respectively. We shall refer to these modes as the fast and slow shear waves, respectively.

To facilitate comparison with experiment we define B_e, the experimentally determined position of the edge, as the position of the minimum of $d\gamma/dB_0$. In Figs. 2, 3, and 4 we display the attenuation coefficient γ as a function of B_0 for frequencies of 10, 50, and 150 Mc/sec and a value of τ corresponding to $ql=10$ at 50 Mc/sec for the fast shear wave. In Figs. 5, 6, and 7 we plot $d\gamma/dB_0$ as a function of B_0 for the same cases. In each of these figures two graphs are displayed. The dashed line is appropriate to the free-electron model, the solid one to the SDW model. Figure 8 shows the position of the edge B_e as a function of frequency $\nu=\omega/2\pi$ for both models and the same value of τ as given above. It should be noted that at low frequencies the values of B_e do not differ by much. Finally, in Fig. 9, the magnitude of $d\gamma/dB_0$ at the edge is plotted as a function of frequency.[19]

The [110] direction in potassium was selected in our discussion because the large difference in the velocities of the fast and slow shear modes permits considerable simplification in the interpretation of experimental results. The behavior in this case is in contrast to the results for the elastically isotropic solid where care had to be exercised to measure both the apparent attenuation and the rotation of the plane of polarization.

[19] Similar calculations have been carried out for the slow shear mode but, since the results are rather similar, we do not display them here.

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New Inversion Scheme for Obtaining Fermi-Surface Radii from de Haas–van Alphen Areas

F. M. MUELLER*
Department of Physics, and Institute for the Study of Metals, University of Chicago, Chicago, Illinois
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The existing solution to this problem has not been useful in practice. We have found a new and much simpler method based on an expansion of both the area and of the radius squared in appropriate spherical harmonics. The integral equation between these objects yields the simple result

$$a_L^m = b_L^m \pi P_L(0),$$

where $P_L(0)$ is the Legendre function of order L and a_L^m and b_L^m are the coefficients of expansion.

MEASUREMENT of the periodic variation of the magnetic susceptibility of pure metals at low temperatures gives detailed information about the Fermi surfaces of these materials. This de Haas–van Alphen effect has long been used as an experimental tool for finding the Fermi surface extremal cross-sectional areas. Recent improvements in experimental technique now yield results accurate to a few parts in 10^4. We have developed a simple method which, while maintaining mathematical exactness and high experimental accuracy, converts the extremal-area measurements into Fermi-surface radii. The conversion of extremal areas to radii is a purely geometrical problem. By an elegant piece of differential geometry, Lifshitz and Pogorelov[1] (LP) found a formal solution and gave sufficient conditions for the inversion to be unique: These are that the surface be closed, have a center of inversion symmetry, and have a unique radius vector from that center. Their technique depends on a complete knowledge of the extremal area, $A(\theta,\varphi)$. This is impracticable experimentally, so that the technique has never been applied successfully. There is also a mathematical difficulty because one is required to evaluate a principal-value integral of the data, a necessarily discontinuous function.

These two requirements are so stringent that, in practice, accurate data have been fitted by trial and error. The method of expansion in spherical harmonics outlined below needs only a small number of independent data points, performs the principal value integral implicitly, and automatically provides a least-squares fit to the data. It also provides a prescription for finding orientations of the external field which determine the Fermi surface most efficiently, and avoids duplication of effort.

We shall write the equatorial area, $\sigma(\hat{\xi})$, as the integral of the square of the radius over the unit sphere,

$$\sigma(\hat{\xi}) = \frac{1}{2}\int \rho^2(\hat{\epsilon})\delta(\hat{\epsilon}\cdot\hat{\xi})d\Omega(\hat{\epsilon}), \qquad (1)$$

where $\hat{\xi}$ and $\hat{\epsilon}$ are unit vectors. This equation, first considered by LP, compactly states the formal problem. The Dirac delta function selects those directions of $\rho^2(\hat{\epsilon})$, the square of the radius vector, perpendicular to $\hat{\xi}$, the magnetic field direction. Thus, the surface integral

* Advanced Research Projects Agency Research Assistant.

[1] I. M. Lifshitz and A. V. Pogorelov; Dokl. Akad. Nauk SSSR **96**, 1143 (1954).

over $d\Omega$ is reduced to a line integral on the equatorial plane.

The mathematical problem is to invert Eq. (1). Lifshitz and Pogorelev do this by a Green's-function technique, whereas we prefer to solve the related eigenfunction problem by expansion in spherical harmonics.

Thus, we have

$$\sigma(\hat{\epsilon})=\sum_{\lambda,\mu} a_\lambda{}^\mu Y_\lambda{}^\mu(\hat{\epsilon}),$$
$$\rho^2(\hat{\xi})=\sum_{L,m} b_L{}^m Y_L{}^m(\hat{\xi}), \qquad (2)$$

where the a's and b's are the coefficients of expansion of the area and the square of the radius of the spherical harmonics, $Y_L{}^m(\hat{r})$, in the $\hat{\epsilon}$ and $\hat{\xi}$ coordinate systems, respectively.

The δ function has a well-known integral representation in terms of plane waves. By using the Rayleigh expansion of these plane waves we obtain sums of products of two spherical harmonics:

$$\delta(\hat{\epsilon}\cdot\hat{\xi})=\frac{1}{2\pi}\int_{-\infty}^{\infty}\exp\left(ik(\hat{\epsilon}\cdot\hat{\xi})\right)dk$$
$$=2\sum_{L,n} i^L Y_L{}^{n*}(\hat{\xi})Y_L{}^n(\hat{\epsilon})I_L, \qquad (3)$$

where

$$I_L=\int_{-\infty}^{\infty} j_L(k)dk. \qquad (4)$$

Using the integral representation of the spherical Bessel function in terms of Legendre polynominals, we have

$$I_L=\int_{-\infty}^{\infty} dk\frac{1}{2i^L}\int_{-1}^{+1} e^{i\mu k}P_L(\mu)d\mu. \qquad (5)$$

Reversing the order of integration, we regain our δ function on the right-hand side of (3):

$$\delta(\hat{\epsilon}\cdot\hat{\xi})=2\pi\sum_{L,n} P_L(0)Y_L{}^{n*}(\hat{\xi})Y_L{}^n(\hat{\epsilon}). \qquad (6)$$

Placing the results of (2), (3), and (6) into (1), we find

$$\sum_{\lambda,\mu} a_\lambda{}^\mu Y_\lambda{}^\mu(\hat{\epsilon})=\sum_{L,m} b_L{}^m Y_L{}^m(\hat{\epsilon})\pi P_L(0). \qquad (7)$$

A final integration over the $\hat{\epsilon}$ coordinate gives

$$a_L{}^m=b_L{}^m\pi P_L(0). \qquad (8)$$

We note that (8) requires that L be even since $P_L(0)=0$ if L is odd. This was also demanded by the inversion symmetry.

The practical advantage of the expansion method[2,3] is that one can represent the series, to a given order, as a system of linear equations and solve for the coefficients by matrix techniques. No integrations need be performed. This is especially useful when information is available only on widely spaced lines of the unit sphere rather than on a comprehensive net.

In the following paper,[4] we show the advantage of using, not the ordinary spherical harmonics in the expansions (2), but symmetrized crystal harmonics. In addition, we show the relationship between the availability of data, the number of allowed expansion coefficients, and the accuracy desired. Finally, we note that similar expansion techniques will be useful in the study of other properties of the Fermi surface such as the effective mass or the Fermi velocity.

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[2] Superficially the present technique looks similar to a method developed by Shoenberg and Stiles (Ref. 3). The difference is that they expand the radius rather than the square of the radius in harmonics. This means that their equation corresponding to (1) involves products of three spherical harmonics—hence, Wigner coefficients. They avoid this difficulty by ignoring all cross terms except the first-order ones. This restricts their method to very nearly spherical Fermi surfaces.

[3] D. Shoenberg and P. J. Stiles, Proc. Roy. Soc. (London) **A281**, 62 (1964).

[4] F. M. Mueller and M. G. Priestly, following paper, Phys. Rev. **148**, 638 (1966).