neglect of momentum transfer in the adiabatic theory. The conclusion that momentum transfer becomes important in this energy range is consistent with the theoretical results of McCarroll.[31]

A basic assumption in both the Born and impulse approximations is that the capture probability is small. This assumption is not consistent with results of McCarroll[31] which indicate for energies of about 25 keV a high capture probability in a range of impact parameters which contributes heavily to the total charge transfer cross section.

In conclusion, it appears that no existing theoretical calculation gives comprehensive agreement with the measurement over the entire energy range of this experiment, but several calculations agree over a part of the energy range.

ACKNOWLEDGMENTS

The author is greatly indebted to D. L. Allensworth for constructing the apparatus and conducting the measurements and to G. J. Lockwood for valuable advice on the construction of the collision chamber. Thanks are also due to T. A. Green and J. M. Peek for calling the author's attention to a number of relevant theoretical papers and for numerous discussions concerning the various theoretical methods.

PHYSICAL REVIEW VOLUME 148, NUMBER 1 5 AUGUST 1966

Cross Section for Energy Transfer between Two Moving Particles*

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(Received 4 January 1966)

The classical cross section $\sigma_{\Delta E}$, for producing a specified energy transfer ΔE in the collision of two particles 1,2 having arbitrary masses and velocities $\mathbf{v}_1,\mathbf{v}_2$ in the laboratory system, is derived. The effective average (for fixed speeds v_1,v_2) of $\sigma_{\Delta E}$ over all directions of the particle velocities $\mathbf{v}_1$ and/or $\mathbf{v}_2$ is then computed. These results are required in the classical calculations of atomic-collision cross sections via the procedures recently proposed by Gryzinski. The method will yield the average of any function $F(v,V,\cos\bar{\theta})$ over all directions of the particle velocities, where $\mathbf{v}=\mathbf{v}_1-\mathbf{v}_2$, $\mathbf{V}$ is the velocity of the center of mass, and $\bar{\theta}$ is the angle between $\mathbf{v}$ and $\mathbf{V}$.

I. INTRODUCTION

RECENTLY, Gryzinski has published three papers[1–3] detailing his procedures for performing classical (nonquantum) calculations of atomic-collision cross sections. The utility of these procedures in electron-atom and electron-molecule collisions has been examined by Bauer and Bartky.[4] For such collisions, one requires the cross section $\sigma_{\Delta E}(\mathbf{v}_1,\mathbf{v}_2)$ for producing an energy transfer ΔE in the collision of two electrons moving with arbitrary velocities $\mathbf{v}_1$, $\mathbf{v}_2$ in the laboratory system. There also is required $\sigma_{\Delta E}{}^{\text{eff}}(v_1,v_2)$ the effective average of $\sigma_{\Delta E}(\mathbf{v}_1,\mathbf{v}_2)$ over all orientations of $\mathbf{v}_1$ and/or $\mathbf{v}_2$ for fixed speeds v_1, v_2. Gryzinski has derived expressions for these quantities, but use of these formulas is complicated by an extremely awkward notation; moreover Gryzinski's expressions involve some subsidiary approximations. For these reasons, Stabler[5] has rederived—and obtained in much simpler form—the exact expressions for $\sigma_{\Delta E}$ and $\sigma_{\Delta E}{}^{\text{eff}}$ in electron-electron collisions. Similar expressions have been obtained by Ochkur and Petrun'kin.[6] However, these authors[5,6] have rederived $\sigma_{\Delta E}$ only for electron-electron collisions, i.e., for colliding particles of equal mass, whereas for calculations of, e.g., ion-atom collisions by Gryzinski's procedures, one requires $\sigma_{\Delta E}$ and $\sigma_{\Delta E}{}^{\text{eff}}$ for collisions of unequally massive charged particles.

This paper derives the required exact formulas for $\sigma_{\Delta E}$ and $\sigma_{\Delta E}{}^{\text{eff}}$ in the unequal-mass case. Application of these formulas to examination of the utility of Gryzinski's procedures in charge-transfer reactions is under way (in cooperation with Hsiang Tai and Jean Welker). This paper obtains the final formula for $\sigma_{\Delta E}{}^{\text{eff}}(v_1,v_2)$ in only one case, namely, Coulomb collisions; it will be clear, however, that the method of performing the average over all orientations is applicable to arbitrary interactions, as well as to the averages of quantities other than $\sigma_{\Delta E}(\mathbf{v}_1,\mathbf{v}_2)$.

II. CALCULATION OF $\sigma_{\Delta E}$

I consider a collision between particles 1 and 2, whose initial velocities in the laboratory system are $\mathbf{v}_1=v_1\mathbf{n}_1$

* Supported by the National Aeronautics and Space Administration under Contract NGR-39-011-35 and Research Grant N-s-6-416.

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and $\mathbf{v}_2=v_2\mathbf{n}_2$, respectively. Their laboratory velocities after the collision will be $\mathbf{v}_1'=v_1'\mathbf{n}_1'$ and $\mathbf{v}_2'=v_2'\mathbf{n}_2'$. Correspondingly, the velocities of these particles measured by an observer moving with the center of mass are $\mathfrak{v}_1=\mathfrak{v}_1\mathfrak{n}_1$, $\mathfrak{v}_2=\mathfrak{v}_2\mathfrak{n}_2$ (initial) and $\mathfrak{v}_1'=\mathfrak{v}_1'\mathfrak{n}_1'$, $\mathfrak{v}_2'=\mathfrak{v}_2'\mathfrak{n}_2'$ (final).[7] It is presumed that the coordinate axes of the laboratory and center-of-mass observers are parallel, so that the components of the vectors defined above are consistent with

$$\mathbf{v}_1=\mathbf{V}+\mathfrak{v}_1\,,\ \text{etc.}, \tag{1}$$

where $\mathbf{V}$ is the center-of-mass velocity measured by the laboratory observer.

$$\mathbf{V}=V\mathbf{n}_V=M^{-1}(m_1\mathbf{v}_1+m_2\mathbf{v}_2)\,,\quad M=m_1+m_2\,. \tag{2}$$

Also,

$$\mathfrak{v}_1=m_2M^{-1}\mathbf{v}\,,\quad \mathfrak{v}_2=-m_1M^{-1}\mathbf{v}\,, \tag{3a}$$

$$\mathfrak{v}_1'=m_2M^{-1}\mathbf{v}'\,,\quad \mathfrak{v}_2'=-m_1M^{-1}\mathbf{v}'\,, \tag{3b}$$

where

$$\mathbf{v}=\mathbf{v}_1-\mathbf{v}_2=v\mathfrak{n}\,,\quad \mathbf{v}'=\mathbf{v}_1'-\mathbf{v}_2'=v\mathfrak{n}'\,, \tag{4}$$

are the relative velocities before and after the collision.

For given $\mathbf{v}_1$, $\mathbf{v}_2$ the vectors $\mathbf{V}$, $\mathbf{v}$ are determined, so that for given $\mathbf{v}_1$, $\mathbf{v}_2$ the polar axis of a fixed system of spherical coordinates can be chosen along $\mathbf{V}$; in this system the polar and azimuth angles of $\mathfrak{n}$ and $\mathfrak{n}'$ are $\bar{\theta}$, $\bar{\phi}$ and $\bar{\theta}'$, $\bar{\phi}'$, respectively. Now suppose 1 is regarded as the "incident" particle. Then the energy gain ΔE by particle 2 (as seen in the laboratory system) is[8]

$$\begin{aligned}\Delta E&=\tfrac{1}{2}m_2v_2'^2-\tfrac{1}{2}m_2v_2^2=\tfrac{1}{2}m_1v_1^2-\tfrac{1}{2}m_1v_1'^2\\ &=m_2\mathbf{V}\cdot(\mathfrak{v}_2'-\mathfrak{v}_2)=\mu vV(\cos\bar{\theta}-\cos\bar{\theta}')\,,\end{aligned} \tag{5}$$

where $\mu=m_1m_2M^{-1}$ is the reduced mass. Equation (5) shows that for given $\mathbf{v}_1$, $\mathbf{v}_2$ the quantity ΔE is a function only of $\bar{\theta}'$. In fact

$$d(\Delta E)=\mu vV\sin\bar{\theta}'d\bar{\theta}'. \tag{6}$$

Let $\sigma(\mathbf{v}_1,\mathbf{v}_2)$ be the total cross section for given $\mathbf{v}_1$, $\mathbf{v}_2$. Then the quantity $\sigma_{\Delta E}(\mathbf{v}_1,\mathbf{v}_2)$ is defined by

$$\sigma(\mathbf{v}_1,\mathbf{v}_2)=\int d(\Delta E)\sigma_{\Delta E}(\mathbf{v}_1,\mathbf{v}_2)\,. \tag{7}$$

But if $\bar{\sigma}(v;\mathfrak{n}\rightarrow\mathfrak{n}')$ is the corresponding differential cross section for scattering in the center-of-mass system (wherein the collision can change only the direction but not the magnitude of the relative velocity), it also is true that

$$\sigma(\mathbf{v}_1,\mathbf{v}_2)=\int d\mathfrak{n}'\bar{\sigma}(v;\mathfrak{n}\rightarrow\mathfrak{n}') \tag{8a}$$

$$=\int d\bar{\phi}'d\bar{\theta}'\,\sin\bar{\theta}'\bar{\sigma}(v;\mathfrak{n}\rightarrow\mathfrak{n}') \tag{8b}$$

$$=\frac{1}{\mu vV}\int d(\Delta E)d\bar{\phi}'\bar{\sigma}(v;\mathfrak{n}\rightarrow\mathfrak{n}')\,, \tag{8c}$$

using (6).

Equations (7) and (8c) imply

$$\sigma_{\Delta E}(\mathbf{v}_1,\mathbf{v}_2)=\frac{1}{\mu vV}\int d\bar{\phi}'\bar{\sigma}(v;\mathfrak{n}\rightarrow\mathfrak{n}')\,. \tag{9}$$

For fixed $\mathbf{v}_1$, $\mathbf{v}_2$, i.e., for fixed $\bar{\theta}$, $\bar{\phi}$, the right side of (9) is a function of $\bar{\theta}'$ and, therefore, by (6), of ΔE. For every value of $\bar{\theta}'$ the integral in Eq. (9) runs over all values of $\bar{\phi}'$ from 0 to 2π, because (for any initial $\bar{\theta}$, $\bar{\phi}$) the final relative velocity $\mathbf{v}'$ can have any direction in space. The cross section $\bar{\sigma}(v;\mathfrak{n}\rightarrow\mathfrak{n}')$, though dependent only on the angle between $\mathfrak{n}$ and $\mathfrak{n}'$, can be a function of $\bar{\phi}'$.

The results so far hold for any $\bar{\sigma}$. For definiteness, I now specialize to the Coulomb case

$$\bar{\sigma}(v;\mathfrak{n}\rightarrow\mathfrak{n}')=\left(\frac{Z_1Z_2e^2}{2\mu v^2}\right)^2\csc^4(\tfrac{1}{2}\chi)\,, \tag{10}$$

where the center-of-mass-system scattering angle χ is the angle between $\mathfrak{n}$ and $\mathfrak{n}'$; and Z_1e, Z_2e are the charges carried by particles 1, 2. Substituting Eq. (10) in Eq. (9), and employing

$$\sin^4(\tfrac{1}{2}\chi)=\tfrac{1}{4}(1-\cos\chi)^2 \tag{11a}$$

$$\cos\chi=\cos\bar{\theta}\,\cos\bar{\theta}'+\sin\bar{\theta}\,\sin\bar{\theta}'\,\cos(\bar{\phi}-\bar{\phi}')\,, \tag{11b}$$

one finds

$$\sigma_{\Delta E}(\mathbf{v}_1,\mathbf{v}_2)=\frac{1}{\mu vV}\left(\frac{Z_1Z_2e^2}{\mu v^2}\right)^2\int_0^{2\pi}d\phi\frac{1}{(a-b\,\cos\phi)^2}\,, \tag{12}$$

where

$$a=1-\cos\bar{\theta}\,\cos\bar{\theta}'\,,\quad b=\sin\bar{\theta}\,\sin\bar{\theta}'. \tag{13}$$

When[9] $a^2\geqslant b^2$, as is the case for a, b of (13)

$$\int_0^{2\pi}d\phi\frac{1}{(a-b\,\cos\phi)^2}=\frac{2\pi a}{(a^2-b^2)^{3/2}}\,. \tag{14}$$

Thus,

$$\sigma_{\Delta E}(\mathbf{v}_1,\mathbf{v}_1)=\frac{2\pi}{\mu vV}\left(\frac{Z_1Z_2e^2}{\mu v^2}\right)^2\frac{(1-\cos\bar{\theta}\,\cos\bar{\theta}')}{|\cos\bar{\theta}-\cos\bar{\theta}'|^3}\,. \tag{15}$$

[7] These German symbols have been used here to designate vectors—and related scalars—measured in the center-of-mass system because barred boldface Latin symbols were not available. Where possible, however, the paper follows the customary procedure of denoting center-of-mass quantities by barred symbols.

[8] This result, and some other equations obtained in this paper, can be found in Gryzinski's papers (Refs. 1–3), or in the earlier work of Chandrasekhar. S. Chandrasekhar, Astrophys. J. **93**, 285, 323 (1941); R. E. Williamson and S. Chandrasekhar, *ibid.* **93**, 305 (1941).

[9] E. T. Whittaker and G. N. Watson, *A Course of Modern Analysis* (Cambridge University Press, New York, 1940), p. 113.

Or, using (5),

$$\sigma_{\Delta E}(\mathbf{v}_1,\mathbf{v}_2)=\frac{2\pi(Z_1Z_2e^2)^2V^2}{v^2|\Delta E|^3}\left(1-\cos^2\bar{\theta}+\frac{\Delta E}{\mu vV}\cos\bar{\theta}\right), \quad (16a)$$

with the restriction, also from (5), that

$$-1\leqslant\cos\bar{\theta}-\Delta E/\mu vV\leqslant 1\,, \quad (16b)$$

which guarantees $\sigma_{\Delta E}\geqslant 0$. For given $\mathbf{v}_1$, $\mathbf{v}_2$, if (16b) is not satisfied, then

$$\sigma_{\Delta E}(\mathbf{v}_1,\mathbf{v}_2)=0\,, \quad (16c)$$

i.e., values of ΔE for which (16b) fails cannot occur.

Equations (16) are the desired result for $\sigma_{\Delta E}$, in what proves to be a convenient form for calculating $\sigma_{\Delta E}{}^{\mathrm{eff}}$. In terms of $\mathbf{v}_1$, $\mathbf{v}_2$, the quantities v, V, $\cos\bar{\theta}$ are, using Eqs. (2) and (4),

$$v=(v_1{}^2+v_2{}^2-2v_1v_2\mathbf{n}_1\cdot\mathbf{n}_2)^{1/2}\,, \quad (17a)$$

$$V=M^{-1}(m_1{}^2v_1{}^2+m_2{}^2v_2{}^2+2m_1m_2v_1v_2\mathbf{n}_1\cdot\mathbf{n}_2)^{1/2}\,, \quad (17b)$$

$$\cos\bar{\theta}=(vV)^{-1}\mathbf{v}\cdot\mathbf{V}=(MvV)^{-1}\times[m_1v_1{}^2-m_2v_2{}^2+(m_2-m_1)v_1v_2\mathbf{n}_1\cdot\mathbf{n}_2]\,. \quad (17c)$$

It can be shown that in the special case $m_1=m_2=m$, Eqs. (16) reduce to the seemingly very different expression for $\sigma_{\Delta E}$ given by Stabler,[5] namely,[10] his Eq. (8).

III. CALCULATION OF $\sigma_{\Delta E}{}^{\mathrm{eff}}$

Suppose the target particle 2 has an isotropic velocity distribution in the laboratory system. Then for any actual $\mathbf{v}_1$ the effective $\sigma_{\Delta E}$ is defined by

$$v_1\sigma_{\Delta E}{}^{\mathrm{eff}}=\frac{1}{4\pi}\int d\mathbf{n}_2|\mathbf{v}_1-v_2\mathbf{n}_2|\sigma_{\Delta E}(\mathbf{v}_1,\mathbf{v}_2)\,. \quad (18)$$

This definition of the effective $\sigma_{\Delta E}$ is appropriate when, e.g., the particles 2 are bound electrons in stationary atoms being ionized by a beam of protons (particles 1). If the atoms have velocity $\mathbf{v}_a\neq 0$ in the laboratory system, e.g., if the atoms form a beam, the velocity distribution of 2, though isotropic in a coordinate system moving with the atoms, is not isotropic in the laboratory system. In this event, realizing that the total reaction rate (e.g., the total rate of ionization) is independent of the observer's velocity, the simplest procedure is to compute the total reaction rate in the system where the velocity of 1 now is $\mathbf{v}_1-\mathbf{v}_a$.

Once, as in (18), the distribution of $\mathbf{v}_2$ is accepted as isotropic, the value of $\sigma_{\Delta E}{}^{\mathrm{eff}}$ obviously cannot depend on the direction of $\mathbf{n}_1$. In other words, $\sigma_{\Delta E}{}^{\mathrm{eff}}$ now depends only on the magnitudes of $\mathbf{v}_1$, $\mathbf{v}_2$, and so can be averaged over $\mathbf{n}_1$ as well as $\mathbf{n}_2$. For the Coulomb case, therefore, using (16)

$$\sigma_{\Delta E}{}^{\mathrm{eff}}(v_1,v_2)=\frac{(Z_1Z_2e^2)^2}{8\pi|\Delta E|^3v_1}\int d\mathbf{n}_1d\mathbf{n}_2\times\frac{V^2}{v}\left(1-\cos^2\bar{\theta}+\frac{\Delta E}{\mu vV}\cos\bar{\theta}\right), \quad (19)$$

where v, V, $\cos\bar{\theta}$ are given by Eqs. (17), and the allowed ranges of $\mathbf{n}_1$, $\mathbf{n}_2$ must be consistent with (16b), i.e., in (19) appear only those $\mathbf{n}_1$, $\mathbf{n}_2$ for which $\sigma_{\Delta E}(\mathbf{v}_1,\mathbf{v}_2)\neq 0$. Specifically, for given v_1, v_2, ΔE the integral (19) runs only over those directions $\mathbf{n}_1$, $\mathbf{n}_2$ for which

$$-1+\Delta E/\mu vV\leqslant\cos\bar{\theta}\leqslant 1\,,\quad \Delta E\geqslant 0 \quad (20a)$$

$$-1\leqslant\cos\bar{\theta}\leqslant 1+\Delta E/\mu vV\,,\quad \Delta E\leqslant 0. \quad (20b)$$

Despite its apparent complexity, the integral (19) can be evaluated in closed form. For any integrand $F(\mathbf{n}_1,\mathbf{n}_2,v_1,v_2)$

$$\int Fd\mathbf{n}_1d\mathbf{n}_2=\frac{1}{v_1{}^2v_2{}^2}\int v_1{}^2d\mathbf{n}_1v_2{}^2d\mathbf{n}_2F(\mathbf{n}_1,\mathbf{n}_2,v_1,v_2)$$
$$=\frac{1}{v_1{}^2v_2{}^2}\int\dot{v}_1{}^2d\mathbf{n}_1d\dot{v}_1\dot{v}_2{}^2d\mathbf{n}_2d\dot{v}_2\times\delta(\dot{v}_1-v_1)\delta(\dot{v}_2-v_2)F(\mathbf{n}_1,\mathbf{n}_2,\dot{v}_1,\dot{v}_2)\,.$$

But[11] $\dot{v}_1{}^2d\mathbf{n}_1d\dot{v}_1$ is the volume element $d\dot{\mathbf{v}}_1$ in the space formed by the components of the vector $\dot{\mathbf{v}}_1=\dot{v}_1\mathbf{n}_1$. Thus, Eq. (19) can be replaced by

$$\sigma_{\Delta E}{}^{\mathrm{eff}}=\frac{(Z_1Z_2e^2)^2}{8\pi|\Delta E|^3v_1{}^3v_2{}^2}\int d\dot{\mathbf{v}}_1d\dot{\mathbf{v}}_2\delta(\dot{v}_1-v_1)\delta(\dot{v}_2-v_2)\times\frac{V^2}{v}\left(1-\cos^2\bar{\theta}+\frac{\Delta E}{\mu vV}\cos\bar{\theta}\right) \quad (21)$$

with the understanding that under the integral sign $\dot{v}_1$, $\dot{v}_2$ now replace v_1, v_2 in Eqs. (17) for v, V, $\cos\bar{\theta}$. Consequently, recalling (2) and (4), the equations relating $\dot{\mathbf{v}}_1$, $\dot{\mathbf{v}}_2$ to $\mathbf{v}$, $\mathbf{V}$ in (21) must be

$$\dot{\mathbf{v}}_1=\dot{v}_1\mathbf{n}_1=\mathbf{V}+m_2M^{-1}\mathbf{v}\,,\quad \dot{\mathbf{v}}_2=\dot{v}_2\mathbf{n}_2=\mathbf{V}-m_1M^{-1}\mathbf{v}. \quad (22)$$

With (22), the Jacobian of the transformation from $d\dot{\mathbf{v}}_1d\dot{\mathbf{v}}_2$ to $d\mathbf{v}d\mathbf{V}$ is unity. Hence,

$$\sigma_{\Delta E}{}^{\mathrm{eff}}=\frac{(Z_1Z_2e^2)^2}{8\pi|\Delta E|^3v_1{}^3v_2{}^2}\int d\mathbf{v}d\mathbf{V}\delta(\dot{v}_1-v_1)\delta(\dot{v}_2-v_2)\times\frac{V^2}{v}\left(1-\cos^2\bar{\theta}+\frac{\Delta E}{\mu vV}\cos\bar{\theta}\right), \quad (23)$$

[10] Note that although Stabler regards 1 as the "target" electron and 2 as the "incident" electron, his ΔE, defined by his Eq. (1), is identical with my ΔE of Eq. (5).

[11] Again the exigencies of the printer have determined the choice of symbols. As used here, the dot in no way is related to the time derivative. Instead, the quantities $\dot{v}_1$, $\dot{v}_2$ here merely denote dummy variables to be distinguished from v_1, v_2.

wherein, recalling $\bar{\theta}=\cos^{-1}(\mathbf{n}\cdot\mathbf{n}_V)$,

$$\dot{v}_1=(V^2+m_2^2M^{-2}v^2+2m_2M^{-1}vV\cos\bar{\theta})^{1/2}, \quad (24a)$$

$$\dot{v}_2=(V^2+m_1^2M^{-2}v^2-2m_1M^{-1}vV\cos\bar{\theta})^{1/2}. \quad (24b)$$

Since (20) and the integrand in (23) do not involve $\mathbf{n}_V$ or the azimuth angle $\bar{\phi}$, Eq. (23) simplifies to

$$\sigma_{\Delta E}^{\mathrm{eff}}(v_1,v_2)=\frac{\pi(Z_1Z_2e^2)^2}{|\Delta E|^3v_1^3v_2^2}\int_0^\infty dv\int_0^\infty dV\int d\bar{\theta}\sin\bar{\theta}$$
$$\times vV^4\delta(\dot{v}_1-v_1)\delta(\dot{v}_2-v_2)[1-\cos^2\bar{\theta}+(\Delta E/\mu vV)\cos\bar{\theta}], \quad (25)$$

where the limits of integration over $\bar{\theta}$ are determined by (20).

Integrate (25) over the allowed range of $\cos\bar{\theta}$, recalling that

$$\int dx f(x)\delta[g(x)]=\sum_i\left\{\left|\left(\frac{dg}{dx}\right)^{-1}\right|f(x)\right\}_{x=x_i}, \quad (26)$$

where x_i are the roots of $g(x)=0$ in the integration interval. Because of (24a), the quantity $\dot{v}_1-v_1$ as a function of $\cos\bar{\theta}$ vanishes only at

$$\cos\bar{\theta}_i=(2m_2M^{-1}vV)^{-1}(v_1^2-V^2-m_2^2M^{-2}v^2). \quad (27)$$

Thus,

$$\sigma_{\Delta E}^{\mathrm{eff}}(v_1,v_2)=\frac{\pi(Z_1Z_2e^2)^2}{|\Delta E|^3v_1^3v_2^2}\int dv\int dV$$
$$\times vV^4\left(\frac{m_2M^{-1}vV}{v_1}\right)^{-1}\left(1-\cos^2\bar{\theta}_i+\frac{\Delta E}{\mu vV}\cos\bar{\theta}_i\right)$$
$$\times\delta\left[\left(\frac{MV^2}{m_2}+\frac{m_1v^2}{M}-\frac{m_1v_1^2}{m_2}\right)^{1/2}-v_2\right] \quad (28)$$

integrated over that portion of the first quadrant of the v, V plane for which $\cos\bar{\theta}_i$ from (27) lies within the limits on $\cos\bar{\theta}$ specified by (20). These restrictions on v, V implied by substituting (27) in (20) take the form, for positive or negative ΔE,

$$(V-m_2M^{-1}v)^2\leqslant v_{1s}^2, \quad (29a)$$

$$v_{1g}^2\leqslant(V+m_2M^{-1}v)^2, \quad (29b)$$

where, recalling Eq. (5),

$$v_{1s}=\text{smaller of } v_1,\ v_1',$$
$$v_{1g}=\text{greater of } v_1,\ v_1'. \quad (30a)$$

Of course

$$v_1'=[v_1^2-(2/m_1)(\Delta E)]^{1/2},$$
$$v_2'=[v_2^2+(2/m_2)(\Delta E)]^{1/2}. \quad (30b)$$

Equations (29) imply that (28) is integrated over the portion of the first quadrant of the v, V plane lying below the line [termed line (a)]

$$V-m_2M^{-1}v=v_{1s}; \quad (31a)$$

lying above the line [termed line (b)]

$$m_2M^{-1}v-V=v_{1s}; \quad (31b)$$

and lying above the line [termed line (c)]

$$V+m_2M^{-1}v=v_{1g}. \quad (31c)$$

The shaded region in Fig. 1 is this allowed portion of the v, V plane.

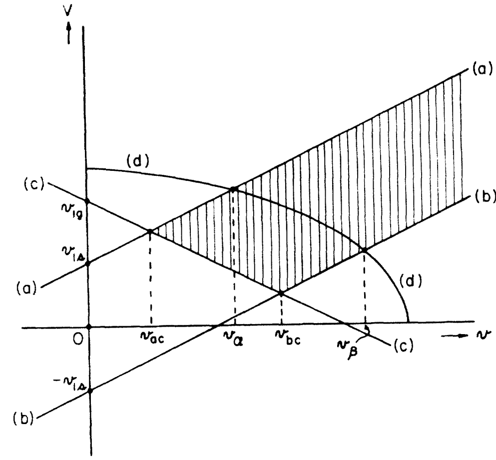

FIG. 1. Integration region (shaded) in the v, V plane for Eq. (28). Lines (a), (b), (c) are plots of Eqs. (31a), (31b), (31c), respectively. Lines (a), (c) intersect at $v=v_{ac}$; lines (b), (c) at $v=v_{bc}$. The ellipse (d) is a plot of Eq. (32), for the case that its intersections with the boundaries of the shaded region occur on lines (a), (b), at $v=v_\alpha$, v_β, respectively. In this case, the limits of integration in (33) are $v_l=v_\alpha$ and $v_u=v_\beta$.

The δ function in (28) vanishes unless

$$\tfrac{1}{2}MV^2+\tfrac{1}{2}\mu v^2=\tfrac{1}{2}m_1v_1^2+\tfrac{1}{2}m_2v_2^2=E$$
$$=\tfrac{1}{2}m_1v_1'^2+\tfrac{1}{2}m_2v_2'^2, \quad (32)$$

where E is the total energy in the laboratory system. In other words, the quantities v, V in (28) indeed must have values consistent with conservation of energy. Equation (32) is an ellipse in the v, V plane. Then, integrating (28) over V, and again using (26),

$$\sigma_{\Delta E}^{\mathrm{eff}}(v_1,v_2)=\frac{\pi(Z_1Z_2e^2)^2}{|\Delta E|^3v_1^2v_2}$$
$$\times\int_{v_l}^{v_u}dv\,V_i^2\left(1-\cos^2\bar{\theta}_i+\frac{\Delta E}{\mu vV_i}\cos\bar{\theta}_i\right) \quad (33)$$

integrated in the range $v_l\leqslant v\leqslant v_u$ for which points v, V on the ellipse (32) lie in the shaded region of Fig. 1. Here, for given v_1, v_2

$$V_i(v)=[M^{-1}(2E-\mu v^2)]^{1/2}$$
$$=[M^{-1}(m_1v_1^2+m_2v_2^2-\mu v^2)]^{1/2} \quad (34a)$$

and, in (33), V_i replaces V in the definition of (27), i.e., now

$$\cos\bar{\theta}_i = (2vV_i)^{-1}[v_1^2 - v_2^2 + M^{-1}(m_1 - m_2)v^2], \quad (34b)$$

as one expects from Eqs. (17a) and (17c).

Equations (34) reduce (33) to a simple integral over v, yielding finally

$$\sigma_{\Delta E}^{\text{eff}}(v_1,v_2) = \frac{\pi(Z_1Z_2e^2)^2}{4|\Delta E|^3 v_1^2 v_2}[(v_1^2 - v_2^2)(v_2'^2 - v_1'^2)(v_l^{-1} - v_u^{-1}) + (v_1^2 + v_2^2 + v_1'^2 + v_2'^2)(v_u - v_l) - \tfrac{1}{3}(v_u^3 - v_l^3)], \quad (35)$$

where v_1', v_2' are given by (30b). The integration limits v_l, v_u in (33) and (35) remain to be determined. Otherwise, (35) is the desired result for $\sigma_{\Delta E}^{\text{eff}}$.

IV. DETERMINATION OF v_l, v_u

Evidently, v_l, v_u are the values of v at which the ellipse (32) intersects the boundaries of the shaded region in Fig. 1. From Eqs. (31c) and (32) one sees that the ellipse always has two real intersections with line (c), of which both, or only one, or neither may lie on the boundary of the shaded region, depending on the values of v_1, v_2. These intersections occur at $v = v_\gamma$ and $v = v_\delta$, given by

$$\begin{aligned} v_\gamma &= v_1 - v_2, \\ v_\delta &= v_1 + v_2, \end{aligned} \quad \Delta E \geqslant 0, \quad \text{i.e.,} \quad v_{1g} = v_1, \quad (36a)$$

$$\begin{aligned} v_\gamma &= v_1' - v_2', \\ v_\delta &= v_1' + v_2', \end{aligned} \quad \Delta E \leqslant 0, \quad \text{i.e.,} \quad v_{1g} = v_1', \quad (36b)$$

where $v_\gamma \leqslant v_\delta$. Similarly, in the first quadrant of the v, V plane, lines (a) and (b) each have at most one intersection with the ellipse, at $v = v_\alpha$ and $v = v_\beta$, respectively, given by

$$\begin{aligned} v_\alpha &= v_2' - v_1', \\ v_\beta &= v_2' + v_1', \end{aligned} \quad \Delta E \geqslant 0, \quad \text{i.e.,} \quad v_{1s} = v_1', \quad (37a)$$

$$\begin{aligned} v_\alpha &= v_2 - v_1, \\ v_\beta &= v_2 + v_1, \end{aligned} \quad \Delta E \leqslant 0, \quad \text{i.e.,} \quad v_{1s} = v_1. \quad (37b)$$

Because the ellipse (32) is everywhere concave downward in the first quadrant, it must intersect the boundary of the shaded region no more than twice; it may not intersect the boundary of the shaded region at all. Thus, referring to Fig. 1, it is clear that the only possible limits of integration in (33) are

$$\begin{aligned} &\text{(i)}\ v_l = v_\alpha, \quad v_u = v_\beta, \\ &\text{(ii)}\ v_l = v_\alpha, \quad v_u = v_\delta, \\ &\text{(iii)}\ v_l = v_\gamma, \quad v_u = v_\delta, \\ &\text{(iv)}\ v_l = v_\gamma, \quad v_u = v_\beta, \\ &\text{(v)}\ \text{no intersections}, \quad \sigma_{\Delta E}^{\text{eff}}(v_1,v_2) = 0. \end{aligned} \quad (38)$$

The conditions for the above cases to occur are[12] (again referring to Fig. 1)

$$\text{(i)}\ v_\gamma \leqslant v_{ac}, \quad v_{bc} \leqslant v_\delta; \quad \text{or equivalently,} \quad v_{ac} \leqslant v_\alpha, \quad v_{bc} \leqslant v_\beta \quad (39a)$$

$$\text{(ii)}\ v_\gamma \leqslant v_{ac} \leqslant v_\delta \leqslant v_{bc}; \quad \text{or equivalently,} \quad v_{ac} \leqslant v_\alpha, \quad v_\beta \leqslant v_{bc} \quad (39b)$$

$$\text{(iii)}\ v_{ac} \leqslant v_\gamma, \quad v_\delta \leqslant v_{bc} \quad (39c)$$

$$\text{(iv)}\ v_{ac} \leqslant v_\gamma \leqslant v_{bc} \leqslant v_\delta, \quad \text{or equivalently,} \quad v_\alpha \leqslant v_{ac}, \quad v_{bc} \leqslant v_\beta \quad (39d)$$

$$\text{(v)}\ \text{either} \quad v_\delta \leqslant v_{ac} \quad \text{or} \quad v_{bc} \leqslant v_\gamma, \quad (39e)$$

where v_{ac} is the value of v at the intersection of lines (a) and (c); v_{bc} is the value of v at the intersection of lines (b) and (c). These values are, for positive or negative ΔE,

$$v_{ac} = (2m_2)^{-1}M(v_{1g} - v_{1s}) = (2m_2)^{-1}M|v_1 - v_1'|, \quad (40a)$$

$$v_{bc} = (2m_2)^{-1}M(v_{1g} + v_{1s}) = (2m_2)^{-1}M(v_1 + v_1'). \quad (40b)$$

Equations (36) and (40) imply[12] that cases (i)–(v) of (38) correspond to the following limits in Eq. (35), and occur under the following circumstances:

$$\text{(i)}\ v_l = v_2' - v_1', \quad v_u = v_1' + v_2', \quad \Delta E \geqslant 0 \quad (41a)$$

$$v_l = v_2 - v_1, \quad v_u = v_1 + v_2, \quad \Delta E \leqslant 0 \quad (41b)$$

when

$$\Delta E \geqslant \frac{4m_1m_2}{M^2}\left(E_1 - E_2 + \left|E_1\frac{v_2}{v_1} - E_2\frac{v_1}{v_2}\right|\right), \quad (41c)$$

provided also $2m_2v_2 \geqslant |m_1 - m_2|v_1$ $\Delta E \geqslant 0$ only (41d)

$$\text{(ii)}\ v_l = v_2' - v_1', \quad v_u = v_1 + v_2, \quad \Delta E \geqslant 0 \quad (42a)$$

$$v_l = v_2 - v_1, \quad v_u = v_1' + v_2', \quad \Delta E \leqslant 0 \quad (42b)$$

when $m_1 > m_2$ and

$$\frac{4m_1m_2}{M^2}\left(E_1 - E_2 - E_1\frac{v_2}{v_1} + E_2\frac{v_1}{v_2}\right) \leqslant \Delta E \leqslant \frac{4m_1m_2}{M^2}\left(E_1 - E_2 + E_1\frac{v_2}{v_1} - E_2\frac{v_1}{v_2}\right). \quad (42c)$$

$$\text{(iii)}\ v_l = v_1 - v_2, \quad v_u = v_1 + v_2, \quad \Delta E \geqslant 0 \quad (43a)$$

$$v_l = v_1' - v_2', \quad v_u = v_1' + v_2', \quad \Delta E \leqslant 0 \quad (43b)$$

when

$$\Delta E \leqslant \frac{4m_1m_2}{M^2}\left(E_1 - E_2 - \left|E_1\frac{v_2}{v_1} - E_2\frac{v_1}{v_2}\right|\right), \quad (43c)$$

provided also

$$2m_1v_1 \geqslant |m_1 - m_2|v_2, \quad \Delta E \leqslant 0 \text{ only}. \quad (43d)$$

$$\text{(iv)}\ v_l = v_1 - v_2, \quad v_u = v_1' + v_2', \quad \Delta E \geqslant 0 \quad (44a)$$

$$v_l = v_1' - v_2', \quad v_u = v_1 + v_2, \quad \Delta E \leqslant 0 \quad (44b)$$

[12] See Sec. V of E. Gerjuoy, University of Pittsburgh Space Research Coordination Center Report No. 25, 1965 (unpublished). The present paper is a condensed version of this report.

when $m_1 < m_2$ and

$$\frac{4m_1m_2}{M^2}\left(E_1 - E_2 - E_2\frac{v_1}{v_2} + E_1\frac{v_2}{v_1}\right) \leqslant \Delta E \leqslant \frac{4m_1m_2}{M^2}\left(E_1 - E_2 + E_2\frac{v_1}{v_2} - E_1\frac{v_2}{v_1}\right). \quad (44c)$$

(v) $$\sigma_{\Delta E}{}^{\mathrm{eff}}(v_1,v_2) = 0 \quad (45a)$$

when Eq. (41c) holds and Eq. (41d) fails, provided

$$\Delta E \geqslant 0; \quad (45b)$$

when Eq. (43c) holds and Eq. (43d) fails, provided

$$\Delta E \leqslant 0. \quad (45c)$$

Of course, because v_1', v_2' in (30b) must be real, Eq. (45a) holds unless

$$-\tfrac{1}{2}m_2v_2{}^2 \leqslant \Delta E \leqslant \tfrac{1}{2}m_1v_1{}^2, \quad (45d)$$

which expresses the fact that the particle losing energy in the collision cannot lose more than its initial kinetic energy.

The above conditions for the occurrence of cases (i)–(v) can be expressed in a variety of alternative forms. For example, when $\Delta E \geqslant 0$, each of the following conditions is implied by and implies[12] (i.e., is equivalent to) the pair (41c), (41d):

$$(2m_2)^{-1}(Mv_1' + |m_1 - m_2|v_1) \leqslant v_2, \quad (46a)$$

$$(2m_2)^{-1}(Mv_1 + |m_1 - m_2|v_1') \leqslant v_2', \quad (46b)$$

$$0 \leqslant v_1' \leqslant (2m_1)^{-1}(Mv_2 - |m_1 - m_2|v_2'). \quad (46c)$$

The equivalence of (46a) and the pair (41c), (41d) when $\Delta E \geqslant 0$ is easily demonstrated.[12] Actually, (46a) is the condition which follows directly from the first set of conditions for case (i) in (39a), namely, from $v_\gamma \leqslant v_{ac}$, $v_{bc} \leqslant v_\delta$; Eq. (46b) is obtained from the equivalent set $v_{ac} \leqslant v_\alpha$, $v_{bc} \leqslant v_\beta$. Thus, Eqs. (46a) and (46b) must be equivalent statements of the same restriction on the values of v_1, v_2, ΔE, i.e., if either of (46a), (46b) holds for given v_1, v_2, ΔE then both of them must hold. Indeed, the equivalence of (46a) and (46b) can be demonstrated[12] directly, without reference to their common genesis in (39a). Conditions (46a) and (46b) arose from the use of (26) to eliminate first the $\delta(\dot{v}_1 - v_1)$ factor in (25); eliminating first the $\delta(\dot{v}_2 - v_2)$ factor in (25) leads[12] to the condition (46c). Similarly—via the procedures which have just been described—one obtains alternative equivalent conditions for the remaining cases (ii)–(v), as well as for case (i) when $\Delta E \leqslant 0$.

Equations (35) and (41)–(45) complete the specification of $\sigma_{\Delta E}{}^{\mathrm{eff}}(v_1,v_2)$ for Coulomb collisions. However, comparing Eqs. (19) and (33), it is clear that the calculation of $\sigma_{\Delta E}{}^{\mathrm{eff}}(v_1,v_2)$ for any (central) interaction would not be essentially difference from the Coulomb case. Whenever, as in (10), the angular variation of $\bar{\sigma}$ depends solely on the angle χ between $\mathbf{n}$ and $\mathbf{n}'$, $\sigma_{\Delta E}(\mathbf{v}_1,\mathbf{v}_2)$ defined by (9) will depend only on $\cos\bar{\theta}$ and $\cos\bar{\theta}'$. But $\cos\bar{\theta}'$ then can be eliminated in favor of ΔE via (5), so that $\sigma_{\Delta E}{}^{\mathrm{eff}}(v_1,v_2)$ defined by (18) will be an average over all $\mathbf{n}_1$, $\mathbf{n}_2$ of a $\sigma_{\Delta E}(\mathbf{v}_1,\mathbf{v}_2)$ depending only on V, v and $\cos\bar{\theta}$, where Eqs. (17) and (20) continue to hold. Thus, one will be led to a single integral involving $\sigma_{\Delta E}$ of form (33), between upper and lower limits v_l, v_u given by precisely the formulas developed in this section. Similar remarks pertain to an average over all $\mathbf{n}_1$, $\mathbf{n}_2$ of any function of v, V, $\cos\bar{\theta}$, where these quantities obey Eqs. (17). Of course, only in special cases, such as the Coulomb case, will the aforementioned integral from v_l to v_u be doable in closed form.

I mention that the result (35) does reduce to Stabler's[5,10] when $m_1 = m_2$.

ACKNOWLEDGMENT

Finally, I wish to thank Hsiang Tai for checking the algebra and helping proofread the manuscript.