. Rev. Letters **9**, 6 (1962); J. L. Hirshfield and J. M. Wachtel, *ibid.* **12**, 533 (1964); Abraham Bers and Sheldon Gruber, Appl. Phys. Letters **6**, 27 (1965); F. W. Crawford, Nucl. Fusion **5**, 75 (1965) and Radio Sci. **69D**, 789 (1965); S. Gruber, M. W. Klein, and P. L. Auer, Phys. Fluids **8**, 1504 (1965). See also E. Canobbio and R. Croci, Phys. Rev. **9**, 549, (1966) and Y. Furutani and G. Kalman, Plasma Phys. **7**, 381 (1965) which appeared after the completion of this paper.

[2] E. E. Canobbio and R. Croci, in *Proceedings of the Sixth International Conference on Ionization Phenomena in Gases, Paris, 1963* (Serma Publishing Company, Paris, 1964).

[3] G. Landauer, in *Proceedings of the Fifth International Conference on Ionization Phenomena in Gases, Munich, Germany, July 1961* (North-Holland Publishing Company, Amsterdam, 1961), Vol. I, p. 389.

[4] R. A. Dory, G. E. Guest, and E. G. Harris, Phys. Rev. Letters **14**, 131 (1965).

[5] F. W. Crawford and J. A. Tataronis, J. Appl. Phys. **36**, 2930 (1965).

[6] Hideo Ikegami, Institute for Plasma Research, Stanford University, Stanford, California, Report No. SUIPR 29, 1965 (unpublished).

intensity vector **E** rotates in a plane perpendicular to the static magnetic field $\mathbf{B}_0$ (the magnetic induction **B** associated with these waves is parallel to $\mathbf{B}_0$); (b) the phase velocity is of the order of magnitude of, or exceeds, the velocity of light in vacuum; (c) the frequency is equal to the electron gyrofrequency or to a multiple of the gyrofrequency.

To illustrate the conditions under which the excitation of such waves can occur, we have chosen a particularly simple dynamical system in which a stream of gyrating electrons (a helical electron beam) interacts with a cold plasma in the presence of the field $\mathbf{B}_0$. Each electron in the beam has a velocity component $v_{\parallel}$ parallel to $\mathbf{B}_0$ and a velocity component $v_\perp$ perpendicular to $\mathbf{B}_0$. Using rectangular coordinates v_x, v_y, and v_z in which the v_z axis is aligned along $\mathbf{B}_0$ and using

$$v_r^2=v_x^2+v_y^2, \tag{1.1}$$

the distribution function for a helical beam can be represented as

$$f(v_r,v_r)=(1/2\pi v_r)\delta(v_z-v_{\parallel})\delta(v_z-v_\perp). \tag{1.2}$$

Although the temperature effects have been neglected, the results obtained are applicable to warm plasmas and play an important role in thermonuclear instabilities, in the sun-earth environment, and in the generation and amplification of microwaves.

II. FORMULATION OF THE PROBLEM

(1) Assumptions

The stationary plasma and the helical electron beam are uniformly distributed in space. The system is charge equilibrated, and the ions in the plasma constitute a fixed, uniform, and neutralizing background. The analysis is based on a small-beam-density approach where only the first-order effects are considered. Thus, the ratio of the beam density to the density of the plasma, designated as σ, is assumed to be very small when compared to one. The effect of the beam is that of a small perturbation and in such a linearized treatment the perturbed fields are assumed to vary as $\exp i(\mathbf{kr}-\omega t)$ where ω is frequency, t is time, and $\mathbf{r}$ is the position vector.

(2) Dispersion Equation

The dispersion equation for the plasma-beam system is expressed as

$$F_{\mathrm{pb}}=F_{\mathrm{p}}(\mathbf{k},\omega)+\sigma F_{\mathrm{b}}(\mathbf{k},\omega). \tag{2.1}$$

The first term depends on the parameters of the plasma, and the second term represents the perturbation produced by the helical beam. In the absence of the beam, i.e., when $\sigma=0$(2.1) reduces to the dispersion equation for the cold, unperturbed plasma.

Following the customary procedure,[7] Eq. (2.1) will be solved for ω assuming **k** is real. The roots are expressed as $\omega=\bar{\omega}+\delta$, where $\bar{\omega}$, designated as "characteristic frequency," is real while δ can be complex. The quantity δ is due to the perturbation produced by the beam. Therefore,

$$\lim_{\sigma\to 0}\delta=0 \quad\text{and}\quad \lim_{\sigma\to 0}\omega=\bar{\omega}. \tag{2.2}$$

When $\mathrm{Im}\delta<0$ there is an instability, i.e., the excited waves grow exponentially with time until they reach magnitudes for which the linear treatment is not valid. The quantity $|\mathrm{Im}\delta|$ is the growth parameter, and it is assumed that $|\mathrm{Im}\delta|\ll\bar{\omega}$.

Equation (2.1) is derived by initially expressing the dispersion equation for the plasma-beam system in a standard form as[8]

$$F_{\mathrm{pb}}=Pn^4+Qn^2+R=0, \tag{2.3}$$

where

$$\begin{aligned}P&=\epsilon_{11}\sin^2\theta+2\epsilon_{13}\cos\theta\sin\theta+\epsilon_{33}\cos^2\theta;\\ Q&=2(\epsilon_{23}\epsilon_{12}-\epsilon_{13}\epsilon_{22})\cos\theta\sin\theta+\epsilon_{13}^2-\epsilon_{11}\epsilon_{33}\\ &\quad-(\epsilon_{22}\epsilon_{33}+\epsilon_{23}^2)\cos^2\theta-(\epsilon_{11}\epsilon_{22}+\epsilon_{12}^2)\sin^2\theta;\\ R&=\epsilon_{32}(\epsilon_{11}\epsilon_{22}+\epsilon_{12}^2)+\epsilon_{11}\epsilon_{23}^2+2\epsilon_{12}\epsilon_{13}\epsilon_{23}-\epsilon_{22}\epsilon_{13}^2.\end{aligned} \tag{2.4}$$

$n=ck/\omega$ (c is the velocity of light), θ is the angle between **k** and $\mathbf{B}_0$, and ϵ_{ij} is a component of the "modified"[9] dielectric tensor of the plasma-beam medium. The term ϵ_{ij} can be expressed as

$$\epsilon_{ij}=\epsilon_{ij}^{(0)}(\omega)+4\pi\sigma\chi_{ij}(\mathbf{k},\omega), \tag{2.5}$$

where the first term is a component of the dielectric tensor of the stationary plasma and the second is a component of the modified susceptibility tensor of the helical beam.

A rectangular coordinate system is used in which the z axis is aligned along $\mathbf{B}_0$ and the x axis is in the plane of **k** and $\mathbf{B}_0$. Subscripts 1, 2, and 3 designate x, y, and z, respectively, and $k_1=k\sin\theta$, $k_3=k\cos\theta$.

Putting $\theta=\pi/2$ in (2.3) and neglecting terms of order σ^2, we obtain

$$F_{\mathrm{pb}}=(n^2-\epsilon_{33})(\epsilon_{11}n^2-\epsilon_{11}\epsilon_{22}-\epsilon_{12}^2)=0. \tag{2.6}$$

The components involved are found from a general

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[7] See, for instance, A. I. Akhiezer and Ya. B. Fainberg, Zh. Eksperim. i Teor. Fiz. **21**, 1262 (1951).

[8] See, for instance, T. H. Stix, *The Theory of Plasma Waves* (McGraw-Hill Book Company, Inc., New York, 1962).

[9] We use the term "modified" dielectric tensor since the tensor (2.5) does not represent the dielectric properties of the medium as interpreted in the light of Maxwell-Lorentz theory. A plasma beam system as well as other systems having plasma-like behavior are "Tellegen media" which, in the Maxwell-Lorentz sense, are phenomenologically described by macroscopic parameters other than the dielectric tensor (2.5). See in that connection Jacob Neufeld, Phys. Rev. **123**, 1 (1961) and J. Appl. Phys. **34**, 2549 (1963).

formulation by Kitzenko and Stepanov[10] as

$$\epsilon_{11}^{(0)}=\epsilon_{22}^{(0)}=1-\frac{\omega_e^2}{\omega^2-\Omega_e^2};\quad \epsilon_{12}^{(0)}=\frac{-i\Omega_e\omega_e}{\omega(\omega^2-\Omega_e^2)};$$

$$\epsilon_{33}^{(0)}=1-\frac{\omega_e^2}{\omega^2};$$

$$\chi_{11}=-\frac{\omega_e^2}{4\pi\omega^2}-\frac{\omega_e^2}{2\pi\omega^2}\sum_{s=-\infty}^{\infty}\frac{\Omega_e^2s^3J_sJ_s'}{a(\omega+s\Omega_e)};$$

$$\chi_{22}=-\frac{\omega_e^2}{4\pi\omega^2}+\frac{\omega_e^2}{4\pi\omega^2}\sum_{s=-\infty}^{\infty}\frac{\Omega_e(a^2J_s'^2)'}{a(\omega+s\Omega_e)};$$

$$\chi_{33}=-\frac{\omega_e^2}{4\pi\omega^2}+\frac{\omega_e^2}{2\pi\omega^2}\sum_{s=-\infty}^{\infty}\frac{v_{\parallel}^2\omega k_{\perp}J_sJ_s'}{\Omega_e v_{\perp}(\omega+s\Omega_e)};$$

$$\chi_{12}=\frac{i\sigma\omega_e^2}{4\pi\omega^2}\sum_{s=-\infty}^{\infty}\frac{\Omega_e s^2(aJ_sJ_s')'}{a(\omega+s\Omega_e)};$$

$$\omega_e=(4\pi Ne^2/m)^{1/2};\quad \Omega_e=|eB_0/me|; \tag{2.7}$$

where m and e designate, respectively, the mass and charge of an electron, N is the electron density, ω_e is the plasma frequency, Ω_e is the magnitude of the electron gyrofrequency, $J_s=J_s(a)$ is the Bessel function of order s and argument a and the prime denotes differentiation with respect to a. The argument $a=kv_{\perp}/\Omega_e$ is the ratio of the Larmor radius of gyrating electrons to the wavelength.

Equation (2.6) is satisfied either by the linearly polarized ordinary wave determined by

$$F_{\rm pb}{}^{\rm ORD}=n^2-\epsilon_{33}=0, \tag{2.8}$$

($\mathbf{E}\|\mathbf{B}_0$), or the extraordinary wave determined by

$$F_{\rm pb}{}^{\rm EX}=\epsilon_{11}n^2-\epsilon_{11}\epsilon_{12}-\epsilon_{13}^2=0, \tag{2.9}$$

($\mathbf{E}$ rotates in a plane perpendicular to $\mathbf{B}_0$).

It is easy to see that under our assumptions an ordinary wave cannot be excited by the helical beam. Our discussion will, therefore, be concerned with instabilities involving excited extraordinary waves and throughout the following we will use Eq. (2.9), dropping the superscript EX.

III. EXCITATION OF EXTRAORDINARY WAVES

(1) Analysis of the Dispersion Equation

Writing (2.9) in the form of (2.1) gives

$$F_{\rm p}=\epsilon_{11}^{(0)}n^2-(\epsilon_{11}^{(0)2}+\epsilon_{12}^{(0)2}), \tag{3.1}$$

as the unperturbed term representing waves in the stationary plasma and

$$\sigma F_{\rm b}=\chi_{11}n^2-\epsilon_{11}^{(0)}(\chi_{22}+\chi_{11})-2\epsilon_{12}^{(0)}\chi_{12}, \tag{3.2}$$

as the perturbation produced by the beam.

The unperturbed term will be written as

$$F_{\rm p}=\epsilon_{11}^{(0)}\phi_{\rm p}, \tag{3.3}$$

where

$$\phi_{\rm p}=n^2-1+\frac{\omega_e^2(\omega^2-\omega_e^2)}{\omega^2(\omega^2-\Omega_e^2-\omega_e^2)}. \tag{3.4}$$

Using (2.7), the perturbation produced by the beam can be expressed as

$$\sigma F_{\rm b}=\sigma\sum_{s=-\infty}^{\infty}F_{\rm bs}, \tag{3.5}$$

where

$$F_{\rm bs}=-\frac{\omega_e^2}{\omega^2}\left\{\left[1+\frac{2\Omega_e s^3J_sJ_s'}{(\omega+s\Omega_e)}\right]n^2 -\sigma\epsilon_{11}^{(0)}\left[2+\frac{\Omega_e(a^2J_s'^2)'-2\Omega_e s^2J_sJ_s'}{a(\omega+s\Omega_e)}\right] +\frac{2\omega_e^2\Omega_e^2s^2(aJ_sJ_s')'}{\omega a(\omega^2-\Omega_e^2)(\omega+s\Omega_e)}\right\}. \tag{3.6}$$

The expression (3.6) is independent of $v_{\parallel}$. Hence the instability behavior is independent of the longitudinal velocity component of gyrating electrons. Thus the frequencies and the rates of growth of excited waves are the same for a circular beam ($v_{\parallel}=0$; $v_{\perp}\neq0$) and for a helical beam ($v_{\parallel}\neq0$; $v_{\perp}\neq0$), provided the transverse velocity component $v_{\perp}$ in both beams is the same.

As shown in (3.5), the effect of the beam is represented by a sum of perturbing terms. Assuming that σ is sufficiently small, the contribution of each perturbing term is negligible if $F_{\rm bs}$ is bounded. Thus the effect of the beam is significant only in the neighborhood of the singularities of $F_{\rm bs}$. These singularities occur at $\omega+s\Omega_e=0$. Consequently, in view of (2.2), the characteristic frequency of a wave perturbed by the beam is equal to the electron gyrofrequency or to a harmonic of the gyrofrequency, i.e.,

$$\tilde{\omega}=-s\Omega_e. \tag{3.7}$$

In the weak-beam approximation ($\sigma\ll1$), we have $|{\rm Im}\delta|\ll\tilde{\omega}$, and therefore $\tilde{\omega}$ cannot be zero. Consequently, $s=0$ has been excluded, i.e., there is no aperiodic instability. If there is an instability when $|s|=1$, the excited wave is in resonance with the gyrofrequency; when $|s|=p$, it is in resonance with the pth harmonic of the gyrofrequency. The frequencies of excited waves will be labeled as $\omega=-s\Omega_e+\delta_s$, where $|{\rm Im}\delta_s|$ is the rate of growth of the sth harmonic.

Since in the neighborhood of a singularity of $F_{\rm bs}$ in (3.5) this term will dominate, the dispersion equation

[10] O. B. Kitzenko and K. M. Stepanov, Ukr. Fiz. Zh. **6**, 297 (1961). See also A. B. Kitzenko and K. N. Stepanov, Zh. Tekhn. Fiz. **31**, 176 (1961) [English transl.: Soviet Phys.—Tech. Phys. **6**, 127 (1961)].

TABLE I. Values of R characterizing various types of plasma.

	Ionosphere	Exosphere	Electric discharges in air	Thermonuclear plasma	Solar corona	Interstellar clouds
$R=\omega_e^2/\Omega_e^2$	0.25	5.43–1.35×10^2	10^{-3}–10	1–10^4	7.6×10^{-2}–7.6×10^2	2.33×10^8

for the sth harmonic is

$$\epsilon_{11}^{(0)}\phi_p+\sigma F_{bs}=0. \tag{3.8}$$

We assume that ϕ_p can be approximated by two terms of its Taylor expansion about $-s\Omega_e$, i.e.,

$$\phi_p(k_1\omega)=\phi(k_1-s\Omega_e)+D(k_1-s\Omega_e), \tag{3.9}$$

where D is the first partial of ϕ_p with respect to ω evaluated at $-s\Omega_e$. The dispersion equation (3.8) can now be written as

$$D(k,\,-s\Omega_e)\delta_s^2+\phi_p(k_1-s\Omega_e)\delta_s+\sigma\omega_e^2/a\Omega_e)T_s=0, \tag{3.10}$$

where

$$D(k_1-s\Omega_e)=\frac{2}{s^3\Omega_e}\left\{\frac{c^2k^2}{\Omega_e}-R\left[\frac{s^2(s^2-R-1)-(s^2-R)(2s^2-R-1)}{(s^2-R^2-1)}\right]\right\};$$

$$R=\omega_e^2/\Omega_e^2; \tag{3.11}$$

$$T_1=(a^2J_1'^2)'-2J_1J_1'-2(aJ_1J_1')';$$

$$T_{-1}=(a^2J_1'^2)'+2J_1J_1'+2(aJ_1J_1')';$$

and

$$T_s=\frac{-1}{s^2\epsilon_{11}^{(0)}}\left\{-2sJ_sJ_s'\frac{c^2k^2}{\Omega_e^2}-\epsilon_{11}^{(0)}[(a^2J_s'^2)-2s^3J_sJ_s']-\frac{2R_s(aJ_sJ_s')'}{s^2-1}\right\}.$$

A separate treatment was required for $|s|=1$ and $|s|>1$. That is, if $|s|=1$, we multiply by δ^2 in the denominator. If $|s|\neq 1$, we multiply by δ, keeping those terms in braces which have δ in the denominator.

(2) Criterion for Instability

An instability occurs if the roots of (3.10) are complex, i.e., when the discriminant is negative. This leads to

$$\phi_p^2(k,-\Omega_e)<4D(k,-\Omega_e)(\sigma\omega_e^2T_s/a\Omega_e). \tag{3.12}$$

Since $\sigma\ll 1$, for this to hold it is necessary that

$$\phi_p(k,-\Omega_e)\approx 0, \tag{3.13}$$

and

$$D(k,-s\Omega_e)T_s>0. \tag{3.14}$$

Each of (3.13) and (3.14) is a necessary condition for instability and both holding is a sufficient condition.

(3) Harmonic Content of Excited Waves

Using the first necessary condition for instability, Eq. (3.13) yields an expression as follows:

$$k=\frac{|s|\Omega_e}{c}\left\{1-\frac{R(s^2-R)}{s^2[(s^2-1)-R]}\right\}^{1/2}. \tag{3.15}$$

The quantity k expressed above represents the wave number of a harmonic wave of order s which can be propagated in a cold, unperturbed plasma transversely to the magnetic field $\mathbf{B}_0$. It is necessary for k to be real since otherwise the wave is evanescent and cannot be excited by a tenuous beam. Consequently, it is necessary that

$$R(s^2-R)/s^2[(s^2-1)-R]<1. \tag{3.16}$$

Thus for a given plasma (R is known), only those harmonics which satisfy the above inequality can be excited by the beam.

The quantity R is a measure of the plasma density with reference to the intensity of the magnetic field. This quantity can be used in classifying various types of plasma which occur in nature or are produced in the laboratory. In our classification (based on the one introduced by Denisse and Delcroix[11]) we shall differentiate between "extremely low density," "low density," and "dense" plasmas. For an extremely low-density plasma $R<m/M_i$, where M_i is the mass of the plasma ion. The low-density plasmas are characterized by $m/M_i<R<1$, whereas in a dense plasma one has $R>1$. Extremely low-density plasmas are relatively uncommon and usually occur in evacuated vessels having pressure of the order of 10^{-5} mm Hg in the presence of a very strong magnetic field (cyclotrons, vacuum gauges, etc.). Low-density and dense plasmas are relatively common. Numerical values of the parameter R for various types of plasma are given in Table I. These values are based on data of Denisse and Delcroix[11] and of Smith.[12]

For each type of plasma, i.e., for each value of R, the inequality (3.16) determines a sequence of harmonic waves excited by the beam. Such a sequence does not always include the fundamental frequency Ω_e and the low-order multiples of Ω_e. Thus if $|s|=1$, the inequality

[11] J. F. Denisse and J. L. Delcroix, *Theorie des Ondes dans les Plasmas* (Dunod Cie., Paris, 1961).
[12] R. L. Smith, J. Geophys. Res. **11**, 3709 (1961).

(3.16) gives

$$R<2. \tag{3.17}$$

This limits the type of plasma in which a helical beam can excite a wave at the fundamental frequency Ω_e. Such a wave can occur only when the plasma is dense.

The inequality (3.16) allows the determination of those harmonics at which an instability can possibly occur. That is, (3.16) gives for $|s|\neq 1$,

$$|s|>0.5+0.5(1+4R)^{1/2}. \tag{3.18}$$

It is clear from this that as R increases (the plasma becomes more dense), the minimum harmonic for which an instability may occur, increases. Hence for the more dense plasmas, lower harmonics cannot be excited. It should be kept in mind that the condition (3.16) is necessary but not sufficient, i.e., satisfying (3.16) does not guarantee instability since (3.14) must also hold.

(4) Rate of Growth of Excited Waves

When an instability occurs, the rate of growth of the excited wave can be determined from (3.10), using the wave number from (3.15) in T_s and D as

$$|\mathrm{Im}\delta_s|=\sigma^{1/2}\omega_e|T_s(-s\Omega_e)/D(-s\Omega_e)|^{1/2}. \tag{3.19}$$

Figures 1(a), 1(b), 1(c), and 2(a) and 2(b) show the dimensionless growth rate $|\mathrm{Im}\delta|/\Omega_e$ as a function of plasma-beam parameters and the harmonic number. These figures are sufficient to indicate the trend of the growth rate with changes in beam velocity and harmonic number as well as changes in the type of plasma considered.

(5) Phase Velocities of Excited Harmonic Waves

Considerations based on phase velocities are of interest since they point out an essential distinction between the waves which are based on quasistatic approximation and the waves discussed in this analysis. The phase velocities of quasistatic waves are very low, i.e., their refractive index is very large when compared to one. On the other hand, the refractive index n_s for waves discussed in this analysis is not very large. This can be ascertained from the relationship

$$n_s^2=\frac{c^2k^2}{s^2\Omega_e^2}=1-\frac{R(s^2-R)}{s^2(s^2-R-1)}. \tag{3.20}$$

For waves at electron gyrofrequency this becomes

$$n_{\pm 1}^2=2-R>0, \tag{3.21}$$

so that $n_s<\sqrt{2}$, i.e., the phase velocity of these waves exceeds $\sqrt{2}/2$ of the velocity of light in vacuum.

For harmonics of the cyclotron frequency it is easy to see that as $|s|$ increases, n_s remains quite small. We can, therefore, conclude that the phase velocities of excited waves cannot be very low, and, therefore, when the temperature effects are neglected, the quasistatic approximation is not applicable.

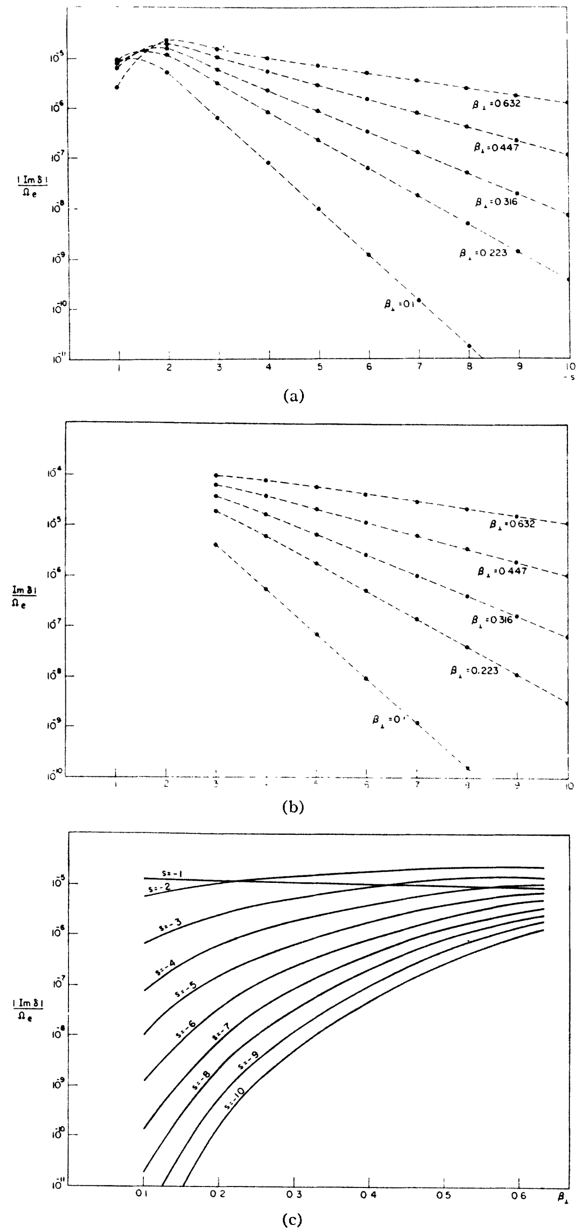

FIG. 1(a) Relationship between $|\mathrm{Im}\delta_s|/\Omega_e$ and s. ($R=0.0136$, $\sigma=10^{-6}$, $\beta_\perp=v_\perp/c=0.1$, 0.223, 0.316, 0.447, and 0.632.) (b) Relationship between $|\mathrm{Im}\delta_s|/\Omega_e$ and s. ($R=1.36$, $\sigma=10^{-6}$, $\beta_\perp=v_\perp/c=0.1$, 0.223, 0.316, 0.447, and 0.632. No excitation for $|s|=1$ and 2.) (c) Relationship between $|\mathrm{Im}\delta_s|/\Omega_e$ and s. ($R=5.444$ $\sigma=10^{-6}$, $\beta_\perp=v_\perp/c=0.1$, 0.223, 0.316, 0.447, and 0.632. No excitation for $|s|<5$.)

(6) Relative Values of the Wavelength with Respect to the Gyroradius

Consider the quantity a representing the ratio of the Larmor radius of gyrating electrons to the wavelength.

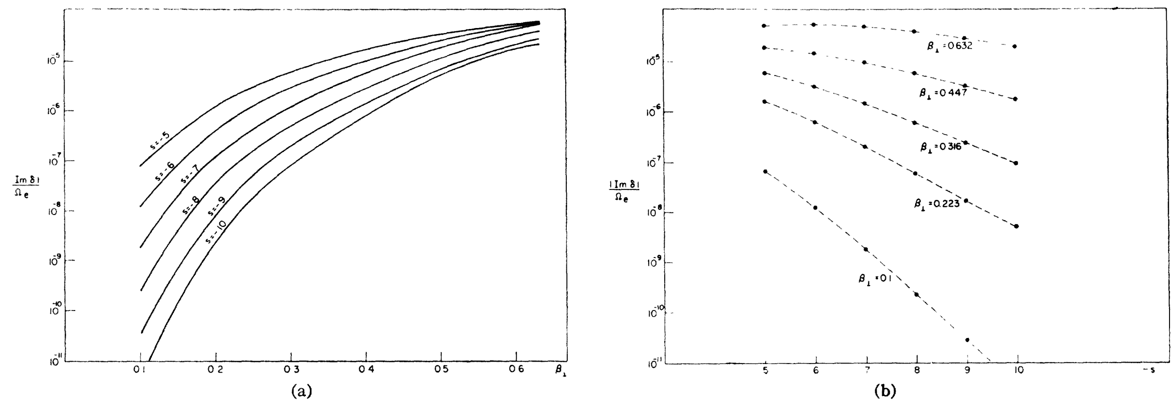

FIG. 2. (a) Relationship between $|\mathrm{Im}\delta_s|/\Omega_e$ and $\beta_\perp$. ($R=0.0136$, $\sigma=10^{-6}$, and $s=-1$ to -10.) (b) Relationship between $|\mathrm{Im}\delta_s|/\Omega_e$ and $\beta_\perp$. ($R=5.444$, $\sigma=10^{-5}$, and s from -1 to -10.)

For fixed $\beta_\perp$ and R there is a unique dependence between the wavelength and the harmonic number of an excited wave. Using (3.15), we obtain

$$a^2 \equiv a^2(s) = s^2\beta_\perp{}^2 - \frac{R(s^2-R)\beta_\perp{}^2}{(s^2-R-1)}. \tag{3.22}$$

Since the beam is nonrelativistic, for lower harmonics, a is relatively small when compared to one. However, with increasing a, $|s|$ monotonically increases. The magnitude a is one of the factors which determines the effectiveness of the interaction between the gyrating particles and the wave.

IV. VALIDITY OF COLD-PLASMA APPROXIMATION

Our analysis is based on cold-plasma approximation, i.e., the temperature effects have been neglected. (It is assumed that the mean thermal velocity v_t of plasma electrons is zero.)

The above assumptions are applicable to a thermal plasma provided[13]

$$\mu = \tfrac{1}{2}(k_\perp v_t/\Omega_e)^{1/2} \ll 1, \tag{4.1}$$

and

$$|(\omega - s\Omega_e)/k_{\parallel} v_t| \gg 1, \tag{4.2}$$

The inequality (4.2) is satisfied for any v_t (since $k_{\parallel} \approx 0$), whereas the inequality (4.1) can be represented as

$$\tfrac{1}{2}(a v_t / v_\perp)^{1/2} \ll 1, \tag{4.3}$$

and, consequently,

$$v_t \ll v_\perp / a. \tag{4.4}$$

For $|s|=1$, a is less than one, and, therefore, the inequality (4.4) is always satisfied when the transverse velocity of gyrating beam electrons exceeds the thermal velocity v_t. For $|s|\neq 1$, the quantity a is larger. However, one can obtain

$$\frac{v_t}{c} \ll \beta_\perp \left[\frac{(s^2-R-1)}{s^4-2Rs-s^2-R^2}\right]^{1/2}. \tag{4.5}$$

The above inequality which limits the permissible values of v_t does not appear to be very stringent for small s.

V. COMPARISON OF RESULTS WITH EXPERIMENTAL DATA

There is a considerable amount of experimental information on wave emission by anisotropic plasma-like media at frequencies which are multiples of electron gyrofrequency. The relevant experiments have been made under such a variety of conditions that one particular mechanism which would control the emission cannot be relied upon. A suggestion has been made that this emission is due to the distortion of the electron orbits in the neighborhood of walls and sheaths.[14] Other suggestions have been made which are based on the radiation by a single gyrating electron, i.e., the collective effect of an assembly of electrons has not been taken into account. There exists, however, sufficient evidence to show that the collective effects play an important role.

According to Landauer,[3] there is a relatively strong emission at harmonic frequencies from a discharge having electron temperatures of only a few electron volts. This can be attributed to collective effects since in a single-particle effect high harmonic emission becomes important only when the electron energies are in a relativistic range. The importance of collective effects has been stressed by Crawford and Tataronis.[5]

We have chosen in our analysis a system comprising a stationary plasma perturbed by a helical beam since such a system represents one of the simplest embodiments of an anisotropic plasma-like medium. The occurrence of an instability in such a system was

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pointed out by Harris[15] in his discussion on the experiments by Bekefi and Hooper[16] and by Gruber, McBee, and Shepherd[17] in which the plasma was generated by an electron beam. It is shown in this investigation that a plasma-beam model in which temperature effects are neglected can be used successfully in explaining some of the aspects of the observed harmonic emissions.

Thus, according to Ikegami,[6] the maximum intensity of the observed radiation occurs at a multiple of the electron gyrofrequency and as the discharge current is increased, the maximum intensity is progressively shifted toward higher harmonics. This observation is supported by our analysis as illustrated in the enclosed graphs. Thus when $R=0.0136$ [Fig. 1(a)] both the fundamental gyrofrequency and its harmonics are excited by the beam. However, for $R=1.36$ [Fig. 1(b)], the lowest harmonic excited by the beam is represented by $|s|=3$, whereas for $R=5.444$ [Fig. 1(c)] the lowest harmonic corresponds to $|s|=5$. The rate of growth generally decreases with increasing $|s|$ except for relatively low values of R. Thus, as shown in Fig. 1(a) when $\beta_\perp=v_\perp/c=0.223$ to 0.632, the rate of growth attains maximum at $|s|=2$ and then decreases when $|s|$ becomes larger.

Both Landauer[3] and Ikegami[6] observed that the harmonic emission is due primarily to the extraordinary wave ($\mathbf{E}\perp\mathbf{B}_0$). This observation is supported by our results. We have found that the extraordinary waves are the only electromagnetic waves which can be excited by the beam, i.e., the ordinary waves ($\mathbf{E}_0\|\mathbf{B}_0$) remain stationary.

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Single-Particle Coupling to Collective Excitations; Sound in He^3

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In order to test the hypothesis that the observed anomalous behavior of the low-temperature specific heat of He^3 is due to the interaction of He^3 atoms with sound, the coupling of the single particles to the collective excitations must be calculated. Based on an infinite-order perturbation theory, a technique is introduced for finding the effective coupling at long wavelengths. Within this approximation scheme the coupling is found to be of the deformation-potential kind. Vertex corrections make the expression used for the single-particle self-energy exact. A singular vertex function, which is ruled out if perturbation theory is valid but which is required in order to change the coupling from deformation-potential to piezoelectric, leads to an inconsistency. All approximation schemes treated result in a deformation-potential coupling between He^3 atoms and sound. Treatments of this coupling have not so far led to results which can explain the observed specific heat.

I. INTRODUCTION

OBSERVATION by Anderson[1] indicates that the recent experiments on the specific heat of He^3 by the Illinois group[2] appear to have a temperature- dependent density-of-states factor which becomes logarithmically singular as the temperature approaches zero. For temperatures ranging from millidegrees to tenths of degrees Kelvin, Anderson pointed out that the specific heat may be fit by a curve of the form $C=\gamma T+AT\ln B/T$.[3] In this temperature range, we may neglect the specific-heat contribution due to phonons, which is cubic in the temperature.

A suggestion proposed by Anderson was interpreted by Balian and Fredkin[4] to mean that the singular behavior of the specific heat could be accounted for by considering fermion self-energy processes of virtual emission and reabsorption of phonons of zero sound. In order to test this hypothesis, the coupling of the single particles to the collective excitations must be calculated.

In Sec. II of this paper we show (in second-order perturbation theory) how the long-wavelength coupling to collective excitations influences the specific heat. In

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