PHYSICAL REVIEW VOLUME 148, NUMBER 1 5 AUGUST 1966

Transverse Ultrasonic Attenuation in Gapless Superconductors*

KAZUMI MAKI†
Department of Physics, University of California, San Diego, La Jolla, California
(Received 28 February 1966)

We present here the calculation of the ultrasonic attenuation coefficient of the transverse waves in the gapless superconductors, such as type-II superconductors in a high magnetic field. Particular attention is paid to the electromagnetic absorption term, which is important in the present case, and the result obtained is a simple generalization of that due to Kadanoff and Falko. We discuss ultrasonic attenuation in type-II superconductors, as well as in a superconducting surface sheath in a field above H_{c2}.

1. INTRODUCTION

ULTRASONIC-attenuation experiments have been successfully conducted in the study of the electronic properties of metals in the superconducting state. It is quite interesting to study the attenuation in gapless region where a rather rigorous calculation is feasible, though there are few experiments[1] on superconductors in the gapless region. (We are concerned here with the gapless superconductors as discussed by Kadanoff and Falko[2] as well as one in type-II superconductors in a high-field region.)

The absorption of the ultrasound in type-II superconductors has been previously studied by Caroli and Matricon[3] and by Cooper *et al.*[4] The former authors confine themselves to the field region $H_{c1}<H\ll H_{c2}$ and treat the absorption due to each vortex line semiquantitatively, while the latter authors consider the attenuation of longitudinal wave near the transition temperature. In both treatments the assumption of infinite electronic mean free path ($l\rightarrow\infty$) is implicit.

In the present paper we restrict ourselves to the dirty limit where the electronic mean free path is short. The method we shall employ here is a simple application of that used successfully for the calculation of the electromagnetic conductivity[5] and the thermal conductivity[6] of the type-II superconductors in the high-field region.

As is well known, the attenuation of the transverse wave is decomposed into two mechanisms[7]: the electromagnetic absorption which is screened out rapidly in the superconducting region because of the sudden appearance of the Meissner effect, and the collision-drag effect associated with the stress tensor which gives rise to a slowly varying residual absorption.

In the work of Kadanoff and Falko[2] the electromagnetic absorption has been discarded. However, in the gapless region where the Meissner screening current is still small, the electromagnetic absorption is by no means less important than the collision drag effect, it is necessary to construct a theory, which allows us to include the effect of the Meissner current in a simple way. In the next section, the theory is formulated in terms of a retarded Green's function where the effect of Meissner screening is introduced into the formalism by an analogy with the random-phase approximation applied in the problem of Coulomb screening.[2] The retarded Green's functions involved are evaluated by using the techniques of a thermal Green's function in Sec. 3. We obtain the following expression for the attenuation coefficient of the transverse wave in the small-frequency limit:

$$\alpha_s{}^T/\alpha_n{}^T=\left[1-\frac{|\Delta(\mathbf{r})|^2}{2(2\pi T)^2}\left(\rho^{-1}\psi'(\tfrac{1}{2}+\rho)-\psi''(\tfrac{1}{2}+\rho)\right)\right]g(ql)+(1-g(ql))\left[1+\frac{1}{\omega^2}\left(\frac{|\Delta(\mathbf{r})|^2}{\pi T}\psi'(\tfrac{1}{2}+\rho)\right)^2\right]^{-1},\quad(1)$$

where $\psi'(z)$ and $\psi''(z)$ are tri-gamma and tetra-gamma functions, $\rho=\alpha/2\pi T$ with α the depairing energy ($\alpha=\tau_{\mathrm{tr}}v^2/3eH_{c2}$ and $=0.59/3\tau_{\mathrm{tr}}v^2eH_{c3}$ for the Abrikosov solution and surface sheath, respectively), and q, ω, and l are the wave vector of the sound, the frequency of the sound, and the electronic mean free path, respectively.

Here $g(z)$ is given by

$$g(z)=\tfrac{3}{2}z^{-3}\{-z+(z^2+1)\arctan z\}.\quad(2)$$

Equation (1) reduces to 1 in the limit $\Delta(r)\rightarrow0$, as it should. The first term in Eq. (1) may be called the collision-drag term which is equivalent to the one obtained by Kadanoff and Falko for superconductors containing magnetic impurities.

In Sec. 4 we discuss the application of the above results for type-II superconductors in the gapless region as well as in the superconducting surface sheath.

* Research sponsored by the Air Force Office of Scientific Research, U. S. Air Force, under AFOSR grant No. AF-AFOSR-610-64.

† On leave of absence from the Research Institute for Mathematical Sciences, Kyoto University, Kyoto, Japan.

[1] Y. Shapira and L. J. Neuringer, Solid State Commun. 2, 349 (1964); Phys. Rev. 140, 1638 (1965).

[2] L. P. Kadanoff and I. I. Falko, Phys. Rev. 136, A1170 (1964); see also V. Ambegaokar and A. Griffin, in *Proceedings of the Ninth Low Temperature Physics Conference, Columbus, Ohio* (Plenum Press, Inc., New York, 1965).

[3] C. Caroli and J. Matricon, Physik Kondensierten Materie 3, 380 (1965).

[4] L. N. Cooper, A. Houghton, and H. J. Lee, Phys. Rev. Letters 15, 584 (1965).

[5] K. Maki, Phys. Rev. 141, 331 (1966).

[6] C. Caroli and M. Cyrot, Physik Kondensierten Materie 4, 285 (1965).

[7] L. T. Claiborne, Jr., and R. W. Morse, Phys. Rev. 136, A893 (1964).

2. ATTENUATION COEFFICIENT OF TRANSVERSE WAVE

Following Tsuneto[8] we shall describe the interaction between the imposed sound wave and the conduction electrons in metal by the Hamiltonian (in the moving system with sound wave)

$$\mathcal{H}_I=\int d^3r\,\boldsymbol{\phi}(\mathbf{r},t)\{i\omega m\mathbf{j}(\mathbf{r},t)-i\boldsymbol{\tau}(r,t)\cdot\mathbf{q}\}\,, \tag{3}$$

where the current operator, stress-tensor operator, and displacement field are given as

$$j_i(\mathbf{r},t)=\sum_{\text{spin}}\{((\boldsymbol{\nabla}-\boldsymbol{\nabla}')_i/2im)\psi^\dagger(\mathbf{r}',t)\psi(\mathbf{r},t)\}\,, \tag{4}$$

$$\tau_{ij}(\mathbf{r},t)=\sum_{\text{spin}}\left\{\frac{(\boldsymbol{\nabla}-\boldsymbol{\nabla}')_i}{2i}\frac{(\boldsymbol{\nabla}-\boldsymbol{\nabla}')_j}{2im}\psi^\dagger(\mathbf{r}',t)\psi(\mathbf{r},t)\right\}\,, \tag{5}$$

and

$$\boldsymbol{\phi}(\mathbf{r},t)=\boldsymbol{\phi}(\mathbf{q},\omega)e^{i\mathbf{q}\cdot\mathbf{r}-i\omega t}\,, \tag{6}$$

respectively.

The attenuation coefficient for the transverse wave is expressed in terms of the retarded Green's function,[9,2]

$$\alpha^T=\mathrm{Re}\{(\omega^2/i\omega\rho_{\text{ion}}v_s)\langle[h_I{}^T,h_I{}^T]\rangle(\mathbf{q},\omega)\}\,, \tag{7}$$

where

$$h_I{}^T(\mathbf{r},t)=(q/\omega)\tau_{xz}(\mathbf{r},t)-mj_x(\mathbf{r},t)\,. \tag{8}$$

Here we take the direction of q as the z axis. The retarded product is defined as usual by

$$\begin{aligned}\langle[A,B]\rangle(q,\omega)=\frac{1}{i}\int_{-\infty}^{t}dt'\int d^3r'&\\ \times\exp[i\omega(t-t')-i\mathbf{q}\cdot(\mathbf{r}-\mathbf{r}')]&\\ \times\langle[A(\mathbf{r},t),B(\mathbf{r}',t')]\rangle\,.&\end{aligned} \tag{9}$$

We note here that the above product should be taken on the system where the Hamiltonian contains the current-current interaction term (see Appendix)

$$\mathcal{H}_j=e^2\int\int d^3r d^3r'\frac{\mathbf{j}(\mathbf{r},t)\mathbf{j}(\mathbf{r}',t)}{|\mathbf{r}-\mathbf{r}'|}\,. \tag{10}$$

Although, in the usual situation, this term gives rise only to a relativistic correction to the Coulomb interaction, in the present case, because of the small momentum, this term q is no longer negligible. We can include this effect using a similar procedure to the one used by Kadanoff and Falko in the problem of Coulomb screening. We arrive at the results

$$\begin{aligned}\langle[A,B]\rangle(\mathbf{q},\omega)&=\langle[A,B]\rangle'(\mathbf{q},\omega)\\ &\times\frac{\langle[A,j_x]\rangle'(\mathbf{q},\omega)4\pi e^2q^{-2}\langle[j_x,B]\rangle'(\mathbf{q},\omega)}{1-4\pi e^2q^{-2}\langle[j_x,j_x]\rangle'(\mathbf{q},\omega)}\,,\end{aligned} \tag{11}$$

where $\langle\cdots\rangle'$ indicates the average over the fictitious system without current-current interaction. In the derivation of the above equation we have assumed

$$\left.\begin{aligned}\langle[A,j_i]\rangle'(\mathbf{q},\omega)=0\,,\\ \langle[B,j_i]\rangle'(\mathbf{q},\omega)=0\,,\end{aligned}\right.\quad\text{for}\quad i\neq x\,. \tag{12}$$

Thus we can rewrite Eq. (7) in terms of expectation values in the fictitious system as

$$\begin{aligned}\alpha^T=\mathrm{Re}\Bigg\{\frac{\omega^2}{i\omega\rho_{\text{ion}}v_s}\Bigg[&\langle[h_I{}^T,h_I{}^T]\rangle'(\mathbf{q},\omega)\\ &+\frac{\langle[h_I{}^T,j_x]\rangle'(\mathbf{q},\omega)4\pi e^2q^{-2}\langle[j_x,h_I{}^T]\rangle'(\mathbf{q},\omega)}{1-4\pi e^2q^{-2}\langle[j_x,j_x]\rangle'(\mathbf{q},\omega)}\Bigg]\Bigg\}\,.\end{aligned} \tag{13}$$

In the limit of small frequency ($\omega\ll\Delta_{00}$) the above equation reduces to

$$\begin{aligned}\alpha^T=\mathrm{Re}\Bigg\{\frac{q^2}{i\omega\rho_{\text{ion}}v_s}\Bigg(&\langle[\tau_{xz},\tau_{xz}]\rangle'(\mathbf{q},\omega)\\ &-\frac{(\langle[\tau_{xz},j_x]\rangle'(\mathbf{q},\omega))^2}{\langle[j_x,j_x]\rangle'(\mathbf{q},\omega)}\Bigg)\Bigg\}\,,\end{aligned} \tag{14}$$

where we made use of Eq. (8) and

$$\langle[\tau_{xz},j_x]\rangle(\mathbf{q},\omega)=\langle[j_x,\tau_{xz}]\rangle(\mathbf{q}\cdot\omega)\,. \tag{15}$$

3. CALCULATION OF RETARDED PRODUCTS

Following the usual convention we shall calculate the above retarded products by analytical continuation of thermal products. Furthermore, since we are interested here in the gapless region we expand thermal products in powers of $\Delta(\mathbf{r})$. Since the calculation is a simple repetition of the one used in the case of electromagnetic con-

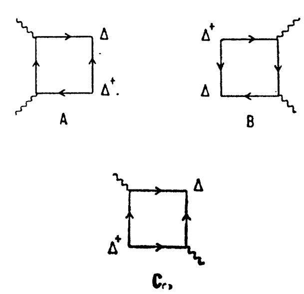

FIG. 1. The diagrams which give rise to the contribution of the terms of order $|\Delta|^2$.

[8] T. Tsuneto, Phys. Rev. **121**, 402 (1961).
[9] T. Tsuneto, Rutgers, the State University, 1964 (unpublished).

ductivity,[5] we leave out the details and we have

$$\langle[\tau_{xz},\tau_{xz}]\rangle'(\mathbf{q},\omega_v)$$

$$=2\pi T\sum_m\frac{1}{m^2}\int\frac{d^3p}{(2\pi)^3}p_z{}^2p_x{}^2\left\{\frac{1}{i\tilde{\omega}-\xi}\frac{1}{i\tilde{\omega}'-\xi-\mathbf{v}\cdot\mathbf{q}}-\frac{1}{i\tilde{\omega}-\xi}\eta_{\mathbf{q},\omega}\Delta_{\mathbf{q}1}\frac{1}{i\tilde{\omega}+\xi+\mathbf{v}\cdot\mathbf{q}_1}\eta_{\mathbf{q}'\omega}\Delta_{\mathbf{q}2}{}^\dagger\frac{1}{i\tilde{\omega}-\xi-\mathbf{v}\cdot(\mathbf{q}_1\cdot\mathbf{q}_2)}\right.$$

$$\times\frac{1}{i\tilde{\omega}'-\xi-\mathbf{v}\cdot(\mathbf{q}+\mathbf{q}_1+\mathbf{q}_2)}-\frac{1}{i\omega-\xi}\frac{1}{i\tilde{\omega}'-\xi-\mathbf{v}\cdot\mathbf{q}}\eta_{\mathbf{q}'\omega'}\Delta_{\mathbf{q}1}{}^\dagger\frac{1}{i\tilde{\omega}'+\xi+\mathbf{v}\cdot(\mathbf{q}+\mathbf{q}_2)}\eta_{\mathbf{q},\omega'}\Delta_{\mathbf{q}2}\frac{1}{i\tilde{\omega}'-\xi-\mathbf{v}\cdot(\mathbf{q}+\mathbf{q}_1+\mathbf{q}_2)}$$

$$\left.-\frac{1}{i\tilde{\omega}-\xi}\eta_{\mathbf{q}'\omega}\Delta_{\mathbf{q}2}\frac{1}{i\tilde{\omega}+\xi+\mathbf{v}\cdot\mathbf{q}_1}\frac{1}{i\tilde{\omega}'+\xi+\mathbf{v}\cdot(\mathbf{q}+\mathbf{q}_1)}\eta_{\mathbf{q}'\omega'}\Delta_{\mathbf{q}1}{}^\dagger\frac{1}{i\tilde{\omega}'-\xi-\mathbf{v}\cdot(\mathbf{q}+\mathbf{q}_1+\mathbf{q}_2)}\right\}$$

$$=\frac{p_0{}^4}{3\pi q}(lq)^{-1}(1-g(ql))\pi T\sum_m\left\{1-\frac{\omega\omega'}{|\omega||\omega'|}-\frac{|\Delta(\mathbf{r})|^2}{2}\left[\frac{\omega\omega'}{|\omega||\omega'|}\left(\frac{1}{(|\omega|+\alpha)^2}+\frac{1}{(|\omega'|+\alpha)^2}\right)-\frac{2}{(|\omega|+\alpha)(|\omega'|+\alpha)}\right]\right\},$$

$$=\frac{p_0{}^4}{3\pi q}(lq)^{-1}(1-g(ql))\omega_\nu\left\{1-\frac{|\Delta(r)|^2}{\omega_\nu}\left[\frac{1}{2\pi T}\psi'\left(\frac{1}{2}+\frac{\omega_\nu}{2\pi T}+\rho\right)-\left(\frac{1}{\omega_\nu}+\frac{1}{\omega_v+2\alpha}\right)\left(\psi\left(\frac{1}{2}+\frac{\omega_\nu}{2\pi T}+\rho\right)-\psi(\tfrac{1}{2}+\rho)\right)\right]\right\}, \quad (16)$$

$$\langle[\tau_{xz},j_x]\rangle(\mathbf{q},\omega_v)$$

$$=2\pi T\sum_n\frac{1}{m^2}\int\frac{d^3p}{(2\pi)^3}p_zp_x{}^2\left\{\frac{1}{i\tilde{\omega}-\xi}\frac{1}{i\tilde{\omega}'-\xi-\mathbf{v}\cdot\mathbf{q}}-\frac{1}{i\tilde{\omega}-\xi}\eta_{\mathbf{q}'\omega}\Delta_{\mathbf{q}1}\frac{1}{i\tilde{\omega}+\xi+\mathbf{v}\cdot\mathbf{q}_1}\eta_{\mathbf{q}'\omega}\Delta_{\mathbf{q}2}{}^\dagger\frac{1}{i\tilde{\omega}-\xi-\mathbf{v}\cdot(\mathbf{q}_1+\mathbf{q}_2)}\right.$$

$$\left.\times\frac{1}{i\tilde{\omega}'-\xi-\mathbf{v}\cdot(\mathbf{q}+\mathbf{q}_1+\mathbf{q}_2)}\quad\frac{1}{i\tilde{\omega}-\xi}\frac{1}{i\tilde{\omega}'-\xi-\mathbf{v}\cdot\mathbf{q}}\eta_{q1\omega'}\Delta_{q1}\frac{1}{i\tilde{\omega}'+\xi+\mathbf{v}\cdot(\mathbf{q}+\mathbf{q}_1)}\eta_{q2\omega'}\Delta_{q2}{}^\dagger\frac{1}{i\tilde{\omega}'-\xi-\mathbf{v}\cdot(\mathbf{q}+\mathbf{q}_1+\mathbf{q}_2)}\right\}$$

$$=\frac{ip_0{}^3}{3\pi q}(1-g(ql))\pi T\sum_n\left\{1-\frac{\omega\omega'}{|\omega||\omega'|}-\frac{|\Delta(r)|^2}{2}\left[\frac{\omega\omega'}{|\omega||\omega'|}\left(\frac{1}{(|\omega|+\alpha)^2}-\frac{1}{(|\omega'|+\alpha)^2}\right)\right]\right\},$$

$$=\frac{ip_0{}^3}{3\pi q}(1-g(ql))\omega\nu, \quad (17)$$

and

$$\langle[\mathbf{j}_x,\mathbf{j}_x]\rangle(\mathbf{q},\omega\nu)$$

$$=2\pi T\sum_n\frac{1}{m^2}\int\frac{d^3p}{(2\pi)^3}p_x{}^2\left\{\frac{1}{i\omega-\xi}\frac{1}{i\tilde{\omega}+\xi-\mathbf{v}\cdot\mathbf{q}}-\frac{1}{i\omega-\xi}\eta_{q1\omega}\Delta\mathbf{q}_1\frac{1}{i\tilde{\omega}+\xi+\mathbf{v}\cdot\mathbf{q}_1}\eta_{q2\omega}\Delta q_2{}^\dagger\frac{1}{i\tilde{\omega}-\xi-\mathbf{v}\cdot(\mathbf{q}_1+\mathbf{q}_2)}\right.$$

$$\times\frac{1}{i\tilde{\omega}'-\xi-\mathbf{v}\cdot(\mathbf{q}+\mathbf{q}_1+\mathbf{q}_2)}-\frac{1}{i\tilde{\omega}-\xi}\frac{1}{i\tilde{\omega}'-\xi-\mathbf{v}\cdot\mathbf{q}}\eta_{q2\omega'}\Delta q_2{}^\dagger\frac{1}{i\tilde{\omega}'+\xi+\mathbf{v}\cdot(q+q_2)}\eta_{q,\omega'}\Delta q_1\frac{1}{i\tilde{\omega}'-\xi-\mathbf{v}\cdot(\mathbf{q}+\mathbf{q}_1+\mathbf{q}_2)}$$

$$\left.+\frac{1}{i\tilde{\omega}-\xi}\eta_{q1\omega}\Delta q_1\frac{1}{i\tilde{\omega}+\xi+\mathbf{v}\cdot\mathbf{q}_1}\cdot\frac{1}{i\tilde{\omega}'+\xi+\mathbf{v}\cdot(\mathbf{q}+\mathbf{q}_1)}\eta_{q2\omega'}\Delta q_2{}^\dagger\frac{1}{i\tilde{\omega}'-\xi-\mathbf{v}\cdot(\mathbf{q}+\mathbf{q}_1+\mathbf{q}_2)}\right\}$$

$$=\frac{p_0{}^2}{3\pi q}(ql)g(ql)\pi T\sum\left\{1-\frac{\omega\omega'}{|\omega||\omega'|}-\frac{|\Delta(\mathbf{r})|^2}{2}\left[\frac{\omega\omega'}{|\omega||\omega'|}\left(\frac{1}{(|\omega|+\alpha)^2}+\frac{1}{(|\omega'|+\alpha)^2}\right)+\frac{2}{(|\omega|+\alpha)(|\omega'|+\alpha)}\right]\right\}$$

$$=\frac{p_0{}^2}{3\pi q}(ql)g(ql)\omega_\nu\left\{1-\frac{|\Delta(\mathbf{r})|^2}{\omega_\nu}\left[\frac{1}{2\pi T}\psi'\left(\frac{1}{2}+\frac{\omega_\nu}{2\pi T}+\rho\right)+\left(\frac{1}{\omega_\nu}+\frac{1}{\omega_\nu+2\alpha}\right)\left(\psi\left(\frac{1}{2}+\frac{\omega_\nu}{2\pi T}+\rho\right)-\psi(\tfrac{1}{2}+\rho)\right)\right]\right\}, \quad (18)$$

where $g(lq)$ is defined in Eq. (2) and $\mathbf{v}=\mathbf{p}/m$, $\omega'=\omega-\omega_\nu$ and $\xi=p^2/2m-\mu$.

Here we introduce the effect of impurity scattering by renormalizations[10] $\omega \to \tilde{\omega} = \omega\eta_\omega$, $\Delta_q \to \tilde{\Delta}_q = \eta_{q\omega}\Delta_q$ [Δ_q is the Fourier transform of $\Delta(\mathbf{r})$] where

$$\eta_\omega = (1 + 1/2\tau|\omega|), \tag{19}$$

and

$$\eta_{q\omega} = \{1 - (1/2\tau|\tilde{\omega}|)(1 - \tfrac{1}{3}\tau\tau_{\mathrm{tr}}v^2q^2)\}^{-1}, \tag{20}$$

and τ, τ_{tr} are the collision lifetime and the transport lifetime of electrons. Furthermore, we set $\tau_{\mathrm{tr}}v^2q^2/6 = \alpha$ by making use of the relation[10]

$$(\tau_{\mathrm{tr}}v^2q^2/6)\Delta(\mathbf{r}) = \alpha\Delta(\mathbf{r}). \tag{21}$$

The parameter $\rho = \alpha/2\pi T$, and $\psi(z)$, $\psi'(z)$ are the di-gamma and tri-gamma functions. We also note here that all three of the diagrams given in Fig. 1 contribute to $\langle[\tau_{x2},\tau_{x2}]\rangle$ and $\langle[j_x,j_x]\rangle$, while only A and B contribute to $\langle[\tau_{x2},j_x]\rangle$.

After putting $\omega_\nu \to i\omega$ we substitute the above expressions for the products in Eq. (14) and we obtain the general expressions for the attenuation coefficient at arbitrary frequency.

In the following we shall confine ourselves to the case of low-frequency limit ($\omega < \pi T_{c0}$), which is the usual situation met in experiments. In this limit, Eqs. (16), (17), and (18) reduce to (we have already performed analytical continuation)

$$\langle[\tau_{xz},\tau_{xz}]\rangle(q,\omega) = (p_0^4/3\pi q)(ql)^{-1}(1-g(ql))i\omega\{1 - (|\Delta(r)|^2/2(2\pi T)^2)(\rho^{-1}\psi'(\tfrac{1}{2}+\rho) - \psi''(\tfrac{1}{2}+\rho))\}, \tag{22}$$

$$\langle[\tau_{xz},j_x]\rangle(q,\omega) = -(p_0^3/3\pi q)(1-g(ql))\omega, \tag{23}$$

$$\langle[j_x,j_x]\rangle(q,\omega) = (p_0^2/3\pi q)(ql)g(ql)i\omega\left\{1 - \frac{|\Delta(\mathbf{r})|^2}{2(2\pi T)^2}(\rho^{-1}\psi'(\tfrac{1}{2}+\rho) + 3\psi''(\tfrac{1}{2}+\rho)) - \frac{1}{i\omega}\frac{2|\Delta(\mathbf{r})|^2}{2\pi T}\psi'(\tfrac{1}{2}+\rho)\right\}. \tag{24}$$

Finally we get the attenuation coefficient for the transverse wave

$$\alpha^T = \frac{p_0^4}{\rho_{\mathrm{ion}}v_s l}(1-g(ql))\left\{1 - \frac{|\Delta(\mathbf{r})|^2}{2(2\pi T)^2}(\rho^{-1}\psi'(\tfrac{1}{2}+\rho) - \psi''(\tfrac{1}{2}+\rho)) + (g^{-1}(ql)-1)\left[1 + \frac{1}{\omega^2}\left(\frac{|\Delta(\mathbf{r})|^2}{\pi T}\psi'(\tfrac{1}{2}+\rho)\right)^2\right]^{-1}\right\}. \tag{25}$$

The first term in the bracket may be interpreted as the collision drag term previously calculated by Kadanoff and Falko[2] while the second term drops off rapidly in the superconducting region because of the sudden appearance of the Meissner effect. It is easily seen that the above expression gives a correct result for the normal state in the limit $\Delta(\mathbf{r}) \to 0$.

We note here that in an inhomogeneous superconductor where $\Delta(\mathbf{r})$ is not constant there is a term which scatters the sound wave of momentum $\mathbf{q}$ to $\mathbf{q}' = \mathbf{q} + \mathbf{k}$, where $\mathbf{k}$ is the characteristic momentum of Abrikosov's structure. However, it is shown that such a term is only a small correction of the order of τT_{c0}. Thus we conclude that in the field region close to the upper critical field in the dirty limit, the measurement of the transverse sound attenuation does not give the detailed configuration of Abrikosov's structure but the amplitude of $\Delta(\mathbf{r})$.

4. ULTRASOUND ATTENUATION IN SUPERCONDUCTORS IN HIGH FIELDS

In this section we shall discuss the attenuation of the sound wave in type-II superconductors in a high-field region ($H_0 \le H_{c2}$, where H_0 is the external field) and in the superconducting surface sheath where in both cases the order parameter is small.

(*a*) *Abrikosov's state in a gapless region.* We shall consider the situation where the attenuation coefficient is measured through a bulk type-II superconductor in a gapless region. In this case the experimentally measurable attenuation coefficient is

$$\alpha_s^T/\alpha_n^T = \left[1 - \frac{\langle|\Delta(\mathbf{r})|^2\rangle_{\mathrm{av}}}{2(2\pi T)^2}(\rho^{-1}\psi'(\tfrac{1}{2}+\rho) - \psi''(\tfrac{1}{2}+\rho))\right]g(ql) + \left\langle\frac{1}{1 + (1/\omega^2)((|\Delta(\mathbf{r})|^2/\pi T)\psi'(\tfrac{1}{2}+\rho))^2}\right\rangle(1-g(ql)), \tag{26}$$

where $\langle A\rangle_{\mathrm{av}}$ means a space average of A.

In the present situation the above average is safely replaced by

$$\alpha_s^T/\alpha_n^T = \left[1 - \frac{\langle|\Delta(\mathbf{r})|^2\rangle_{\mathrm{av}}}{2(2\pi T)^2}(\rho^{-1}\psi'(\tfrac{1}{2}+\rho) - \psi''(\tfrac{1}{2}+\rho))\right]g(ql) + (1-g(ql))\left[1 + \frac{1}{\omega^2}\left(\frac{\langle|\Delta(\mathbf{r})|^2\rangle_{\mathrm{av}}}{\pi T}\psi'(\tfrac{1}{2}+\rho)\right)^2\right]^{-1}, \tag{27}$$

[10] K. Maki, Physics 1, 21 (1964). [Note that the temperature dependence of $\kappa_2(t)$ is in error. The corrected $\kappa_2(t)$ behaves almost identically to $\kappa_1(t)$; see C. Caroli, M. Cyrot, and P. G. de Gennes, Solid State Commun. 4, 17 (1966).]

where $\rho=\tau_{tr}v^2eH_{c2}/6\pi T$, and $\langle|\Delta(\mathbf{r})|^2\rangle_{av}$ is given[10] by

$$\langle|\Delta(\mathbf{r})|^2\rangle_{av}=\frac{eT}{\sigma}\frac{(H_{c2}-H_0)}{(2\kappa_2^2(t)-1)\beta}(\psi'(\tfrac{1}{2}+\rho))^{-1}, \quad (28)$$

$\beta=1.16$ and $\sigma=\tau_{tr}e^2N/m$ the conductivity of the metal in the normal state.

Making use of Eq. (28), we can further simplify the second term in Eq. (27) as

$$\frac{\alpha_{s\ \mathrm{elec}}{}^T}{\alpha_n{}^T}=(1-g(ql))\left[1+\frac{1}{\omega^2\beta^2}\left(\frac{e(H_{c2}-H_0)}{\pi\sigma(2\kappa_2^2(t)-1)}\right)^2\right]^{-1}$$
$$=(1-g(ql))\left[1+\frac{1}{\omega^2}\left(\frac{eM}{4\sigma}\right)^2\right]^{-1}, \quad (29)$$

where M is the magnetization.

(*b*) *Superconducting surface sheath.* Another interesting application of Eq. (25) is to the superconducting surface sheath region of Saint-James and de Gennes. In order to simplify the situation, we consider the film of thickness d [$>(l\xi_0)^{1/2}$] is in a parallel field larger than H_{c2}. Both edges of this film remain still in the superconducting state. In this case if we measure the attenuation coefficient through the film, then Eq. (26) is applicable. We set here $\rho=0.59\tau_{tr}v^2eH_{c3}/6\pi T$, where H_{c3} is the critical field of Saint-James and de Gennes and the normalization of $\langle|\Delta(r)|^2\rangle_{av}$ can be obtained from the general Ginzburg-Landau equation[10] as

$$\langle|\Delta(\mathbf{r})|^2\rangle_{av}=\frac{2}{d}\sqrt{\frac{\pi}{1.18eH_{c3}}}\frac{eT}{\sigma}$$
$$\times\frac{(H_{c3}-H_0)}{(2\kappa_2^2(t)-3.12)}(\psi'(\tfrac{1}{2}+\rho))^{-1}, \quad (30)$$

where we made use of the result due to Abrikosov[11] on the normalization of a surface sheath.

In the present situation we cannot approximate the second term in Eq. (26) by a more simple average and it requires a detailed knowledge of the function $\Delta(\mathbf{r})$ all over the space to evaluate this term.

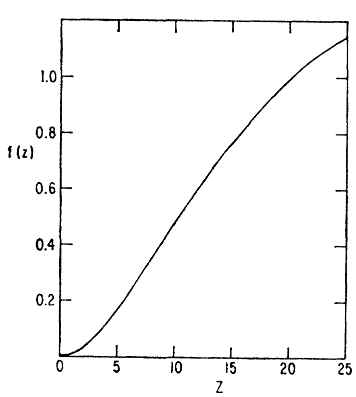

FIG. 2. Plot of $f(z)$.

If H_0 the external field is close to H_{c3}, we can approximate $\Delta(\mathbf{r})$ by a Gaussian function,[11,12] and we can estimate the field dependence of the second term:

$$\left\langle\left[1+\frac{1}{\omega^2}\left(\frac{|\Delta(\mathbf{r})|^2}{\pi T}\psi'(\tfrac{1}{2}+\rho)\right)^2\right]^{-1}\right\rangle_{av}$$
$$=\frac{2}{0.59eH_{c3}d}\int_0^{0.59eH_{c3}d/2}dx(1+z^2e^{-x^2})^{-1}$$
$$\cong 1-\frac{2}{0.59eH_{c3}d}f(z), \quad (31)$$

where

$$f(z)=\int_0^\alpha dx(z^{-2}e^{-x^2}+1)^{-1}, \quad (32)$$

and

$$z=\left[\frac{\sqrt{2}eT}{\omega\sigma}\frac{(H_{c3}-H_0)}{(2\kappa_2^2(t)-3.12)}(\psi'(\tfrac{1}{2}+\rho))^{-1}\right]; \quad (33)$$

$f(z)$ is numerically calculated and drawn in Fig. 2.

5. CONCLUDING REMARKS

In the section above we have calculated the attenuation coefficient of the transverse wave in gapless superconductors. It is shown that the measurement of the attenuation of the transverse wave serves not only to determine the critical field (H_{c2} and/or H_{c3}), but also to find the amplitude of $\Delta(\mathbf{r})$ in the gapless region.

ACKNOWLEDGMENTS

The author would like to thank Dr. Allan Griffin for calling his attention to the importance of electromagnetic terms discussed in the text and for useful discussions.

APPENDIX

We shall discuss here how to incorporate the effect of current-current interaction which is important in the present problem. There is a close similarity between the present problem and the problem of Coulomb screening which is discussed by Kadanoff and Falko.[2]

The current-current interaction is described by a Hamiltonian

$$\mathcal{H}_j=e^2\int\int d^3rd^3r'\frac{\mathbf{j}(\mathbf{r},t)\mathbf{j}(\mathbf{r}',t)}{|\mathbf{r}-\mathbf{r}'|},$$

where $\mathbf{j}(r)$ is the current operator.

[11] A. A. Abrikosov, Zh. Eksperim. i Teor. Fiz. **47**, 720 (1964) [English transl.: Soviet Phys.—JETP **20**, 480 (1965)]; see also M. J. Zuckerman, Phys. Letters **13**, 277 (1964).

[12] M. Tinkham, report at the Conference on Type-II Superconductivity, Cleveland, Ohio, 1964 (unpublished).

In the spirit of the random-phase approximation we can set up the equations for the retarded correlation function[2]

$$\langle[A,B]\rangle(\mathbf{q},\omega)=\langle[A,B]\rangle'(\mathbf{q},\omega)+\langle[A,j_x]\rangle'(\mathbf{q},\omega)\times 4\pi e^2q^{-2}\langle[j_x,B]\rangle(\mathbf{q},\omega)\,, \quad \text{(A1)}$$

$$\langle[j_x,B]\rangle(q,\omega)=\langle[j_x,B]\rangle'(\mathbf{q},\omega)+\langle[j_x,j_x]\rangle'(\mathbf{q}\omega)\times 4\pi e^2q^{-2}\langle[j_x,B]\rangle(\mathbf{q},\omega)\,, \quad \text{(A2)}$$

and

$$\langle[j_x,j_x]\rangle(\mathbf{q},\omega)=\langle[j_x,j_x]\rangle'(\mathbf{q},\omega)\times\{1+4\pi e^2q^{-2}\langle[j_x,j_x]\rangle(\mathbf{q},\omega)\}\,, \quad \text{(A3)}$$

where $\langle\cdots\rangle$ and $\langle\cdots\rangle'$ are the retarded products over the real system (containing current-current interaction) and the fictitious system, respectively. We assumed here that $\langle[A,j_i]\rangle=0$ $\langle[B,j_i]\rangle=0$ for $i\neq x$ for simplicity. Solving the above equations we have

$$\langle[j_x,j_x]\rangle(\mathbf{q},\omega)=\frac{\langle[j_x,j_x]\rangle'(\mathbf{q},\omega)}{1-4\pi e^2q^{-2}\langle[j_x,j_x]\rangle'(\mathbf{q},\omega)}\,, \quad \text{(A4)}$$

$$\langle[j_x,B]\rangle(\mathbf{q},\omega)=\frac{\langle[j_x,B]\rangle'(\mathbf{q},\omega)}{1-4\pi e^2q^{-2}\langle[j_x,j_x]\rangle'(\mathbf{q},\omega)}\,, \quad \text{(A5)}$$

and

$$\langle[A,B]\rangle(\mathbf{q},\omega)=\langle[A,B]\rangle'(q,\omega)+\frac{\langle[A,j_x]\rangle'(\mathbf{q},\omega)4\pi e^2q^{-2}\langle[j_x,B]\rangle'(\mathbf{q},\omega)}{1-4\pi e^2q^{-2}\langle[j_x,j_x]\rangle'(\mathbf{q},\omega)}\,. \quad \text{(A6)}$$

PHYSICAL REVIEW VOLUME 148, NUMBER 1 5 AUGUST 1966

Diffusion Constants near the Critical Point for Time-Dependent Ising Models. II*

KYOZI KAWASAKI†
Department of Chemistry, Massachusetts Institute of Technology, Cambridge, Massachusetts
(Received 7 March 1966)

The nature of the local-equilibrium approximation employed in our previous work is examined with the use of the correlation-function expression for the general frequency-dependent transport coefficient for systems described by master equations. The transport coefficient is found to be a nondecreasing function of frequency which in the high-frequency limit reduces to the transport coefficient calculated in the local-equilibrium approximation. Thus the local-equilibrium approximation provides an upper bound to the zero-frequency transport coefficient. For the case of spin diffusion, combined with the result of our previous work, this limit implies that the diffusion constant must vanish at least as fast as the inverse of the magnetic susceptibility near the Curie point, which establishes the existence of the critical slowing down for our model of spin diffusion.

1. INTRODUCTION

IN our earlier work of the same title,[1] we have studied the diffusion constant near the critical point for time-dependent Ising models which are described by appropriate master equations. The sole approximation in that treatment of the model system is that at each instant of time the reduced probability distribution function for spin configurations is replaced by its value in local thermal equilibrium with given inhomogeneous average spin density at that time. Within the limitation of this approximation, the dynamics of the problem is completely separated from the complicated statistics of the Ising spin problem. Thus it is natural to investigate the validity of this approximation, which is our main concern in this paper.

Since the question raised here is not restricted to the diffusion process, we shall treat the more general problem of transport coefficients for systems described by master equations, restricting ourselves to cases in which only one macroscopic variable is involved in the transport process. The main result of this investigation is that the transport coefficient calculated in the local equilibrium approximation provides a rigorous upper bound to the true transport coefficient. For the case of spin diffusion, combined with the result of I, the existence of critical slowing down is thus rigorously established for the model of I.

Our argument proceeds with the use of the correlation function expression for the transport coefficient, which in turn is expressed in terms of the transition probability appearing in the master equation (Sec. 1). Then we obtain a spectral representation of the general frequency-dependent transport coefficient by making use of eigenfunctions of the symmetrized master equation. The transport coefficient is then found to be a nondecreasing function of frequency; we identify the high-frequency limit of this transport coefficient with

* A portion of this work was supported by the National Science Foundation.
† Present address: Atomic Energy Research Establishment Harwell, Didcot, England. Permanent address (after November 1966): Department of Physics, Kyushu University, Fukuoka, Japan.
[1] K. Kawasaki, Phys. Rev. 145, 224 (1966). Hereafter referred to as I. Equations in I are cited, for instance, as I (2.11).