Theory of Bound States in a Random Potential*

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A systematic procedure is proposed for calculation of the energy-level density of low-lying bound states in a random potential. The procedure is based on a function-space formulation of the problem; and certain features of the method may be useful in solving more general problems in functional integration. The method works in one, two, or three dimensions. In the one-dimensional case, the result checks exactly with the known solution.

I. INTRODUCTION

WE wish to present a reformulation of some recent work by Halperin and Lax[1,2] on the theory of low-lying energy levels in a random potential. This problem originates physically in analyses of the impurity band "tail" in highly doped semiconductors. But certain mathematical features of the problem indicate that a thorough understanding of its solution may be valuable in a much broader range of applications. Specifically, the problem is one of performing an integration in a function space—the Hilbert space of potential-energy functions. The solution apparently is not accessible to perturbation—theoretic techniques. We hope to show in a later publication that similar mathematical problems arise in the theory of first-order phase transitions, and that the methods discussed here lead to new results in that subject.

The problem to be solved is the following. Consider the Schrödinger equation

$$-\tfrac{1}{2}\nabla^2\psi_n+V\psi_n=E_n\psi_n\,, \tag{1.1}$$

and suppose that the potential $V(\mathbf{r})$ is a random function obeying Gaussian statistics such that

$$\langle V(\mathbf{r})\rangle=0\,, \tag{1.2}$$

and

$$\langle V(\mathbf{r})V(\mathbf{r}')\rangle=w(\mathbf{r}-\mathbf{r}')\,. \tag{1.3}$$

Here $w(\mathbf{r}-\mathbf{r}')$ is a correlation function of finite range. The angular brackets in Eqs. (1.2) and (1.3) are defined for any functional $A\{V(\mathbf{r})\}$ by the functional integral

$$\langle A\rangle\equiv\int\delta V(\mathbf{r})\exp\left[-\tfrac{1}{2}\int d\mathbf{r}d\mathbf{r}'V(\mathbf{r})K(\mathbf{r}-\mathbf{r}')V(\mathbf{r}')\right]\cdot A\{V(\mathbf{r})\}\,, \tag{1.4}$$

where the differential $\delta V(\mathbf{r})$ should be understood to include a normalization factor so that

$$\langle 1\rangle=1\,. \tag{1.5}$$

The kernel $K(\mathbf{r}-\mathbf{r}')$ has the property

$$\int d\mathbf{r}'K(\mathbf{r}-\mathbf{r}')w(\mathbf{r}'-\mathbf{r}'')=\delta(\mathbf{r}-\mathbf{r}'')\,, \tag{1.6}$$

which assures the validity of Eq. (1.3). The energy levels E_n are defined by, say, periodic boundary conditions on the sides of a very large box. The problem is to compute the average energy-level density $\langle\rho(E)\rangle$ with particular emphasis on large negative values of E corresponding to low-lying bound states in the random potential.

In the following work we shall be concerned primarily with the so-called "white noise" case, in which the correlation function w may be approximated by a delta function:

$$w(\mathbf{r}-\mathbf{r}')=\gamma\delta(\mathbf{r}-\mathbf{r}')\,. \tag{1.7}$$

The effect of this approximation is to restrict our calculations to a certain range of energies, low enough to correspond to localized bound states but not so low that the characteristic wave functions change appreciably across the correlation length of the potential. The alternative limit, in which the potential is very smooth compared to the wave functions, has been discussed by Kane.[3] In this case a Thomas-Fermi approximation seems to give reasonable results. We shall retain the finite-ranged correlation function w throughout part of our formulation in order to clarify the distinction between these two cases.

It is very convenient for our purposes that, for a one-dimensional system and a white-noise potential, the density of states may be computed exactly. This was first pointed out by Frisch and Lloyd,[4] and has been studied in detail by Halperin.[5] At high energies it turns out that $\langle\rho(E)\rangle$ is well approximated by conventional perturbation-theoretic methods. In the limit $E\to-\infty$, however, the exact result is

$$\langle\rho(E)\rangle\sim\frac{4|E|}{\pi\gamma}\exp\left\{-\frac{2}{3\gamma}(2|E|)^{3/2}\right\}. \tag{1.8}$$

Equation (1.8) provides us with an exact expression

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[3] E. O. Kane, Phys. Rev. **131**, 79 (1963).
[4] H. L. Frisch and S. P. Lloyd, Phys. Rev. **120**, 1175 (1960).
[5] B. Halperin, Phys. Rev. **139**, A104 (1965).

against which to check the validity of more general approximation schemes.

The considerations of Halperin and Lax (HL) are ingeniously intuitive, and appear to be very special to the problem at hand. Our main effort in this paper is to formulate these ideas more systematically. In Sec. II we show that a nonlinear equation, derived by HL as a self-consistent determination of a characteristic bound-state wave function, may be interpreted as a determination of a class of stationary points of the functional integrand. Section III contains some formalism necessary for an accurate evaluation of the integrand near a stationary point. The usual method for evaluating functional integrals is to write the integrand in exponential form and expand the argument of the exponential up to terms quadratic in the function-space variables. In Sec. IV we point out that this procedure is inadequate for the present problem, and then describe a correct calculation leading to a general expression almost identical to that of HL. The functional integration is performed in Sec. V, with special attention paid to the energy dependence of the various factors involved. As a slight improvement on the work of HL, we are able here to retain all terms contributing to the multiplicative constant in the asymptotic expression for $\langle\rho(E)\rangle$. Explicit one-dimensional calculations leading to Eq. (1.8) are presented in Sec. VI.

II. FUNCTION-SPACE FORMULATION

The formal calculation of the density of states proceeds in the usual way via the Green's function. For a particular $V(\mathbf{r})$ we define $G_E(\mathbf{r},\mathbf{r}'|V)$ to satisfy

$$(-\tfrac{1}{2}\nabla_r^2+V(\mathbf{r})-E)G_E(\mathbf{r},\mathbf{r}'|V)=\delta(\mathbf{r}-\mathbf{r}'). \quad (2.1)$$

Then define

$$\mathcal{G}_E(\mathbf{r}-\mathbf{r}')\equiv\langle G_E(\mathbf{r},\mathbf{r}'|V)\rangle, \quad (2.2)$$

where the angular brackets are defined by Eq. (1.4), and we have invoked translational symmetry. Finally,

$$\langle\rho(E)\rangle=\lim_{\delta\to+0}(1/\pi)\mathcal{G}_{E+i\delta}(0). \quad (2.3)$$

Next consider the right-hand side of Eq. (2.2) as an integration over functions V. According to Eq. (1.4) the most probable V's are those which are small everywhere; and presumably the dominant contribution to the real part of $\mathcal{G}$ comes from the region of function space near $V=0$. On the other hand, for large negative E, the only functions V which can contribute an imaginary part to $\mathcal{G}$ are those which have a negative fluctuation deep enough to bind a particle at energy E near the points $\mathbf{r},\mathbf{r}'$. The low-energy "tail" given by Eq. (1.8) describes the small but finite probability of such fluctuations for very low E.

Our reformulation of the HL theory is based on the observation that the functional integrand defining $\mathcal{G}$ is stationary at points $\bar{V}$ in the function space such that $\bar{V}$ binds—or very nearly binds—a particle at energy E. To see this, write Eq. (2.2) in the form

$$\mathcal{G}(\mathbf{r}-\mathbf{r}')=\int\delta V e^{-H(\mathbf{r},\mathbf{r}'|V)}, \quad (2.4)$$

where

$$H(\mathbf{r},\mathbf{r}'|V)=\tfrac{1}{2}\int d\mathbf{s}d\mathbf{s}'V(\mathbf{s})K(\mathbf{s}-\mathbf{s}')V(\mathbf{s}')-\ln G(\mathbf{r},\mathbf{r}'|V). \quad (2.5)$$

The stationarity condition,

$$\delta H/\delta V(\mathbf{s})|_{V=\bar{V}}=0, \quad (2.6)$$

leads, via direct functional differentiation, to the equation:

$$\bar{V}(\mathbf{s})=-\int d\mathbf{s}'w(\mathbf{s}-\mathbf{s}')\frac{G(\mathbf{r},\mathbf{s}'|\bar{V})G(\mathbf{s}',\mathbf{r}'|\bar{V})}{G(\mathbf{r},\mathbf{r}'|\bar{V})}. \quad (2.7)$$

Now, for any potential V, the Green's function may be written in the form

$$G_E(\mathbf{s},\mathbf{s}'|V)=\sum_n\frac{\psi_n(\mathbf{s}|V)\psi_n^*(\mathbf{s}'|V)}{E_n(V)-E}, \quad (2.8)$$

where the ψ_n and E_n are defined in Eq. (1.1) and we have emphasized the V dependence of these quantities. We suppose—and shall verify later—that for V near $\bar{V}$ there exists an isolated ground state ψ_0 with energy E_0 near E. In this case the right-hand side of (2.8) may be approximated by the $n=0$ term alone:

$$G_E(\mathbf{s},\mathbf{s}'|\bar{V})\cong\frac{\psi_0(\mathbf{s}|\bar{V})\psi_0(\mathbf{s}'|\bar{V})}{E_0(\bar{V})-E}\equiv g(\mathbf{s})g(\mathbf{s}'). \quad (2.9)$$

Substituting (2.9) into (2.7), we obtain

$$\bar{V}(\mathbf{s})=-\int d\mathbf{s}'w(\mathbf{s}-\mathbf{s}')g^2(\mathbf{s}'). \quad (2.10)$$

Then the combination of Eqs. (1.1), (2.9), and (2.10) gives

$$-\tfrac{1}{2}\nabla^2\psi_0-\frac{1}{E_0-E}\int d\mathbf{r}'w(\mathbf{r}-\mathbf{r}')\psi_0^2(\mathbf{r}')\psi_0(\mathbf{r})=E_0\psi_0, \quad (2.11)$$

which is a nonlinear equation determining ψ_0 and E_0, identical in form to an analogous equation obtained by HL.

We now may apply purely dimensional arguments to check the validity of the approximation in Eq. (2.9) and to obtain some qualitative information about the various functions involved. Certainly ψ_0 is a smoothly varying function of $\mathbf{r}$. As discussed in the Introduction, we confine our attention to the case in which the range of the correlation function w is very much smaller than the distance over which ψ_0 changes appreciably. In this case there is only one fundamental length in the problem, which we shall denote by K^{-1}. That is, ψ_0 is a function of the variable $K\mathbf{r}$. Then each term in Eq.

(2.11) must be of order $K^2\psi_0$. Specifically,

$$E_0 \propto -K^2, \tag{2.12}$$

and

$$\frac{1}{E_0-E}\int d\mathbf{r}'w(\mathbf{r}-\mathbf{r}')\psi_0^2(\mathbf{r}')\cong\frac{\gamma\psi_0^2(\mathbf{r})}{E_0-E}\propto K^2, \tag{2.13}$$

where

$$\gamma=\int d\mathbf{r}\, w(\mathbf{r}). \tag{2.14}$$

Because ψ_0 must be normalized, we have

$$\psi_0(0)\propto K^{d/2}, \tag{2.15}$$

where $d=1, 2, 3$ is the dimensionality. Thus Eq. (2.13) implies

$$E_0-E\propto\gamma K^{d-2}. \tag{2.16}$$

There being only one fundamental length, the bound-state energy-level spacings in $\bar{V}$ must be of order K^2. (A potential well in which the spacings behave otherwise as the binding increases requires at least two fundamental lengths for its description, a radius and a surface thickness.) Thus, we may conclude that Eq. (2.9) is a valid approximation for $G(\bar{V})$ as long as $d\leq 3$.

From these dimensional considerations we also may deduce the general form of the exponential factor in the density of states. According to (2.13) we have

$$\bar{V}(\mathbf{r})\propto K^2\times\text{function of }(K\mathbf{r}). \tag{2.17}$$

Then, at the point $V=\bar{V}$, the statistical weight in Eq. (1.4) is

$$\exp\left\{-\tfrac{1}{2}\int d\mathbf{r}d\mathbf{r}'\bar{V}(\mathbf{r})K(\mathbf{r}-\mathbf{r}')\bar{V}(\mathbf{r}')\right\}$$

$$\cong\exp\left\{-\text{const}\times\frac{|E|^{2-d/2}}{\gamma}\right\}, \tag{2.18}$$

where we have used $|E|\propto K^2$ from (2.12) and (2.16). Clearly (2.18) checks with (1.8) for $d=1$.

One other aspect of the solution for $\bar{V}$ and ψ_0 which will turn out to be very important is that the variables $\mathbf{r}$ and $\mathbf{r}'$ which appeared on the right-hand side of Eq. (2.7) have disappeared in Eq. (2.10). Of course, Eq. (2.9) is valid only for $\mathbf{s}$ and $\mathbf{s}'$ within the region where $\bar{V}$ is large; and the location of this region is determined by $\mathbf{r}$ and $\mathbf{r}'$. But, within this region, there is a sort of translational invariance of the solutions of (2.6). Accordingly, we shall use the notation $\bar{V}_\mathbf{z}(\mathbf{r})$ to denote the function centered at the point $\mathbf{z}$; i.e. ,$\bar{V}_\mathbf{z}(\mathbf{r})$ is a function of $|\mathbf{r}-\mathbf{z}|$.

III. PERTURBATION EXPANSION OF G FOR V NEAR $V_\mathbf{z}$

It turns out to be convenient to let $\bar{V}_\mathbf{z}(\mathbf{r})$ be defined by Eqs. (2.9),-(2.11); although this function is not quite an exact stationary point of the functional integrand in Eq. (2.4). We must then make an accurate evaluation of the integrand in the neighborhood of $\bar{V}$.

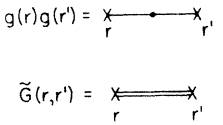

FIG. 1. Diagrammatic notation for the two kinds of basic Green's functions.

For any potential V, we can write

$$G(\mathbf{r},\mathbf{r}'|V)=G(\mathbf{r},\mathbf{r}'|\bar{V}_\mathbf{z})$$

$$-\int d\mathbf{s}\,G(\mathbf{r},\mathbf{s}|\bar{V}_\mathbf{z})\varphi(\mathbf{s})G(\mathbf{s},\mathbf{r}'|V), \tag{3.1}$$

where

$$\varphi(\mathbf{s})=V(\mathbf{s})-\bar{V}_\mathbf{z}(\mathbf{s}). \tag{3.2}$$

Using $G(\bar{V}_\mathbf{z})$ as the zeroth-order approximation, we may obtain an infinite series of contributions to $G(V)$ by the usual iteration process for solving Eq. (3.1). To do this correctly, however, we must improve the approximation (2.9) for $G(\bar{V}_\mathbf{z})$. We write

$$G(\mathbf{r},\mathbf{r}'|\bar{V}_\mathbf{z})=g(\mathbf{r})g(\mathbf{r}')+\tilde{G}(\mathbf{r},\mathbf{r}'), \tag{3.3}$$

where $g(\mathbf{r})$ is defined in Eq. (2.9) and $\tilde{G}$ contains the rest of the sum in the representation (2.8).

It is now useful to introduce a diagrammatic notation. We denote the factors $g(\mathbf{r})g(\mathbf{r}')$ by a line with a dot in the middle and a factor $\tilde{G}(\mathbf{r},\mathbf{r}')$ by a double line as shown in Fig. 1. An interaction φ is denoted by a wavy line. Figure 2 shows a typical diagram in the expansion of $G(V)$. Consider first all diagrams which have at least one line with a dot. Starting from the left at point $\mathbf{r}$, we may sum all insertions between $\mathbf{r}$ and the first dot. This gives

$$h(\mathbf{r})=g(\mathbf{r})-\int d\mathbf{s}M(\mathbf{r},\mathbf{s})g(\mathbf{s})$$

$$+\int d\mathbf{s}d\mathbf{s}'M(\mathbf{r},\mathbf{s})M(\mathbf{s},\mathbf{s}')g(\mathbf{s}')+\cdots$$

$$=\int d\mathbf{s}\left(\frac{1}{1+M}\right)(\mathbf{r},\mathbf{s})g(\mathbf{s}), \tag{3.4}$$

where the matrix M is

$$M(\mathbf{r},\mathbf{s})=\tilde{G}(\mathbf{r},\mathbf{s})\varphi(\mathbf{s}). \tag{3.5}$$

Next sum all insertions between two dots. This leads to

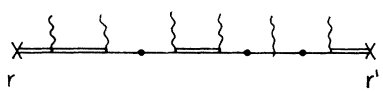

FIG. 2. A typical diagram in the expansion of $G(V)$.

the total contribution

$$R=-\int d\mathbf{s}g(\mathbf{s})\varphi(\mathbf{s})g(\mathbf{s}) + \int d\mathbf{s}d\mathbf{s}'g(\mathbf{s})\varphi(\mathbf{s})M(\mathbf{s},\mathbf{s}')g(\mathbf{s}')-\cdots = -\int d\mathbf{s}g(\mathbf{s})\varphi(\mathbf{s})h(\mathbf{s}), \quad (3.6)$$

where we have used (3.4) and (3.5). We are left with the sum over all insertions between the last dot and the end of the diagram $\mathbf{r}'$. In the same way as before we obtain

$$\bar{h}(\mathbf{r}')=\int dsg(\mathbf{s})\left(\frac{1}{1+M}\right)^T(\mathbf{s},\mathbf{r}'), \quad (3.7)$$

with the transposed matrix M^T.

Summing over all diagrams with one, two, three, etc. dots leads to

$$h(\mathbf{r})\cdot[1+R+R^2+\cdots]\cdot\bar{h}(\mathbf{r}')=h(\mathbf{r})\bar{h}(\mathbf{r}')/(1-R). \quad (3.8)$$

The sum of all diagrams without any dots (i.e., only $\tilde{G}$ lines) gives

$$\tilde{G}_{\text{ren}}(\mathbf{r},\mathbf{r}')=\tilde{G}(\mathbf{r},\mathbf{r}') - \int d\mathbf{s}\tilde{G}(\mathbf{r},\mathbf{s})\varphi(\mathbf{s})\tilde{G}(\mathbf{s},\mathbf{r}')+\cdots = \int d\mathbf{s}\left(\frac{1}{1+M}\right)(\mathbf{r},\mathbf{s})\tilde{G}(\mathbf{s},\mathbf{r}'). \quad (3.9)$$

By adding (3.8) and (3.9), we obtain the final result for G:

$$G(\mathbf{r},\mathbf{r}'\,|\,V)=h_{\mathbf{z}}(\mathbf{r})\bar{h}_{\mathbf{z}}(\mathbf{r}')/(1-R(V))+\tilde{G}_{\text{ren.}}(\mathbf{r},\mathbf{r}'\,|\,V;\mathbf{z}). \quad (3.10)$$

Here we have re-emphasized the $\mathbf{z}$ dependence of the functions occurring on the right-hand side.

IV. INTEGRATION OVER SITES z

Having obtained Eq. (3.10), which is formally an exact representation for $G(V)$, we must compute its imaginary part for $\mathbf{r}=\mathbf{r}'$ and then perform the functional integration over $\varphi=V-\bar{V}_{\mathbf{z}}$ to obtain $\langle\rho(E)\rangle$. The right-hand side of Eq. (3.10), however, refers to the site $\mathbf{z}$ via the choice of $\bar{V}_{\mathbf{z}}$ which defines the perturbation expansion. One's first guess might be simply to set $\mathbf{z}=\mathbf{r}$; but this turns out to be inadequate because of the translational invariance discussed at the end of Sec. II. The point is that the functional integrand remains constant for small displacements in function space of the form

$$\bar{V}_{\mathbf{z}}(\mathbf{r})\rightarrow\bar{V}_{\mathbf{z}+\Delta\mathbf{z}}(\mathbf{r})\cong\bar{V}_{\mathbf{z}}(\mathbf{r})-\Delta\mathbf{z}\cdot\nabla_{\mathbf{r}}\bar{V}_{\mathbf{z}}(\mathbf{r}), \quad (4.1)$$

at least when $\mathbf{z}$ is near $\mathbf{r}$. This implies that we must pay special attention to perturbations φ proportional to components of $\nabla\bar{V}$ when we perform the functional integration.

The proper treatment is closely related to the considerations of HL. Note that, for each V in the function space, we are free to choose a special $\bar{V}_{\mathbf{z}}$ with which to compute $G(V)$. [Remember that Eq. (3.10) is formally independent of $\bar{V}_{\mathbf{z}}$.] In this spirit, for each V, we choose $\mathbf{z}$ so that $\bar{V}_{\mathbf{z}}$ most nearly fits V in the neighborhood of $\mathbf{r}$. More precisely, we demand that the mean-square deviation,

$$D(\mathbf{z}\,|\,V)=\int d\mathbf{s}[V(\mathbf{s})-\bar{V}_{\mathbf{z}}(\mathbf{s})]^2, \quad (4.2)$$

be a minimum as a function of $\mathbf{z}$. Let $\mathbf{z}_{\mathbf{r}}(V)$ be the minimum of D which lies closest to $\mathbf{r}$, and evaluate $\langle\rho(E)\rangle$ in the form

$$\langle\rho(E)\rangle=(1/\pi)\langle\text{Im}G(\mathbf{r},\mathbf{r}\,|\,V;\mathbf{z}_{\mathbf{r}}(V))\rangle, \quad (4.3)$$

where the explicit $\mathbf{z}_{\mathbf{r}}$ dependence refers in an obvious way to the right-hand side of Eq. (3.10).

Now imagine a region σ centered at the point $\mathbf{r}$. This region is defined so that its linear dimensions are very much larger than K^{-1}, the range of $\bar{V}$, but very much smaller than the average distance between those minima of D corresponding to V's deep enough to bind particles at energies E or lower. Specifically,

$$K^{-d}\ll\text{volume of }\sigma\ll\left[\int_{-\infty}^{E}dE'\langle\rho(E')\rangle\right]^{-1}. \quad (4.4)$$

In what follows we shall consider only energies low enough such that a σ can be found which easily satisfies the inequalities (4.4). In principle, a systematic improvement of the calculation might be obtained by considering the possibility of regions in which two or more deep minima of D overlap.

With the above understanding, we can write

$$\langle\rho(E)\rangle=-\frac{1}{\pi}\left\langle\int_{\sigma}d\mathbf{z}\,\text{Im}G(\mathbf{r},\mathbf{r}\,|\,V;\mathbf{z})\delta(\mathbf{z}-\mathbf{z}_{\mathbf{r}}(V))\right\rangle = -\frac{1}{\pi}\left\langle\int_{\sigma}d\mathbf{z}\,\text{Im}G(\mathbf{r},\mathbf{r}\,|\,V;\mathbf{z}) \cdot\delta(\nabla D(\mathbf{z}\,|\,V))\,|\det(\nabla\nabla D)|\right\rangle. \quad (4.5)$$

Here, the last factor inside the angular brackets denotes a Jacobian, i.e., the absolute value of the determinant of the matrix whose elements are $\partial^2D/\partial z_i\partial z_j$. Note that we do not have to worry about subsidiary shallow minima (or maxima) of D within σ because ImG will vanish unless the potential has a deep well.

To complete the formalism we must insert into Eq. (4.5) the proper approximation for ImG. For any

particular V, $G(V)$ as a function of E will have a discrete set of poles associated with bound states. Each of these poles contributes an imaginary part via the usual relation

$$1/(x-i\delta)=P(1/x)+i\pi\delta(x). \tag{4.6}$$

According to our development in Sec. III, the functions $h(\mathbf{r})$, R, and $\tilde{G}_{\text{ren.}}$ are to be computed by a sort of perturbation expansion in powers of $\varphi=V-\bar{V}_{\mathbf{z}}$ in which ψ_0, the lowest lying state in V, does not appear as an intermediate state. Thus we may guess that no poles will occur in these functions for energies E near E_0. More precisely, any poles in h, R, or $\tilde{G}_{\text{ren.}}$ for E near E_0 must correspond to large potential fluctuations φ such that V has a deep well outside of σ. The ground state in such a well must be degenerate with, but orthogonal to ψ_0. Such poles will not contribute to Eq. (4.5); nor will they show up in any finite order of the perturbation expansion in powers of φ.

The conclusion is that the relevant imaginary part must come from a zero of the denominator $1-R$ in Eq. (3.10). Accordingly, we write

$$\mathrm{Im}G(\mathbf{r},\mathbf{r}|\,V;\,\mathbf{z})\cong\pi h_{\mathbf{z}}(\mathbf{r})\bar{h}_{\mathbf{z}}(\mathbf{r})\cdot\delta(1-R(V)). \tag{4.7}$$

Furthermore, if we evaluate $h_{\mathbf{z}}(\mathbf{r})$ by perturbation theory, the right-hand side of (4.7) will vanish for $|\mathbf{r}-\mathbf{z}|>K^{-1}$; and the restriction of the $\mathbf{z}$ integration to the region σ is superfluous. Finally, it is convenient to reverse the order of the $\mathbf{z}$ integration and the functional integration denoted by the angular brackets in (4.5). The result is

$$\langle\rho(E)\rangle\cong\int d\mathbf{z}\langle h_{\mathbf{z}}(\mathbf{r})\bar{h}_{\mathbf{z}}(\mathbf{r})\delta(1-R(V))\delta(\boldsymbol{\nabla}D(\mathbf{z}|\,V))\times|\det(\boldsymbol{\nabla}\boldsymbol{\nabla}D)|\rangle. \tag{4.8}$$

This formula is essentially identical to that of HL, the main difference being that our derivation will enable us to avoid certain inconsistencies in the subsequent evaluation of $\langle\rho(E)\rangle$.

V. THE FUNCTIONAL INTEGRATION

In order to perform the functional integration over potentials V, we expand $V(\mathbf{r})$ in some complete set of orthonormal functions $\varphi_n(\mathbf{r})$:

$$V(\mathbf{r})=\sum_{n=0}^{\infty}\xi_n\varphi_n(\mathbf{r}). \tag{5.1}$$

The ξ_n form an infinite but discrete set of Gaussian random variables. Having moved the $\mathbf{z}$ integration outside of the functional integration in Eq. (4.8), we are free to choose the set of φ_n differently for each $\mathbf{z}$. Accordingly, it is natural to choose φ_0 proportional to $\bar{V}_{\mathbf{z}}(\mathbf{r})$:

$$\varphi_0(\mathbf{r}-\mathbf{z})=-a\bar{V}_{\mathbf{z}}(\mathbf{r});\quad a^{-2}=\int d\mathbf{r}\bar{V}^2(\mathbf{r}). \tag{5.2}$$

We also write

$$V(\mathbf{r})=\xi_0\varphi_0(\mathbf{r}-\mathbf{z})+\bar{\varphi}(\mathbf{r}), \tag{5.3}$$

where $\bar{\varphi}$ is orthogonal to $\varphi_0(\mathbf{r}-\mathbf{z})$. Note that, in the white-noise limit, the weight factor in Eq. (1.4) becomes

$$\exp\left\{-\tfrac{1}{2}\int d\mathbf{r}d\mathbf{r}'V(\mathbf{r})K(\mathbf{r}-\mathbf{r}')V(\mathbf{r}')\right\}\rightarrow\exp\left\{-\frac{1}{2\gamma}\sum_{n=0}^{\infty}\xi_n^2\right\}. \tag{5.4}$$

Proper evaluation of the low-energy limit now requires a careful study of the K dependence of the various factors occuring in Eq. (4.8). From previous considerations, we expect the dominant contribution to the functional integral to come from regions of the function space in which $\xi_0\varphi_0\sim\bar{V}\sim K^2$, which implies $\xi_0\sim K^{2-d/2}$ and $a\sim K^{d/2-2}$. Here we have used Eq. (2.17) and have again used the fact that K^{-1} is the only fundamental length in the problem. Thus each spatial integration contributes a factor K^{-d}; and each φ_n must carry a factor $K^{d/2}$ for normalization. If ξ_0 is the only anomalous Gaussian variable, the other ξ_n will be of order $\gamma^{1/2}$—independent of K—according to Eq. (5.4). From the discussion of Sec. II we have $g\sim\gamma^{-1/2}K$ and $\tilde{G}(\bar{V})\sim K^{d-2}$.

The most interesting term to consider is the quantity $1-R$ which enters as the argument of a delta function in Eq. (4.8). Let us examine separately the first two terms in the perturbation expansion for R given in Eq. (3.6). These terms are

$$-\int d\mathbf{r}g^2(\mathbf{r})[V(\mathbf{r})-\bar{V}(\mathbf{r})]=-\frac{1}{\gamma a}\left(\xi_0+\frac{1}{a}\right); \tag{5.5}$$

and

$$\int d\mathbf{r}d\mathbf{r}'g(\mathbf{r})[V(\mathbf{r})-\bar{V}(\mathbf{r})]\tilde{G}(\mathbf{r},\mathbf{r}'|\,\bar{V})[V(\mathbf{r}')-\bar{V}(\mathbf{r}')]g(\mathbf{r}')$$
$$=\int d\mathbf{r}d\mathbf{r}'g(\mathbf{r})\left[\left(\xi_0+\frac{1}{a}\right)\varphi_0(\mathbf{r})+\bar{\varphi}(\mathbf{r})\right]\tilde{G}(\mathbf{r},\mathbf{r}'|\,\bar{V})\times\left[\left(\xi_0+\frac{1}{a}\right)\varphi_0(\mathbf{r}')+\bar{\varphi}(\mathbf{r}')\right]g(\mathbf{r}'). \tag{5.6}$$

If we set Eq. (5.5) equal to unity ($1-R=0$), we deduce that

$$\xi_0+(1/a)\cong-\gamma a\sim-\gamma K^{d/2-2}. \tag{5.7}$$

Counting factors of K in (5.6), we notice that terms containing factors (ξ_0+1/a) vanish as $K\rightarrow\infty$ for $d\leq3$. But the term in (5.6) containing two factors of $\bar{\varphi}$ is of order unity, and must be retained in the low-energy limit. In a similar fashion, we note that higher-order terms in the expansion of R will contribute at most extra factors of the form $\bar{\varphi}\tilde{G}d\mathbf{r}\sim K^{-2+d/2}$, which are

negligible for large K and $d\leq 3$. Thus the correct approximation is

$$1-R\cong 1+\frac{1}{\gamma a}\left(\xi_0+\frac{1}{a}\right) - \int d\mathbf{r}d\mathbf{r}' g(\mathbf{r})\bar{\varphi}(\mathbf{r})\tilde{G}(\mathbf{r},\mathbf{r}'\,|\,\bar{V})\bar{\varphi}(\mathbf{r}')g(\mathbf{r}'). \quad (5.8)$$

It is now trivial to verify that, to leading order in K, one should write

$$h(\mathbf{r})=\bar{h}(\mathbf{r})\cong g(\mathbf{r}). \quad (5.9)$$

The last term on the right-hand side of Eq. (5.8) suggests that the natural choice for the orthonormal set φ_n is one which diagonalizes this quadratic form. That is, the φ_n are the eigenstates of the integral equation

$$\varphi_n(\mathbf{r})=\gamma\lambda_n\int d\mathbf{r}' g(\mathbf{r})\tilde{G}(\mathbf{r},\mathbf{r}'\,|\,\bar{V})g(\mathbf{r}')\varphi_n(\mathbf{r}'). \quad (5.10)$$

We must verify, however, that the φ_0 we already have chosen in Eq. (5.2) is orthogonal to these eigenstates.

To solve Eq. (5.10), it is convenient to introduce the functions $u_n(\mathbf{r})$ such that

$$\varphi_n(\mathbf{r})=g(\mathbf{r})u_n(\mathbf{r}). \quad (5.11)$$

Then Eq. (5.10) becomes

$$u_n(\mathbf{r})=\gamma\lambda_n\int d\mathbf{r}'\tilde{G}(\mathbf{r},\mathbf{r}'\,|\,\bar{V})g^2(\mathbf{r}')u_n(\mathbf{r}'); \quad (5.12)$$

and the orthogonality relation is

$$\int d\mathbf{r} g^2(\mathbf{r})u_n(\mathbf{r})u_m(\mathbf{r})=\delta_{n,m}. \quad (5.13)$$

Now $\tilde{G}$ satisfies

$$(-\tfrac{1}{2}\nabla_r^2-E+\bar{V})\tilde{G}(\mathbf{r},\mathbf{r}'\,|\,\bar{V})=\delta(\mathbf{r}-\mathbf{r}')-\psi_0(\mathbf{r})\psi_0(\mathbf{r}'). \quad (5.14)$$

Operating on both sides of Eq. (5.12) with the operator which appears on the left in Eq. (5.14), we obtain

$$(-\tfrac{1}{2}\nabla^2-E+\bar{V})u_n(\mathbf{r})=\frac{\gamma\lambda_n}{E_0-E}\psi_0^2(\mathbf{r})u_n(\mathbf{r}) - \frac{\gamma\lambda_n}{E_0-E}\left[\int d\mathbf{r}'\psi_0^3 u_n\right]\psi_0(\mathbf{r}). \quad (5.15)$$

If we remember that our analysis is restricted to the Gaussian-white-noise limit, and that we need the λ_n only to leading order in K, we may rearrange (5.15) to read

$$[-\tfrac{1}{2}\nabla^2-E_0+(1+\lambda_n)\bar{V}(\mathbf{r})]u_n(\mathbf{r}) = -\gamma\lambda_n[\int d\mathbf{r}'\psi_0(\mathbf{r}')g^2(\mathbf{r}')u_n(\mathbf{r}')]\psi_0(\mathbf{r}). \quad (5.16)$$

Note that we have replaced E by E_0 on the left-hand side.

It is obvious that one solution of Eq. (5.16) is

$$u_0(\mathbf{r})\propto\psi_0(\mathbf{r}); \quad \lambda_0=0. \quad (5.17)$$

Although the product $g\psi_0\propto\varphi_0$ is not, strictly speaking, an eigenstate of (5.10), Eqs. (5.13) and (5.16) imply that φ_0 is orthogonal to all the $\varphi_n(n\neq 0)$ within the approximation discussed above. Furthermore, the right-hand side of (5.16) vanishes for all n. It follows that the numbers $1+\lambda_n$ can be interpreted as the strengths of potentials of shape $\bar{V}(\mathbf{r})$ which bind eigenstates $u_n(\mathbf{r})$ at the fixed energy E_0.

Before completing the calculation, it will be useful to note that one more solution of Eq. (5.16) may be obtained explicitly. In the Gaussian white-noise limit, Eq. (2.11) becomes

$$[-\tfrac{1}{2}\nabla^2-E_0-(\gamma/(E_0-E))\psi_0^2(\mathbf{r})]\psi_0(\mathbf{r})=0. \quad (5.18)$$

By taking the gradient of this equation,

$$[-\tfrac{1}{2}\nabla^2-E_0-(3\gamma/(E_0-E))\psi_0^2(\mathbf{r})]\nabla\psi_0(\mathbf{r})=0, \quad (5.19)$$

we see that there is a d-fold degenerate set of eigenstates $\mathbf{u}_1\sim\nabla\psi_0$ with eigenvalue $1+\lambda_1=3$. In analogy with Eq. (5.2), we shall write these states in the form:

$$\varphi_{1\nu}(\mathbf{r})=b\frac{\partial\bar{V}}{\partial x_\nu}; \quad b^{-2}=d^{-1}\int d\mathbf{r}(\nabla\bar{V})^2 (\nu=1,\cdots,d). \quad (5.20)$$

We are now ready to evaluate the various terms in Eq. (4.8). We have

$$\langle\rho(E)\rangle\sim\int d\mathbf{z}\, g^2(\mathbf{r}-\mathbf{z})\prod_{n=0}^{\infty}\left(\int_{-\infty}^{\infty}\frac{d\xi_n}{(2\pi\gamma)^{1/2}}\right) \exp\left[-\frac{1}{2\gamma}\sum_{n=0}^{\infty}\xi_n^2\right]\delta(1-R)\delta(\nabla D)\,|\det(\nabla\nabla D)|\,, \quad (5.21)$$

where we have inserted the correct normalizing factor into (1.4) for the white-noise case. Remember that we have referred all functions inside the functional integral to the origin $\mathbf{z}$. Now,

$$1-R=1+(1/\gamma a)\left(\xi_0+\frac{1}{a}\right)-\frac{1}{\gamma}\sum_{n=1}^{\infty}\sum_{\nu=1}^{\bar{\nu}(n)}(\xi_{n\nu}^2/\lambda_n), \quad (5.22)$$

where $\xi_{n\nu}$ represents the set of ξ's corresponding to the, say, $\bar{\nu}(n)$ degenerate states $\varphi_{n\nu}$. Also:

$$(\nabla D)_\nu=2\int d\mathbf{r}V(\mathbf{r})(\nabla\bar{V}(\mathbf{r}))_\nu=\frac{2}{b}\sum_n\xi_n\int d\mathbf{r}\varphi_n(\mathbf{r})\varphi_{1\nu}(\mathbf{r}) =\frac{2}{b}\xi_{1\nu}, \quad \nu=1,\cdots,d; \quad (5.23)$$

and

$$|\det(\nabla\nabla D)| = |2\int d\mathbf{r}V(\mathbf{r})\det[\nabla\nabla\bar{V}(\mathbf{r})]|$$

$$\cong -2\int d\mathbf{r}\bar{V}(\mathbf{r})\det[\nabla\nabla\bar{V}(\mathbf{r})]\equiv C. \quad (5.24)$$

Here it is legitimate to replace V by $\bar{V}$ to leading order in K.

The integration over ξ_0 now can be performed using the delta function whose argument is given by (5.22). That is

$$\xi_0 = -\frac{1}{a} - \gamma a\left(1-\frac{1}{\gamma}\sum_{n=1}^{\infty}\sum_{\nu}(\xi^2_{n\nu}/\lambda_n)\right). \quad (5.25)$$

Note that the second term is smaller than the first by a factor $a^2 \propto K^{d-4}$. With this value of ξ_0, the argument of the exponential weight factor becomes

$$\frac{1}{2\gamma}\sum_{n=0}^{\infty}\sum_{\nu}\xi_{n\nu}^2 \cong \frac{1}{2\gamma a^2}+1+\frac{1}{2\gamma}\sum_{n=2}^{\infty}\left(1-\frac{2}{\lambda_n}\right)\sum_{\nu}\xi_{n\nu}^2, \quad (5.26)$$

where we need to keep only terms up to and including order zero in K. We also have taken note of the fact that $\lambda_1=2$; thus the sum in (5.26) must start with $n=2$. Here we see explicitly the effect of the translational symmetry discussed at the beginning of Sec. IV. The exponential part of the integrand is stationary with respect to functional variations of V away from $\bar{V}$ of the form $\varphi_{1\nu}\sim(\nabla\bar{V})_\nu$, i.e., spatial displacements.

The integration over $\mathbf{z}$ in (5.21) pertains only to the factor $g^2(\mathbf{r}-\mathbf{z})$ and yields $(E_0-E)^{-1}$. A factor $(b/2)^d$ comes from the integrations over the $\xi_{1\nu}$ by virtue of the second delta function in Eq. (5.21) and Eq. (5.23). Integrations over the remaining ξ_n are trivial; and the final result is

$$\langle\rho(E)\rangle\sim\frac{(E_0-E)^{-1}}{(2\pi\gamma)^{(1+d)/2}}\left(\frac{b}{2}\right)^d\gamma ac\left[\prod_{n=2}^{\infty}\left(1-\frac{2}{\lambda_n}\right)^{-\bar{\nu}(n)/2}\right]$$

$$\times\exp\left\{-\left(\frac{1}{2\gamma a^2}+1\right)\right\}. \quad (5.27)$$

VI. EXPLICIT RESULTS FOR ONE DIMENSION

In one dimension, all of the steps leading to Eq. (5.27) can be carried out analytically, and we can check that Eq. (5.27) reduces to the known result given in Eq. (1.8).

We start with the nonlinear Eq. (2.11), which becomes

$$-\tfrac{1}{2}d^2\psi_0/dx^2-(\gamma/(E_0-E))\psi_0^3=E_0\psi_0. \quad (6.1)$$

The solution of Eq. (6.1) is

$$\psi_0(x)=(\tfrac{1}{2}K)^{1/2}\,\mathrm{sech}(Kx), \quad (6.2)$$

where

$$E_0=-\tfrac{1}{2}K^2 \quad (6.3)$$

and

$$E_0-E=\gamma/2K. \quad (6.4)$$

Then

$$\bar{V}(x)=-(\gamma/(E_0-E))\psi_0^2=-K^2\,\mathrm{sech}^2(Kx). \quad (6.5)$$

The three constants a, b, c, defined by Eqs. (5.2), (5.20), and (5.24) respectively, are

$$a^{-2}=\int_{-\infty}^{\infty}dx(\bar{V}(x))^2=\tfrac{4}{3}K^3, \quad (6.6)$$

$$b^{-2}=\int_{-\infty}^{\infty}dx\left[\frac{d\bar{V}}{dx}\right]^2=\frac{16}{15}K^5, \quad (6.7)$$

and

$$c=-2\int_{-\infty}^{\infty}dx\bar{V}(x)\frac{d^2\bar{V}}{dx^2}=\frac{2}{b^2}=\frac{32}{15}K^5. \quad (6.8)$$

We also have to compute the eigenvalues λ_n. If we use Eq. (6.3) and write $Kx=y$, Eq. (5.16) takes the form

$$\left[\frac{d^2}{dy^2}-1+2(1+\lambda_n)\mathrm{sech}^2y\right]u_n(y)=0. \quad (6.9)$$

To solve this equation, we make the transformations

$$u_n(y)=(\mathrm{sech}y)v_n(t), \quad (6.10)$$

$$t=\tanh y. \quad (6.11)$$

Equation (6.9) becomes

$$(1-t^2)(d^2v_n/dt^2)-4t(dv_n/dt)+2\lambda_nv_n=0, \quad (6.12)$$

which implies that the v_n are Gegenbauer polynomials of order $\frac{3}{2}$. The eigenvalues are

$$\lambda_n=\tfrac{1}{2}n(n+3). \quad (6.13)$$

The infinite product in Eq. (5.27) may be evaluated easily:

$$\prod_{n=2}^{\infty}\left(1-\frac{2}{\lambda_n}\right)^{-1/2}=\prod_{n=2}^{\infty}\left[\frac{n-1}{n}\cdot\frac{n+4}{n+3}\right]^{-1/2}$$

$$=\lim_{N\to\infty}\left[\frac{1}{2}\cdot\frac{2}{3}\cdots\frac{N-1}{N}\cdot\frac{6}{5}\cdot\frac{7}{6}\cdots\frac{N+4}{N+3}\right]^{-1/2}=\sqrt{5}. \quad (6.14)$$

This happens to be precisely the factor which is missing in HL.

Returning to Eq. (5.27), note that

$$\exp\left[-\left(\frac{1}{2\gamma a^2}+1\right)\right]$$

$$=\exp\left[-\frac{2}{3\gamma}(2|E|)^{3/2}+O(|E|^{-3/2})\right] \quad (6.15)$$

by virtue of Eqs. (6.4) and (6.6). Then, combining all of the terms in the pre-factor computed above, we obtain exactly Eq. (1.8).