Attenuation of Longitudinal Acoustic Waves in Type-II Superconductors*†

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The attenuation of longitudinal sound waves in a superconductor with a spatially dependent energy gap is studied by a Green's-function method. In order that explicit expressions for the ultrasonic loss may be derived, the calculation is restricted to temperatures $T\sim T_c$, the transition temperature. The results of the calculation are used to study ultrasonic attenuation in a type-II superconductor near T_c, which contains a low density of flux lines, $H\sim H_{c_1}$. It is assumed that under these conditions the flux tubes in the material are fixed, are well separated, and form a periodic array. It is then shown that not only is the attenuation anisotropic, but anomalous absorption or transmission takes place whenever the sound wavelength matches the distance between vortices. Although the above theory is only for $T\sim T_c$ and $H\sim H_{c_1}$, it is expected that this effect will exist over wider ranges of temperature and magnetic field, and, if accessible to experiment, will yield a direct measure of the spacing of flux lines in type-II superconductors.

I. INTRODUCTION

Type-II superconductors are distinguished from those of type I by the fact that, although flux penetrates the sample at a magnetic field $H=H_{c_1}$, it is not until a much higher field $H_{c_2}\gg H_c$ that the material returns to its normal state. This was first explained by Abrikosov[1] who based his arguments on the Landau-Ginzburg[2] theory for $K>1/\sqrt{2}$. Abrikosov predicted that at H_{c_1} flux penetrates the material in the form of flux filaments or vortices having spatial dimensions of the order of ξ, the coherence distance. It was later suggested[3] that associated with each Abrikosov flux line is one discrete quantum of flux $ch/2e$. A further prediction of the theory was that the mixed state of type-II superconductivity is characterized by the flux lines in the material forming a periodic array, the distance between lines being dependent upon the magnetic field strength.

Although the Abrikosov theory has stimulated a great deal of experimental and theoretical work, the information available on the structure of the mixed state is mostly indirect. In this paper we investigate the possibility of obtaining detailed information on the nature of the mixed state from measurement of ultrasonic attenuation for propagation parallel and perpendicular to the magnetic field.

Ultrasonic attenuation has proved, in recent years, a most valuable tool in the study of both normal and superconducting material. The attenuation characteristics in type-I material determined by Morse[4] *et al.* are well described by a theory, based upon the BCS[5] description of superconductors, developed by Tsuneto.[6] Ultrasonic attenuation in type-II superconductors has been investigated theoretically by Caroli[7] *et al.* They show that at very low temperatures $T\ll T_c$, where T_c is the transition temperature, the ultrasonic loss due to the quasinormal cores is very small. Among the reasons ultrasonic attenuation may yield information on the structure of the mixed state is the fact that the normal cores form a periodic array in the material. If we restrict our attention to a superconductor which is near the transition temperature $T\sim T_c$, and which contains a low density of flux lines $H\ll H_{c_2}$, then the effect of the enclosed flux is to produce a slow periodic variation in the energy gap Δ, with period $\sim\delta$, the penetration depth. We also note that in this region the magnetic field is large in the vortex but drops off to a small value in a distance δ. In type-II material the coherence distance ξ is always less than δ.[1]

Taking as our starting point the Gorkov[8] formulation of superconductivity we determine, in Sec. II, the response of a superconductor with a spatially dependent energy gap to a longitudinal sound wave. It proves convenient to use the thermal Green's-function method to calculate a temperature response function; the required causal function is subsequently determined by analytic continuation. In order to clarify ensuing theory the main points of the argument underlying this approach are summarized in Sec. III. Finally, in that section, the response function is expanded in powers of $\Delta(T)$; terms to order $|\Delta(T)|^2$ are retained. This procedure is justified

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as long as we treat the attenuation near T_c. In particular it can be shown that this approach is valid in a type-I superconductor if $|\Delta(T)|^2/kT(v_F q)\ll 1$, where $\mathbf{q}$ is the wave vector of the sound wave.

In Sec. IV we discuss the attenuation using a theory analogous to that used by Gorkov[9] in his derivation of the Landau-Ginzburg equations. This theory is, strictly speaking, only valid in the extreme limit of the conditions stated above, and also when the wavelength of the sound wave $\lambda_S\ll\delta$, the period of the gap. It has, however, the merit of relative simplicity and illustrates some of the features to be expected in sound attenuation under more general conditions. In Sec. V we generalize the theory in order that a wider range of sound wavelengths may be treated; particular attention is paid to the attenuation when $\lambda_S\sim\delta$.

Finally, in Sec. VI we determine the ultrasonic attenuation in a model type-II superconductor. The model is determined by choosing an energy gap which varies periodically in the plane perpendicular to the magnetic field. The period of oscillation of the gap function is the distance between flux lines and is determined by the external magnetic field. It is shown that structure in the attenuation occurs when the sound wave propagates perpendicular to the magnetic field and when its wavelength $\lambda_S\sim d$, the distance between normal cores. Namely anomalous absorption or transmission takes place whenever $v_F|\mathbf{q}\pm(2\pi/d)\mathbf{u}|=\omega$, where $\mathbf{q}$ and ω are the wave vector and frequency of the sound wave and $\mathbf{u}$ is a unit vector in the direction of the gap periodicity. If this effect is accessible to experiment, determination of the attenuation will yield a direct measurement of the distance between flux tubes.

II. THE ATTENUATION COEFFICIENT

We consider a pure superconductor in the presence of a static magnetic field $\mathbf{A}_s$, and subject to an impressed ultrasonic wave. The effect of the sound wave is to induce in the superconductor longitudinal ϕ and transverse $\mathbf{A}$ fields which then drive the system. If we choose as a model for the superconductor the weakly coupled electron gas of BCS,[5] the Hamiltonian of the system in second-quantized formulation is

$$H=H_0+H'(t)\,,\qquad(1)$$

where

$$H_0=\sum_\alpha\int d^3r\,\psi_\alpha^\dagger(\mathbf{r})[h_0(\mathbf{r})-\mu]\psi_\alpha(\mathbf{r})+g\sum_{\alpha\beta}\int d^3r\,\psi_\alpha^\dagger(\mathbf{r})\psi_\beta^\dagger(r)\psi_\beta(\mathbf{r})\psi_\alpha(\mathbf{r})\,,\qquad(2)$$

$$h_0(r)=(1/2m)[\nabla/i-(e/c)\mathbf{A}_s]\,,\qquad(3)$$

and

$$H'(t)=\sum_\alpha\int d^3r\,\psi_\alpha^\dagger(\mathbf{r})h'(\mathbf{r},t)\psi_\alpha(\mathbf{r})\,,\qquad(4)$$

where

$$h'(\mathbf{r},t)=e\phi(\mathbf{r})+(ie/2mc)\{[\nabla-(ie/c)\mathbf{A}_s(\mathbf{r})]\cdot\mathbf{A}(\mathbf{r},t)+\mathbf{A}(\mathbf{r},t)\cdot[\nabla-(ie/c)\mathbf{A}_s(\mathbf{r})]\}\,.\qquad(5)$$

The attenuation constant α is given by the ratio of the power dissipated by the sound wave per unit volume to the energy per unit volume contained in the impressed wave, i.e.

$$\alpha=Q/\tfrac{1}{2}(\rho_{\text{ion}}|u|^2v_S)\,,\qquad(6)$$

where

$$Q=\left\langle\frac{d}{dt}\langle H\rangle(t)\right\rangle_{\text{av}}\qquad(7)$$

is the time-averaged rate of increase of the average energy of the system at time t.

The density matrix $\rho(t)$ of the system in the presence of the perturbing electromagnetic fields satisfies the Von Neumann equation.

$$i\partial\rho/\partial t=[H_0+H'(t),\,\rho]\,.\qquad(8)$$

Using Eq. (8) and the interaction picture for conveince, we can now determine the density matrix to second order in the perturbation $H'(t)$. It is easy to show that

$$i\partial\rho_I(t)/\partial t=[H_I'(t),\rho_I(t)]\,,\qquad(9)$$

where

$$\rho_I(t)=e^{iH_0t}\rho(t)e^{-iH_0t}\,.\qquad(10)$$

Integrating Eq. (9) and keeping only terms to second order in $H'(t)$, we get

$$\rho_I(t)=\rho_0-i\int_{-\infty}^t dt'[H_I'(t'),\rho_0]-\int_{-\infty}^t dt'\int_{-\infty}^{t'}dt''[H'_I(t'),[H'_I(t''),\rho_0]]\,,\qquad(11)$$

where ρ_0 is the density matrix of the system in the absence of the perturbing field.

$$\rho_0=e^{-\beta H_0}/\mathrm{Tr}(e^{-\beta H_0})\,.\qquad(12)$$

The average energy of the system is immediately found by substituting the approximate expression for ρ_I in Eq. (7).

$$\langle H'\rangle(t)=\langle H_0\rangle+\langle H_I'(t)\rangle-i\int_{-\infty}^t dt'\langle[H_0,H_I'(t')]\rangle-\int_{-\infty}^t dt'\langle[H'_I(t),H'_I(t')]\rangle-i\int_{-\infty}^t dt'\int_{-\infty}^{t'}dt''\langle[[H_0,H'_I(t')],H'_I(t'')]\rangle\,.\qquad(13)$$

[9] L. P. Gor'kov, Zh. Eksperim. i Teor. Fiz. **36**, 1918 (1959) [English transl.: Soviet Phys.—JETP **9**, 1364 (1959)].

All statistical averages in Eq. (13) are with respect to ρ_0. Taking the time derivative of Eq. (13) and remembering that

$$dH_I'(t)/dt=[\partial H'(t)/\partial t]_I+i[H_0,H_I'(t)], \quad (14)$$

we have

$$\frac{d}{dt}\langle H\rangle(t)=\frac{d}{dt}\langle H'_I(t)\rangle$$
$$-i\int_{-\infty}^{t}dt'\left\langle\left[\left(\frac{\partial H'(t)}{\partial t}\right)_I, H_I'(t')\right]\right\rangle. \quad (15)$$

Finally, time averaging, the power dissipated per unit volume is given by

$$Q=\langle Q(t)\rangle_{\text{av}}, \quad (16)$$

where

$$Q(t)=-i\int_{-\infty}^{t}dt'\left\langle\left[\left(\frac{\partial H'(t)}{\partial t}\right)_I, H_I'(t')\right]\right\rangle. \quad (17)$$

We now use this general formula for the power dissipation together with the interaction Hamiltonian given by Eq. (5) to determine the attenuation of longitudinal sound waves in a type-II superconductor. Although this calculation could be carried through in all generality, we now make approximations which reduce the complexity of the problem, but which will not significantly affect the general structure of the results.

We choose a gauge in which the magnetic field induced by the sound wave is transverse. For sound frequencies of interest we may neglect any transverse currents which arise when the longitudinal sound wave propagates in the presence of the static magnetic field. It must also be pointed out that, as we do not calculate in a gauge-invariant manner, we are forced into neglecting the dependence of the energy gap on the perturbing fields, and consequently the effects of collective modes. It has been shown[6] however, that the collective modes in a superconductor give a negligible contribution to the ultrasonic attenuation.

When these approximations have been made the interaction Hamiltonian becomes

$$h'(\mathbf{r})=e\phi(\mathbf{r}) \quad (18)$$

and the power dissipated by the ultrasonic wave is

$$Q(t)=\int d^3r\,\rho^c(\mathbf{r},t)\frac{\partial\phi(\mathbf{r},t)}{\partial t}, \quad (19)$$

where $\rho^c(\mathbf{r},t)$ is the causal charge response of the superconductor, in the presence of a static magnetic field A_s, to the scalar field $\phi(\mathbf{r},t)$.

$$\rho^c(\mathbf{r},t)=e^2\int_{-\infty}^{\infty}dt'\int d^3r'\,\phi(\mathbf{r}',t')\sum_{\alpha\beta}G_{\alpha\beta}{}^{\text{II}R}(\mathbf{r},\mathbf{r}':t-t') \quad (20)$$

and

$$G_{\alpha\beta}{}^{\text{II}R}(\mathbf{r},\mathbf{r}':t-t')=-i\Theta(t-t')\langle[n_\alpha(\mathbf{r}),n_\beta(\mathbf{r}')]\rangle \quad (21)$$

is the retarded two-particle Green's function.

Finally, carrying out the time average in Eq. (16) and Fourier analyzing we have

$$Q=\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\int\frac{d^3q}{(2\pi)^3}\omega\rho^c(\mathbf{q},\omega)\phi^*(\mathbf{q},\omega). \quad (22)$$

To determine Q it is convenient to use the finite-temperature Green's-function technique to find ρ^c. Although the prescription for obtaining ρ^c from the two-particle-temperature Green's function is given in the literature,[10] we will sketch briefly the main features of the argument in order that later steps in the calculation may be exhibited clearly.

III. THE RESPONSE FUNCTION

The two-particle Green's function in the temperature representation is defined:

$$G_{\alpha\beta,\gamma\delta}{}^{\text{II}}(12:34)=-\langle T_\tau\psi_\alpha(1)\bar\psi_\gamma(3)\psi_\beta(2)\bar\psi_\delta(4)\rangle, \quad (23)$$

where

$$\psi_\alpha(1)=\psi_\alpha(\mathbf{r},\tau)=e^{(H_0-\mu N)\tau}\psi_\alpha(\mathbf{r})e^{-(H_0-\mu N)\tau},$$
$$\bar\psi_\alpha(1)=e^{(H_0-\mu N)\tau}\bar\psi_\alpha(\mathbf{r})e^{-(H_0-\mu N)\tau}, \quad (24)$$

and

$$0<\tau<\beta.$$

For our purposes it is only necessary to study the special case

$$G_{\alpha\beta,\alpha\beta}{}^{\text{II}}(12:12)=-\langle T_\tau\bar\psi_\alpha(1)\psi_\alpha(1)\bar\psi_\beta(2)\psi_\beta(2)\rangle$$
$$=-\langle n(\mathbf{r}_1,\tau_1)n(\mathbf{r}_2,\tau_2)\rangle,\quad \tau_1>\tau_2. \quad (25)$$

The density operator $n(\mathbf{r},\tau)$ is formally considered as a function of the "time" parameter τ.

Putting in a complete set of states

$$G^{\text{II}}(12,12)=-\sum_{nm}e^{(\Omega+\mu N_n-E_n)\beta}e^{\omega_{nm}\tau}n_{nm}(1)n_{mn}(2). \quad (26)$$

Fourier analyzing the τ dependence of Eq. (25) and using the boundary condition $G^{\text{II}}(\tau)=G^{\text{II}}(\tau+\beta)$, we have

$$G^{\text{II}}(1,2:\omega_0)=-\sum_{mn}A_{nm}(1,2)/(i\omega_0-\omega_{nm}), \quad (27)$$

where

$$\omega_0=2m\pi/\beta,\quad (m=\text{integer})$$

and

$$A_{nm}(1,2)=e^{(\Omega+\mu N_n-E_m)\beta}(1-e^{\omega_{nm}\beta})n_{mn}(1)n_{nm}(2). \quad (28)$$

[10] See, for example, A. A. Abrikosov, L. P. Gor'kov, and I. E. Dzyaloshinski, *Methods of Quantum Field Theory in Statistical Physics* (Prentice-Hall, Inc., Englewood Cliffs, New Jersey, 1963).

Similarly it is easy to show that

$$G^{\mathrm{II}R}(\mathbf{1},\mathbf{2}:\omega)=-\lim_{\delta\to 0}\sum_{mn}\frac{A_{nm}(\mathbf{1},\mathbf{2})}{(\omega+i\delta)-\omega_{nm}}. \quad (29)$$

Equations (27) and (29) give a relation between $G^{\mathrm{II}R}(\omega)$ which occurs in the power dissipation formula and $G^{\mathrm{II}}(12,\omega)$, namely:

$$G^{\mathrm{II}}(\omega_0)=G^{\mathrm{II}R}(i\omega_0) \quad \text{for} \quad \omega_0>0. \quad (30)$$

On the other hand from Eq. (26)

$$G^{\mathrm{II}}(\omega_0)=G^{\mathrm{II}*}(-\omega_0). \quad (31)$$

Thus, from a knowledge of $G^{\mathrm{II}R}(\omega)$ which is analytic in the upper half (ω) plane, Eqs. (30) and (31) can be used to construct the temperature Green's function for all ω_0. We are, however, interested in the inverse problem, namely knowing $G^{\mathrm{II}}(\omega_0)$ how do we construct $G^{\mathrm{II}R}(\omega)$. If $G^{\mathrm{II}}(\omega_0)$ is known for all frequencies ω_0, and if we can construct a function $F(\omega)$ which is analytic in the upper half plane and has the properties

$$F(i\omega_0)=G(\omega_0) \quad \text{for} \quad \omega_0>0,$$

it follows from complex variable theory that $F(\omega)$ coincides with $G^{\mathrm{II}R}(\omega)$ everywhere in the upper half plane.

The procedure for determining the power absorption is now apparent, the response function $G^{\mathrm{II}}(\omega_0)$ in the temperature representation will be determined for the superconducting system in a magnetic field. Then, by continuing this function off the set of points ω_0, $G^{\mathrm{II}R}(\omega)$ and hence Q can be determined.

The two-particle-temperature Green's function for the superconductor is decomposed using the Gorkov[8] factorization procedure:

$$G_{\alpha\beta,\alpha\beta}{}^{\mathrm{II}}(12,12)=-G_{\alpha\alpha}(1,1)G_{\beta\beta}(2,2) +G_{\alpha\beta}(1,2)G_{\beta\alpha}(2,1)+F_{\alpha\beta}(1,2)\bar{F}_{\alpha\beta}(1,2). \quad (32)$$

The term $G_{\alpha\alpha}(1,1)G_{\beta\beta}(2,2)$ is irrelevant in our calculation of the response as it only contributes at precisely zero frequency, the remaining part of the right-hand side of Eq. (32) is exactly the two-particle correlation function we need for our calculation of Q.

Formally regarding the charge response $\rho(\mathbf{1},\tau)$ and the external field $\phi(\mathbf{1},\tau)$ as functions of the "time" parameter τ, the charge response in the temperature representation is given after summing over spins by

$$\rho(\mathbf{1},\tau)=2e^2\int_0^\beta d\tau_2\int d^32\,\phi(\mathbf{2},\tau_2) \times[G(1,2)G(2,1)-F(1,2)\bar{F}(1,2)], \quad (33)$$

where G and F are defined by Gorkov's[8] equations in the temperature representation

$$\bar{F}(1,2)=\int_0^\beta d\tau_3\int d^33\,\Delta^*(3)G(3,2)G^0(3,1),$$

$$F(1,2)=\int_0^\beta d\tau_3\int d^33\,\Delta(3)G(2,3)G^0(1,3), \quad (34)$$

$$G(1,2)=G^0(1,2)-\int_0^\beta d\tau_3\int d^33\,\Delta(3)\bar{F}(3,2)G^0(1,3),$$

and

$$\Delta(3)=\lim_{2\to 3}F(\mathbf{3},\mathbf{2}:\tau_3-\tau_2). \quad (35)$$

Note that all the Green's functions in Eq. (34) are functions of the external static magnetic field A_s the field dependence of the energy gap Δ manifests itself via a spatial dependence of the gap function.

We have now derived formally an expression for the power absorption by a superconductor in a static magnetic field. To make explicit calculation possible Eqs. (34) are iterated keeping terms to lowest order in Δ, an approach which is valid when $T\sim T_c$; these expressions are substituted in Eq. (33) and terms to order Δ^2 are retained giving

$$\rho(\mathbf{1},\tau)=2e^2\int_0^\beta d\tau_2\int d^32\Big[G^0(1,2)G^0(2,1)-\int_0^\beta d\tau_3\int_0^\beta d\tau_4\int d^33\int d^34\Delta^*(3)\Delta(4) \times[G^0(1,2)G^0(4,1)G^0(4,3)G^0(2,3)+G^0(4,2)G^0(4,3)G^0(1,3)G^0(2,1)+G^0(2,3)G^0(1,3)G^0(4,1)G^0(4,2)]\Big]\phi(\mathbf{2},\tau_2). \quad (36)$$

The problem is now how to use this expression such that the absorption is typical of a type-II superconductor: we take the attitude that although the magnetic field penetrates the sample in the form of a periodic array of flux lines, the effect of the presence of such flux lines near $T\sim T_c$ and for $H\ll H_{c_2}$, is to induce a slow periodic variation in the energy gap and an associated slow variation in the field inside the sample. Making these assumptions it is reasonable to proceed using a method analogous to that used by Gorkov[9] in his derivation of the Landau-Ginzburg[2] equations. An outline of this (local) theory is given in the next section.

IV. LOCAL APPROXIMATION

The normal-metal Green's functions, e.g. $G^0(1,2)$ occurring in Eq. (36) decay exponentially for distances $(1-2)>\xi$, where ξ is the coherence distance. Therefore, as the pentration depth δ of the magnetic field $H\ll H_{c_2}$ is such that $\delta>\xi$, then, in integrals such as (35), the magnetic field and the energy gap may be treated as slowly varying functions. To be more precise, the magnetic field varies over a distance of the order of δ; however, when $H\ll H_{c_2}$ the energy gap in the intervortex regions is essentially constant, but in the neighborhood of the vortices it may vary over distances less than δ. We assume then that $\delta>\xi$ (definition of type II) and $\delta>\lambda_S$, the wavelength of the external field. This treatment will give some indication of the effects to be expected in the more interesting case, when $\lambda_S\sim\delta$, which will be treated in the next section.

All slowly varying quantities will then be expanded about the point (1), for example,

$$\Delta(3)\Delta^*(4)=|\Delta(1)|^2+(\mathbf{3}-\mathbf{1})\cdot\left[\frac{\partial\Delta(1)}{\partial(\mathbf{1})}\cdot\Delta^*(1)\right]+(\mathbf{4}-\mathbf{1})\cdot\left[\frac{\partial\Delta^*(1)}{\partial(\mathbf{1})}\cdot\Delta(1)\right], \tag{37}$$

keeping terms only to first order in the derivation of the gap.

Secondly, if the magnetic field is sufficiently weak, $H\ll H_{c_2}$, that (the Larmor radius) $ep_F/H\gg\delta$, i.e., $p_F\gg eH\delta$, then a semiclassical method can be used to determine

$$G^0(12,H)=e^{i\phi(1,2)}G^0(1-2), \tag{38}$$

where

$$\mathbf{n}\cdot\partial\phi(1,2)/\partial(\mathbf{1})=e\mathbf{n}\cdot\mathbf{A}(1). \tag{39}$$

As the field $A(1)$ is slowly varying, the phase $\phi(1,2)$ can be written

$$\phi(1,2)=e\mathbf{A}(1)\cdot(\mathbf{1}-\mathbf{2}) \tag{40}$$

and as near T_c

$$A\sim H\delta\sim[1-T/T_c]^{1/2},$$

the phase is small, and the exponential can be expanded with respect to A giving to first order in the static field

$$G^0(12,H)=G^0(1-2)[1+ie\mathbf{A}(1)\cdot(\mathbf{1}-\mathbf{2})+\cdots], \tag{41}$$

where $G^0(1-2)$ is the normal metal Green's function in the absence of the external field.

Substituting Eqs. (37) and (40) into Eq. (36) and expanding all quantities near (1) up to first-order terms, we have

$$\begin{aligned}\rho(1,\tau_1)=2e^2\Bigg[\rho^N-\int_0^\beta d\tau_2\int_0^\beta d\tau_3\int_0^\beta d\tau_4\int d^32\int d^33\int d^34[&G^0(1-2)G^0(4-1)G^0(4-3)G^0(2-3)\\&+G^0(4-2)G^0(4-3)G^0(1-3)G^0(2-1)+G^0(2-3)G^0(1-3)G^0(4-1)G^0(4-2)]\Bigg]\\\times\left\{|\Delta(1)|^2+(\mathbf{3}-\mathbf{1})\cdot\left[\frac{\partial\Delta(1)}{\partial(\mathbf{1})}\cdot\Delta^*(1)\right]+(\mathbf{4}-\mathbf{1})\cdot\frac{\partial\Delta^*(1)}{\partial(\mathbf{1})}\cdot\Delta(1)+\cdots\right\}&[1+2ie\mathbf{A}(1)\cdot(\mathbf{4}-\mathbf{3})+\cdots]\phi(2,\tau_2).\end{aligned} \tag{42}$$

The leading superconducting term is then the term proportional to $|\Delta(1)|^2$ which would, if $\Delta(1)$ were independent of position, give the charge response of a type-I superconductor in the small gap region. It is also immediately obvious, because the kernel $[G^0(1-2)G^0(4-1)G^0(4-3)G^0(2-3)+\cdots]$ is symmetric under interchange of 3 and 4, that the term proportional to $[\mathbf{A}(1)\cdot(\mathbf{4}-\mathbf{3})|\Delta(1)|^2]$ vanishes. Thus there is no correction to the charge density due to the magnetic-field dependence of the Green's functions to the order considered here. The only remaining terms are those proportional to the gradient of the energy gap. We will now exhibit the gross features of such terms saving a detailed calculation for the next section where this term can be picked out as a special case of a more general calculation.

To illustrate the procedure let us assume for simplicity a real energy gap; then on carrying out the Fourier transforms and some straightforward but tedious algebra we have

$$\begin{aligned}\rho^{\text{grad}}(1,\omega_m)=\frac{ie^2}{(2\pi)^3\beta_m}\int d^3k\sum_\omega\int d^3p[\mathbf{B}(1)\cdot\mathbf{p}]\phi(\mathbf{k},\omega_m)\exp i\mathbf{k}\cdot\mathbf{1}\{&G^0_{\omega-\omega_m}(\mathbf{p}-\mathbf{k})G^0_{-(\omega-\omega_m)}(\mathbf{p}-\mathbf{k})[G^0_{-\omega}(\mathbf{p})]^2G^0_\omega(\mathbf{p})\\&+[G^0_\omega(\mathbf{p})]^2[G^0_{-\omega}(\mathbf{p})]^2G^0_{\omega-\omega_m}(\mathbf{p}-\mathbf{k})+2[G^0_\omega(\mathbf{p})]^3G^0_{\omega-\omega_m}(\mathbf{p}-\mathbf{k})G^0_{-\omega}(\mathbf{p})+\text{c.c.}\},\end{aligned} \tag{43}$$

where

$$\mathbf{B}(1)=[\partial\Delta(1)/\partial(1)]\Delta^*(1). \tag{44}$$

Equation (43) demonstrates, as $\mathbf{B}(1)\cdot\mathbf{p}=(\mathbf{B}(1)\cdot\mathbf{k})(\mathbf{k}\cdot\mathbf{p})$ apart from terms which vanish on averaging over angles, that if the sound wave propagates in a direction perpendicular to $\mathbf{B}(1)$, i.e., parallel to the magnetic field, then there will be no contribution to the absorption from $\rho^{\text{grad}}(1,\omega_m)$. Thus we see immediately that there will be anisotropy in the ultrasonic attenuation arising because of the presence of the magnetic field. The theory developed in this section is only capable of describing the attenuation when the sound wavelength $\lambda_S<\delta$, the distance over which the magnetic field and energy gap vary. In the next section we will generalize the theory in order that a wider range of sound wavelengths may be studied, we will be interested in particular in the effects which arise when $\lambda_S\sim\delta$.

V. NONLOCAL THEORY

It was shown in Sec. II that the power absorbed by a type-II superconductor on applying a compressional wave is given by

$$Q=\frac{1}{2\pi i}\int\frac{d\omega}{2\pi}\int\frac{d^3q}{(2\pi)^3}\omega\rho(\mathbf{q},\omega)\phi^*(\mathbf{q},\omega), \tag{45}$$

where in the temperature representation

$$\rho(1,\omega_m)=\sum_n\int d^32\,\phi(2,\omega_m)[G_{\omega_n}(1,2)G_{\omega_n^{(-)}}(2,1)-F_{\omega_n}(1,2)\bar{F}_{\omega_n^{(-)}}(1,2)]=\sum_{v=0}^{\infty}\rho^{2v}(1,\omega_n), \tag{46}$$

where

$$\rho^{2v}(1,\omega_m)\propto|\Delta|^{2v}$$

and

$$\omega_n^{(-)}=\omega_n-\omega_m.$$

Again making the assumption that we are sufficiently close to the critical temperature T_c: we cut off the series expansion at order $|\Delta|^2$. A consideration of higher order terms shows, that in the case of type-I superconductor, this procedure is valid if

$$|\Delta(T)|^2/kT(v_Fq)\ll1.$$

We now have

$$\rho(1,\omega_m)=\rho_0(1,\omega_m)+\rho_2(1,\omega_m), \tag{47}$$

where

$$\rho_0(1,\omega_m)=\frac{2e^2}{\beta}\sum_n\int d^32\,\phi(2,\omega_m)G^0_{\omega_n}(1,2)G^0_{\omega_n^{(-)}}(2,1) \tag{48}$$

and

$$\begin{aligned}\rho_2(1,\omega_m)=-\frac{2e^2}{\beta}\sum_n\int d^32\int d^33\int d^34\,\phi(2,\omega_m)[G^0_{-\omega_n}(3,1)G^0_{\omega_n}(3,2)\Delta(3)\\ \times G^0_{-\omega_n^{(-)}}(4,2)\Delta^*(4)G^0_{\omega_n^{(-)}}(4,1)+G^0_{\omega_n}(1,3)G^0_{-\omega_n}(4,3)\Delta(3)G^0_{\omega_n}(4,2)\Delta^*(4)G^0_{\omega_n^{(-)}}(2,1)\\ +G^0_{\omega_n}(1,2)G^0_{\omega_n^{(-)}}(2,3)\Delta(3)G^0_{-\omega_n^{(-)}}(4,3)\Delta^*(4)G^0_{\omega_n^{(-)}}(4,1)].\end{aligned} \tag{49}$$

As before both the functions $\Delta(1)$ and $G_{\omega_n}(1,2)$ are subject to the static magnetic field $A_s(1)$.

The magnetic-field dependence of the normal-state Green's function is separated out using Eq. (38). We will again assume that the magnetic field is weak and slowly varying so that the results of Sec. IV are valid. As we have remarked previously, the magnetic field dependence of the energy gap manifests itself in a spatial variation of Δ. The functional form of Δ should be determined self-consistently. In this paper we will choose a physically reasonable form for $\Delta(1)$ emphasizing the fact that if in a type-II superconductor, where $H\ll H_{c2}$, the magnetic field penetrates the sample in the form of a periodic array of flux tubes, then the energy gap must vary periodically in the plane perpendicular to H. It will be assumed that the flux tubes remain stationary under the influence of the perturbing fields.

Fourier transforming Eqs. (48) and (49) we have:

$$\rho_0(q,\omega_m)=2e^2\phi(\mathbf{q},\omega_m)R_N(\mathbf{q},\omega_m) \tag{50}$$

and

$$\rho_2(q,\omega_m)=-2e^2\int\frac{d^3k}{(2\pi)^3}\int\frac{d^3k'}{(2\pi)^3}\phi(\mathbf{q}-\mathbf{k}-\mathbf{k}')\Delta(\mathbf{k})\Delta^*(\mathbf{k}')\sum_{j=0}^{2}R_j(\mathbf{q},\mathbf{k},\mathbf{k}',\omega_m)\,, \tag{51}$$

where

$$R_N(\mathbf{q},\omega_m)=\frac{1}{\beta}\sum_n\int d^3p\; G^0{}_{\omega_n}(\mathbf{p})G^0{}_{\omega_n{}^{(-)}}(\mathbf{p}-\mathbf{q})\,, \tag{52}$$

$$R_0(\mathbf{q},\mathbf{k},\mathbf{k}',\omega_m)=\frac{1}{\beta}\sum_n\int d^3p\; G^0{}_{-\omega_n}(\mathbf{p})G^0{}_{\omega_n}(\mathbf{p}+\mathbf{k})G^0{}_{-\omega_n{}^{(-)}}(\mathbf{p}+\mathbf{q}-\mathbf{k}')G^0{}_{\omega_n{}^{(-)}}(\mathbf{p}+\mathbf{q})\,, \tag{53}$$

$$R_1(\mathbf{q},\mathbf{k},\mathbf{k}',\omega_m)=\frac{1}{\beta}\sum_n\int d^3p\; G^0{}_{\omega_n}(\mathbf{p})G^0{}_{-\omega_n}(\mathbf{p}+\mathbf{k})G^0{}_{\omega_n}(\mathbf{p}-\mathbf{k}-\mathbf{k}')G^0{}_{\omega_n{}^{(-)}}(\mathbf{p}-\mathbf{q})\,, \tag{54}$$

and

$$R_2(\mathbf{q},\mathbf{k},\mathbf{k}',\omega_m)=\frac{1}{\beta}\sum_n\int d^3p\; G^0{}_{\omega_n}(\mathbf{p})G^0{}_{\omega_n{}^{(-)}}(\mathbf{p}-\mathbf{q}+\mathbf{k}+\mathbf{k}')G^0{}_{-\omega_n{}^{(-)}}(\mathbf{p}-\mathbf{q}+\mathbf{k}')G^0{}_{\omega_n{}^{(-)}}(\mathbf{p}-\mathbf{q})\,. \tag{55}$$

Since only electrons with energy β^{-1} from the Fermi surface contribute and as $\Delta(\mathbf{r})$, $\Delta^*(\mathbf{r})$, and $\phi(\mathbf{r},\omega_m)$ are slowly varying functions of r over a Fermi wavelength, then the contributions of R_N, $R_0\cdots R_2$ to the response function $\rho(\mathbf{q},\omega_m)$ will arise from k, k', and q which are negligibly small compared to k_F. Thus

$$G^0{}_{\omega_n}(\mathbf{p}+\mathbf{k})=[i\omega_n-\epsilon-\Gamma_k]^{-1}\,, \tag{56}$$

where

$$\epsilon=(p^2-p_F{}^2)/2m \tag{57}$$

and

$$\Gamma_k=v_Fk(\hat{p}\cdot\hat{k})+O(k^2/2m)\,. \tag{58}$$

We will first determine the normal state contribution $\rho_0(\mathbf{q},\omega_m)$: this will serve as a check on the validity of the formalism and as an illustration of the technique to be used in determining R_j.

$$R_N=\frac{N(0)}{\beta}\sum_n\int\frac{d\Omega_p}{4\pi}\int_{-\infty}^{\infty}d\epsilon\; G^0{}_{\omega_n}(\mathbf{p})G^0{}_{\omega_n{}^{(-)}}(\mathbf{p}-\mathbf{q})\,. \tag{59}$$

Under the conditions stated above Eq. (59) becomes

$$R_N=[N(0)/\beta]\sum_n\int_{-1}^{1}\frac{dx}{2}\int_{-\infty}^{\infty}d\epsilon\frac{1}{(i\omega_n-\epsilon)[i(\omega_n-\omega_m)-\epsilon+\Gamma_q]}\,. \tag{60}$$

To carry the calculation further we must bear in mind that for large ω_n and ϵ, R_N behaves like $\omega_n{}^{-2}$ for $\omega_n\gg\epsilon$ and like ϵ^{-2} for $\epsilon\gg\omega_n$. Therefore, strictly speaking, the integral over ϵ and the sum over frequencies ω_n diverges. It has been shown, however, that the correct result is obtained by performing the sum over ω_n first.[10] The reason being that the sum is only nonzero in a very narrow energy region near the Fermi surface of the order Γ_q. In this region, the integral with respect to the momentum is rapidly convergent, and it is only for this reason we can write the energy of the excitations measured from the Fermi surface in the form

$$\epsilon=(p^2-p_F{}^2)/2m\sim v_F(|p|-p_F)\,. \tag{61}$$

Thus in integrals of the above type we must always calculate the sum first, and only afterwards integrate with respect to ϵ. Otherwise, the integral with respect to ϵ will include the region $||p|-p_F|\sim p_F$ and then it is no longer appropriate to write all quantities as expansions near the Fermi surface. With these remarks in mind, we carry out the sum over n:

$$R_N=\frac{N(0)}{\beta}\int_{-1}^{1}\frac{dx}{2}\int_{-\infty}^{\infty}d\epsilon\sum_n\frac{1}{(\zeta_n-\epsilon)[(\zeta_n-z)-\epsilon+\Gamma_q]}\,, \tag{62}$$

where

$$\zeta_n=(2n+1)\pi i/\beta\,,\quad z=2m\pi i/\beta\,.$$

The summation is performed by noting that $\tanh(\beta\zeta/2)$ has poles at $\zeta_n=(2n+1)\pi i/\beta$, with residue $2/\beta$ at these points, and integrating around the contour of Fig. 1.

$$R_N=\frac{N(0)}{4\pi i}\int_{-1}^{1}\frac{dx}{2}\int_{C_1}\frac{\tanh(\beta\zeta/2)}{(\zeta-\epsilon)[(\zeta-z)-\epsilon+\Gamma_q]}. \quad (63)$$

We now transform the integral along the contour C_1 into an integral along the two contours C_2 and C_3. It is easily seen that the integral along C_3, regarded formally as a function of z, has singularities at the points $z=(2n+1)\pi i/\beta-\epsilon+\Gamma_q$, since the contour of integration goes through the points $\zeta=(2n+1)\pi i/\beta$, where the function $\tanh(\beta\zeta/2)$ becomes infinite. Therefore as the causal charge response is obtained only when $z\to(\omega+i\delta)$ we need the branch of the function $R_N(z)$, which is analytic in the upper half z plane. To obtain this we transform the expression (63) for the special values $z=2m\pi i/\beta$ in such a way that the contour of integration does not go through singularities of the integrand when we later extend (63) to arbitrary values of z.

We make a change of variable in the integral along C_3, and let $\zeta'=\zeta-z$, then as $\tanh[\beta(\zeta'+z)/2]=\tanh(\beta\zeta'/2)$, (63) becomes

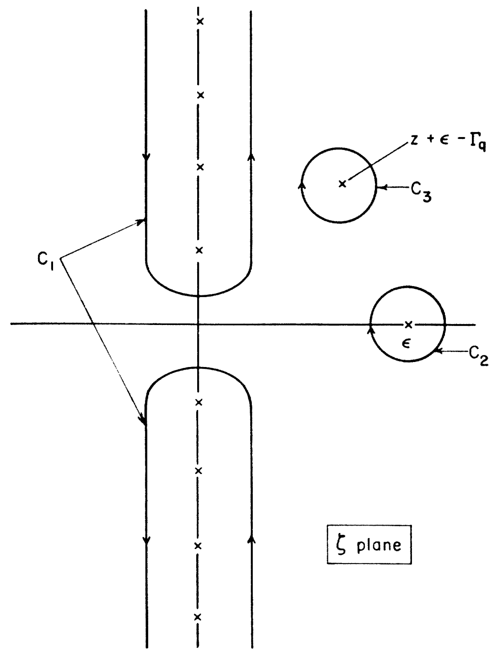

FIG. 1. Contour for Σ_n, the points on the imaginary ζ axis are $\zeta_n=(2n+1)\pi i/\beta$.

$$R_N=\frac{N(0)}{4\pi i}\int_{-1}^{1}\frac{dx}{2}\left[\int_{C_2}\frac{\tanh(\beta\zeta/2)}{(\zeta-\epsilon)[(\zeta-z)-\epsilon+\Gamma_q]}+\int_{C_3'}\frac{\tanh(\beta\zeta/2)}{(\zeta+z-\epsilon)(\zeta-\epsilon+\Gamma_q)}\right], \quad (64)$$

which reduces to

$$R_N=-\frac{N(0)}{2}\int_{-1}^{1}dx\frac{\Gamma_q}{(\Gamma_q-z)}. \quad (65)$$

If we now analytically continue $z\to(\omega+i\delta)$, integrate over angles and expand in powers of v_S/v_F, where v_S is the velocity of sound, we have:

$$R_N=-N(0)[1+i\pi\omega/v_Fq-2(\omega/v_Fq)^2+\cdots] \quad (66)$$

and therefore

$$\rho^N(\mathbf{q},\omega)=-2e^2N(0)[1+\tfrac{1}{2}i\pi(\omega/v_Fq)-O(\omega/v_Fq)^2+\cdots]. \quad (67)$$

This is the result obtained by conventional methods: see for example Tsuneto[6] Eq. (3.27).

The remaining terms R_0, R_1, and R_2 can also be evaluated using the method illustrated above. These contributions to the charge response are dependent on the system being in the superconducting state vanishing when the energy gap goes to zero. If we consider $R_0(\mathbf{q},z)$ as an example, then converting the sum into an integral around the contour C_1 of Fig. 1 and taking care that the resulting expression is an analytic function of z in the upper half z plane we obtain:

$$R_0(\mathbf{q},z)=-N(0)\int\frac{d\Omega_p}{4\pi}\frac{1}{(z-\Gamma_q+\Gamma_{k'})(z+\Gamma_q-\Gamma_k)}\int_{-\infty}^{\infty}d\epsilon\tanh(\beta\epsilon/2)$$
$$\times\left[\frac{2z}{(2\epsilon)^2-\Gamma_k^2}-\frac{2\epsilon+z}{(2\epsilon+z)^2-\Gamma_q^2}+\frac{2\epsilon}{(2\epsilon)^2-\Gamma_{k'}^2}-\frac{2z+\epsilon}{(2\epsilon+z)^2-\Gamma_{q'}^2}\right], \quad (68)$$

where

$$\Gamma_{q'}=\Gamma_q-\Gamma_k-\Gamma_{k'}, \quad (69)$$

which is an analytic function of z for values of z in the upper half z plane.

The ϵ integral is evaluated by noticing that if we close the contour in the upper half ϵ plane in such a way as to avoid the poles of $\tanh(\beta\epsilon/2)$, then the analyticity of $R_0(\mathbf{q},z)$ as a function of z is preserved. Further noting that poles on the real axis have zero residue, we have

$$R_0(\mathbf{q},z)=-N(0)\frac{4\pi i}{\beta}\int\frac{d\Omega_p}{4\pi}\frac{1}{(z+\Gamma_{k'}-\Gamma_q)(z+\Gamma_q-\Gamma_k)}\times\sum_{n=0}\left[\frac{2\zeta_n}{(2\zeta_n)^2-\Gamma_k{}^2}+\frac{2\zeta_n}{(2\zeta_n)^2-\Gamma_{k'}{}^2}-\frac{2\zeta_n+z}{(2\zeta_n+z)^2-\Gamma_q{}^2}-\frac{2\zeta_n+z}{(2\zeta_n+z)^2-\Gamma_{q'}{}^2}\right],\quad(70)$$

where

$$\zeta_n=(2n+1)\pi i/\beta.$$

If we now remember that in the case of interest the minimum value of $\zeta_n\sim kT_c$, and that the wavelength of a typical ultrasonic wave is such that $qv_F\ll kT_c$, then we may expand Eq. (70) in powers of $\beta\Gamma_k$, for example,

$$\sum_{n=0}^{\infty}\frac{2\zeta_n}{(2\zeta_n)^2-\Gamma_k{}^2}=\frac{\beta}{2\pi i}\left[\sum_{n=0}^{\infty}\frac{1}{2n+1}+\sum_{v=1}^{\infty}\left(\frac{\beta\Gamma_k}{2\pi i}\right)^{2v}\sum_{n=0}^{\infty}\frac{1}{(2n+1)^{2v+1}}\right].\quad(71)$$

Now using the fact that

$$\sum_{n=1}^{\infty}1/n^v=\zeta(r),\quad(72)$$

where $\zeta(v)$ is the Riemann zeta function, we have

$$\sum_{n=0}^{\infty}\frac{2\zeta_n}{(2\zeta_n)^2-\Gamma_k{}^2}=\frac{\beta}{2\pi i}\left[\sum_{n=0}^{\infty}\frac{1}{2n+1}+\sum_{v=1}^{\infty}\left(\frac{\beta\Gamma_k}{2\pi i}\right)^{2v}\left(1-\frac{1}{2^{2v+1}}\right)\zeta(2v+1)\right].\quad(73)$$

Similarly

$$\sum_{n=0}^{\infty}\frac{2\zeta_n+z}{(2\zeta_n+z)^2-\Gamma_q{}^2}=\frac{\beta}{2\pi i}\left[\sum_{n=0}^{\infty}\frac{1}{(2n+1)+\beta z/2\pi i}+\tfrac{1}{2}\sum_{v=1}^{\infty}\left(\frac{\beta\Gamma_q}{2\pi i}\right)^{2v}\zeta\left(2v+1,\frac{1}{2}+\frac{\beta z}{2\pi i}\right)\right],\quad(74)$$

where

$$\zeta(x,\alpha)=\sum_{n=0}^{\infty}(n+\alpha)^{-1}x.\quad(75)$$

Substituting Eqs. (74) and (73) in Eq. (70) we have

$$R_0(\mathbf{q},z)=2N(0)\int\frac{d\Omega_p}{4\pi}\frac{2}{(z-\Gamma_q+\Gamma_{k'})(z+\Gamma_q-\Gamma_k)}\left[\sum_{n=0}^{\infty}\left(\frac{1}{2n+1+\beta z/2\pi i}-\frac{1}{2n+1}\right)-\sum_{v=1}^{\infty}\left(\frac{\beta}{2\pi i}\right)^{2v}(\Gamma_k{}^{2v}+\Gamma_{k'}{}^{2v})\left(1-\frac{1}{2^{2v+1}}\right)\zeta(2v+1)+\frac{1}{2}\left(\frac{\beta}{4\pi i}\right)^{2v}(\Gamma_q{}^{2v}+\Gamma_{q'}{}^{2v})\zeta\left(2v+1,\frac{1}{2}+\frac{\beta z}{4\pi i}\right)\right].\quad(76)$$

Further, as we may now analytically continue $z\rightarrow(\omega+i\delta)$ and as $\omega=v_S|q'|\ll v_F|q'|\ll\beta^{-1}$, we can expand $\zeta(2\nu+1,\frac{1}{2}+\beta z/4\pi i)$ in powers of $|\beta z|$ to leading orders. We find

$$R_0(\mathbf{q},z)=\frac{7\zeta(3)}{2}N(0)\left(\frac{\beta}{2\pi}\right)^2\int\frac{d\Omega_p}{4\pi}\frac{(\Gamma_q-\Gamma_{k'})(\Gamma_q-\Gamma_k)+z^2}{(\Gamma_q-\Gamma_{k'}-z)(\Gamma_q-\Gamma_k+z)}-3\zeta(2)N(0)\left(\frac{i\beta z}{2\pi}\right)\int\frac{d\Omega_p}{4\pi}\frac{1}{(\Gamma_q-\Gamma_{k'}-z)(\Gamma_q-\Gamma_k+z)}.\quad(77)$$

Similarly it can be shown that

$$R_{\frac{1}{2}}(\mathbf{q},z)=-\frac{7\zeta(3)}{4}N(0)\left(\frac{\beta}{2\pi}\right)^2\int\frac{d\Omega_p}{4\pi}\frac{\Gamma_q\Gamma_{q'}+z^2\mp z(\Gamma_k-\Gamma_{k'})}{(\Gamma_q-z)(\Gamma_{q'}-z)}+\frac{3\zeta(2)}{2}N(0)\left(\frac{i\beta z}{2\pi}\right)\int\frac{d\Omega_p}{4\pi}\frac{1}{(\Gamma_q-z)(\Gamma_{q'}-z)}\quad(78)$$

and therefore

$$\rho^2(\mathbf{q},z)=-2e^2\int\frac{d^3k}{(2\pi)^3}\int\frac{d^3k''}{(2\pi)^3}\phi(\mathbf{k}'',z)\Delta(\mathbf{k})\Delta(\mathbf{q}-\mathbf{k}-\mathbf{k}'')R(\mathbf{q},\mathbf{k},\mathbf{k}'',z),\quad(79)$$

where

$$R(\mathbf{q},\mathbf{k},\mathbf{k}'',z)=\frac{7\zeta(3)}{2}N(0)\left(\frac{\beta}{2\pi}\right)^2\int\frac{d\Omega_p}{4\pi}\left[\frac{(\Gamma_q+\Gamma_{k''})(\Gamma_q-\Gamma_k)+z^2}{(\Gamma_k+\Gamma_{k''}-z)(\Gamma_q-\Gamma_k+z)}-\frac{\Gamma_q\Gamma_{k''}+z^2}{(\Gamma_q-z)(\Gamma_{k''}-z)}\right]$$
$$-3\zeta(2)\left(\frac{i\beta z}{2\pi}\right)\int\frac{d\Omega_p}{4\pi}\left[\frac{1}{(\Gamma_k+\Gamma_{k''}-z)(\Gamma_q-\Gamma_k+z)}-\frac{1}{(\Gamma_q-z)(\Gamma_{k''}-z)}\right]. \quad (80)$$

In the case of a homogeneous or type-I superconductor $\Delta(\mathbf{k})=\delta^3(\mathbf{k})\Delta$ and Eq. (80) reduces to

$$\rho^{2\mathrm{I}}(\mathbf{q},z)=-2e^2|\Delta|^2\phi(\mathbf{q},z)\left[\frac{7\zeta(3)}{2}N(0)\left(\frac{\beta}{2\pi}\right)^2\int\frac{d\Omega_p}{4\pi}\left[\frac{\Gamma_q^2+z^2}{(\Gamma_q-z)(\Gamma_q+z)}-\frac{\Gamma_q^2+z^2}{(\Gamma_q-z)^2}\right]\right.$$
$$\left.-3\zeta(2)\frac{i\beta z}{(2\pi)}\int\frac{d\Omega_p}{4\pi}\left[\frac{1}{(\Gamma_q-z)(\Gamma_q+z)}-\frac{1}{(\Gamma_q-z)^2}\right]\right]. \quad (81)$$

Carrying out the integrals and expanding in powers of (v_S/v_F) we have:

$$\rho^{2\mathrm{I}}(\mathbf{q},\omega)=-\frac{3\zeta(2)}{2}N(0)e^2\phi(\mathbf{q},\omega)\frac{|\Delta(T)|^2}{v_Fq}\left[1-i\frac{4}{3}\left(\frac{v_S}{v_F}\right)^3+O\left(\frac{v_S}{v_F}\right)^5\right]. \quad (82)$$

Combining Eq. (82) with the expression for $\rho^N(\mathbf{q},\omega)$ given by Eq. (67) we find the response function for a type-I superconductor at a temperature $T\sim T_c$ is

$$\rho(\mathbf{q},\omega)=-2e^2N(0)\phi(\mathbf{q},\omega)\left\{1+\frac{3\zeta(2)}{4}\beta\frac{|\Delta(T)|^2}{v_Fq}+\cdots+\frac{\pi i}{2}\frac{\omega}{v_Fq}\left[1-\frac{2\zeta(2)}{\pi^2}\beta\frac{|\Delta(T)|^2}{\omega}\left(\frac{v_S}{v_F}\right)^3+\cdots\right]\right\}. \quad (83)$$

VI. ATTENUATION IN A MODEL TYPE-II SUPERCONDUCTOR

In Sec. V we have derived a general expression for the absorption of a longitudinal sound wave by a superconductor in the region where $T\sim T_c$ and $H\sim H_{c_1}$. The formalism is adapted to take into account explicitly a spatial variation of the energy gap.

To make further progress we must now construct a model for a type-II superconductor in a magnetic field $H>H_{c_1}$. The choice of model is based upon Abrikosov's suggestion that when $H>H_{c_1}$ magnetic flux penetrates a type-II superconductor in the form of flux filaments which form a periodic array inside the material. We visualize the type-II superconductor as superconducting material threaded by a periodic net of tubes of normal material; the periodicity is represented mathematically by choosing an energy gap $\Delta(x,y)$ which varies periodically in the plane perpendicular to the magnetic field.

$$\Delta(x,y)=\tfrac{1}{2}\Delta(T)[1+\cos(ax)\cos(ay)]. \quad (84)$$

This representation of $\Delta(x)$ specifies a body-centered-cubic array of flux tubes of spacing (d), $a=2\pi/d$. The parameter $\Delta(T)$ is the energy gap attained at the center of the superconducting region at the particular temperature of interest. This parameter $\Delta(T)$ will be equal to the energy gap in bulk material provided the typical distance over which the energy gap varies is greater than the coherence distance.

We find it convenient to consider the case of an energy gap which varies one-dimensionally over the system

$$\Delta(x)=\tfrac{1}{2}\Delta(T)[1+\cos(ax)] \quad (85)$$

or

$$\Delta(\mathbf{k})=\tfrac{1}{4}(2\pi)^3\Delta(T)[2\delta(\mathbf{k})+\delta(\mathbf{k}-a\mathbf{u})+\delta(\mathbf{k}+a\mathbf{u})], \quad (86)$$

where $\mathbf{u}$ is a unit vector in the x direction. This choice simplified considerably the ensuing mathematics, and the results obtained when the energy gap given by Eq. (84) is used can be readily deduced from it. The gap of Eq. (85) could also be realized by periodically superimposing laminae of normal and superconducting material.

The charge response in this two-dimensional array is given by

$$\rho^2(\mathbf{q},\omega)=-2e^2\int\frac{d^3k}{(2\pi)^3}\int\frac{d^3k''}{(2\pi)^3}\phi(\mathbf{k}'',\omega)\times\Delta(\mathbf{k})\Delta^*(\mathbf{q}-\mathbf{k}-\mathbf{k}'')R(\mathbf{q},\mathbf{k},\mathbf{k}'',\omega), \quad (87)$$

where $R(\mathbf{q},\mathbf{k},\mathbf{k}'',\omega)$ is given by Eq. (80).

Integrating over $\mathbf{k}$ and $\mathbf{k}''$ we have

$$\rho^2(\mathbf{q},\omega)=-2e^2[(|\Delta(T)|^2/4)\phi(\mathbf{q})R(\mathbf{q},0,\mathbf{q},\omega) \quad (88a)$$
$$+(|\Delta(T)|^2/8)\phi(\mathbf{q}-a\mathbf{u})R(\mathbf{q},0,\mathbf{q}-a\mathbf{u},\omega) \quad (88b)$$
$$+(|\Delta(T)|^2/8)\phi(\mathbf{q}+a\mathbf{u})R(\mathbf{q},0,\mathbf{q}+a\mathbf{u},\omega) \quad (88c)$$

$$+(|\Delta(T)|^2/8)\phi(\mathbf{q}-a\mathbf{u})R(\mathbf{q},\, a\mathbf{u},\, \mathbf{q}-a\mathbf{u},\, \omega) \quad (88d)$$

$$+(|\Delta(T)|^2/8)\phi(\mathbf{q}+a\mathbf{u})R(\mathbf{q},\, -a\mathbf{u},\, \mathbf{q}+a\mathbf{u},\, \omega) \quad (88e)$$

$$+(|\Delta(T)|^2/16)\phi(\mathbf{q}-2a\mathbf{u})R(\mathbf{q},\, a\mathbf{u},\, \mathbf{q}-2a\mathbf{u},\, \omega) \quad (88f)$$

$$+(|\Delta(T)|^2/16)\phi(\mathbf{q}+2a\mathbf{u})R(\mathbf{q},\, -a\mathbf{u},\, \mathbf{q}+2a\mathbf{u},\, \omega) \quad (88g)$$

$$+(|\Delta(T)|^2/16)\phi(\mathbf{q})R(\mathbf{q},\, -a\mathbf{u},\, \mathbf{q},\, \omega) \quad (88h)$$

$$+(|\Delta(T)|^2/16)\phi(\mathbf{q})R(\mathbf{q},a\mathbf{u},\mathbf{q},\omega)]. \quad (88i)$$

Note that Eq. (88) reduces to the type-I limit, constant gap when $a\rightarrow 0$, i.e., the distance between flux lines becomes very large. The local approximation is obtained from Eq. (80) by expanding the Kernel $R(q,k,k'')$ in powers of k/k''.

Explicit expressions are given for R in the Appendix. Inspection of Eq. (88) shows that the dominant terms in the power absorption will come from

$$\rho^2(\mathbf{q},\omega)=-2e^2[(|\Delta(T)|^2/4)\phi(q)R(\mathbf{q},0,\mathbf{q},\omega) \quad (89a)$$

$$+(|\Delta(T)|^2/16)\phi(q)R(\mathbf{q},\, -a\mathbf{u},\, \mathbf{q},\, \omega) \quad (89b)$$

$$+(|\Delta(T)|^2/16)\phi(q)R(\mathbf{q},a\mathbf{u},\mathbf{q},\omega)]. \quad (89c)$$

The remaining terms in Eq. (88) will give only a small contribution to the power absorption: because of overlap integrals of the form $\int\phi^*(\mathbf{q})\phi(\mathbf{q}\pm a\mathbf{u})d^3q$ and the fact that $\phi(\mathbf{q})$ is peaked around $q\sim q_{\text{ext}}$. We also note that the term (89a) looks like the charge response of a type-I superconductor except that the role of the energy gap is now played by the average $\Delta(T)/2$.

Using the results of the Appendix Eq. (89) becomes

$$\rho^2(\mathbf{q},\omega)=\rho^{2,\text{average}}-\tfrac{1}{8}e^2|\Delta(T)|^2\{7\zeta(3)N(0)\omega(\beta/2\pi)^2\{J_1[|\mathbf{q}-a\mathbf{u}|]+J_1[|\mathbf{q}+a\mathbf{u}|]-2J_1[|\mathbf{q}|]-2\omega I_1(\mathbf{q},\mathbf{q})\}$$
$$+3\zeta(2)(i\beta\omega/2\pi)\{I_1[(\mathbf{q}-a\mathbf{u}),\, -(\mathbf{q}+a\mathbf{u})]+I_1[(\mathbf{q}+a\mathbf{u}),\, -(\mathbf{q}-a\mathbf{u})]+2I_1(\mathbf{q},\mathbf{q})\}\}. \quad (90)$$

We now use the fact mentioned in the Appendix that if I_1 is written $I_1=I_{1p}+I_{1\delta}$, then $I_{1\delta}$ does not contribute to Eq. (90) and I_{1p} contributes only to the terms proportional to $\zeta(2)$. Therefore

$$\rho^2(\mathbf{q},\omega)=\rho^{2,\text{average}}-\tfrac{1}{8}e^2|\Delta(T)|^2\{7\zeta(3)N(0)\omega(\beta/2\pi)^2\{J_1[|\mathbf{q}-a\mathbf{u}|]+J_1[|\mathbf{q}+a\mathbf{u}|]-2J_1[|\mathbf{q}|]\}$$
$$+3\zeta(2)(i\beta\omega/2\pi)\{I_{1p}[(\mathbf{q}-a\mathbf{u}),\, -(\mathbf{q}+a\mathbf{u})]+I_{1p}[(\mathbf{q}+a\mathbf{u}),\, -(\mathbf{q}-a\mathbf{u})\}+2I_1(\mathbf{q},\mathbf{q})+\text{even function of }\omega]\}, \quad (91)$$

where

$$J_1[|\mathbf{q}-a\mathbf{u}|]=\frac{i\pi\Theta[1-\omega/(v_F|\mathbf{q}-a\mathbf{u}|)]}{2v_F|\mathbf{q}-a\mathbf{u}|} \quad (92)$$

and

$$I_{1p}[(\mathbf{q}-a\mathbf{u}),\, -(\mathbf{q}+a\mathbf{u})]=I_{1p}[(\mathbf{q}+a\mathbf{u}),\, (\mathbf{q}-a\mathbf{u})]=\frac{2}{(\sqrt{L})}\left[\tan^{-1}\left(\frac{4v_F^2(\mathbf{q}-a\mathbf{u})\cdot\mathbf{q}}{(\sqrt{L})}\right)+\tan^{-1}\left(\frac{4v_F^2(\mathbf{q}+a\mathbf{u})\cdot\mathbf{q}}{(\sqrt{L})}\right)\right], \quad (93)$$

where

$$L=16v_F^4\{q^2|\mathbf{q}+a\mathbf{u}|^2(1-\omega^2/v_F^2|\mathbf{q}+a\mathbf{u}|^2)-[(\mathbf{q}+a\mathbf{u})\cdot\mathbf{q}]^2\}. \quad (94)$$

The terms of Eq. (93) give rise to an attenuation which is critically dependent on the direction of the acoustic wave. In fact from the discussion in the Appendix it can be seen that the two terms of Eq. (91) become singular when $v_F|\mathbf{q}\pm a\mathbf{u}|=v_Sq$, respectively.

We then expect anomalous attenuation whenever $|\mathbf{q}\pm a\mathbf{u}|=v_S/v_F|\mathbf{q}|$, that is the component of $\mathbf{q}$ perpendicular to the magnetic field is of the order of a, and when the component of $\mathbf{q}$ parallel to the magnetic field is of the order of $(v_S/v_F)a$. It can also be seen that the terms proportional to $\zeta(3)$ namely $J_1(|\mathbf{q}+a\mathbf{u}|)$ and $J_1(|\mathbf{q}-a\mathbf{u}|)$ also become very large when $v_F|q\pm au|\sim v_S/v_F$, although these terms actually vanish in the type-I limit.

To complete this section we analyze in more detail the $\zeta(2)$ terms for sound propagation perpendicular and parallel to $\mathbf{u}$.

(1) $\mathbf{q}\perp\mathbf{u}$: It is easily seen in this case that the anomalous absorption mentioned above does not occur for any value of $\mathbf{q}$.

(2) $\mathbf{q}\|\mathbf{u}$. In this case there is a possibility of structure: for reasons of comparision we will first examine the type-I limit.

We call $I(\mathbf{q},a\mathbf{u})=2[I[(\mathbf{q}-a\mathbf{u}),\, -(\mathbf{q}+a\mathbf{u})]+I(\mathbf{q},\mathbf{q})]$. Using Eq. (93) this becomes

$$I(\mathbf{q},a\mathbf{u})=\frac{2i}{(\sqrt{L})}\left(\ln\left|\frac{(\sqrt{L})+i4v_F^2(\mathbf{q}-a\mathbf{u})\cdot\mathbf{q}}{(\sqrt{L})-i4v_F^2(\mathbf{q}-a\mathbf{u})\cdot\mathbf{q}}\right|+\ln\left|\frac{(\sqrt{L})+i4v_F^2(\mathbf{q}+a\mathbf{u})\cdot\mathbf{q}}{(\sqrt{L})-i4v_F^2(\mathbf{q}-a\mathbf{u})\cdot\mathbf{q}}\right|\right)+2I(\mathbf{q},\mathbf{q}), \quad (95)$$

which becomes, in type-I limit,

$$\lim_{a\to 0} I(\mathbf{q},a\mathbf{u})=\frac{4i}{(\sqrt{L})}\left(\ln\left|\frac{(\sqrt{L})+i4v_F^2q^2}{(\sqrt{L})-i4v_F^2q^2}\right|\right)+2I(\mathbf{q},\mathbf{q}), \quad (96)$$

where

$$\lim_{a\to 0} L = -16 v_F^2 q^2 \omega^2 . \tag{97}$$

Thus

$$\lim_{a\to 0} I(\mathbf{q},a\mathbf{u}) = \frac{1}{v_F|q|\omega} \ln\left|\frac{\omega+v_F q}{\omega-v_F q}\right| + 2I(\mathbf{q},\mathbf{q}) , \tag{98}$$

where

$$I(\mathbf{q},\mathbf{q}) = -(v_F^2 q^2)^{-1}[1-(\omega/v_F q)^2]^{-1} . \tag{99}$$

This yields the result obtained previously in Sec. V. The leading term in $I(q,-q)$ is cancelled by the leading term in $I(q,q)$ leaving

$$\lim_{a\to 0} I(\mathbf{q},a\mathbf{u}) = -\tfrac{4}{3}(v_S/v_F)^2[1/(v_F q)^2] . \tag{100}$$

We now examine Eq. (96) when $\mathbf{q}\|a\mathbf{u}$. In this case $L=-16v_F^2q^2\omega^2$ and the first term in $I(\mathbf{q},a\mathbf{u})$ is

$$I'(\mathbf{q},a\mathbf{u}) = \frac{1}{2v_F q\omega} \ln\left|\frac{\omega+v_F(q-a)}{\omega-v_F(q-a)}\right| . \tag{101}$$

The function $I'(\mathbf{q},a\mathbf{u})$ decreases for increasing q until $q=a-\omega/v_F$, where it becomes negative and infinite; at $q=a$ the logarithm is zero and when $q=a+\omega/v_F$, $I'(\mathbf{q},a\mathbf{u})$ is infinite and positive. That is when $\mathbf{q}\|\mathbf{u}$ the absorption is first of all drastically reduced and then immediately shows a sharp increase. It is also noticed that the terms in Eq. (91) proportional to $\zeta(3)$ become very large when $\mathbf{q}\|\mathbf{u}$ and $q=a+\omega/v_F$ leading to a consequent increase in absorption.

To summarize, it has been shown that ultrasonic attenuation in a type-II superconductor, in the mixed state, depends on the direction of propagation of the sound wave. It has also been shown that when a sound wave propagates perpendicular to the magnetic field, the attenuation will show structure whenever the sound wavelength matches the distance between flux lines. Because of this fact it might be expected that the effect should be present over wider ranges of temperature and magnetic field than can be treated by present theory. If this effect is accessible to experiment, it should provide a direct measure of the distribution of the flux lines in a type-II superconductor.

APPENDIX: ANGULAR INTEGRALS

From Eq. (87) the pure superconducting part of the charge response can be written:

$$\rho^2(\mathbf{q},z) = -2e^2 \int \frac{d^3k}{(2\pi)^3} \int \frac{d^3k''}{(2\pi)^3} \phi(\mathbf{k}'',z)\Delta(\mathbf{k})\Delta^*(\mathbf{q}-\mathbf{k}-\mathbf{k}'')R(\mathbf{q},\mathbf{k},\mathbf{k}'',z) , \tag{A1}$$

where

$$R(z) = \frac{7\zeta(3)}{2} N(0)\left(\frac{\beta}{2\pi}\right)^2 \int \frac{d\Omega_p}{4\pi}\left[\frac{(\Gamma_k+\Gamma_{k''})(\Gamma_q-\Gamma_k)+z^2}{(\Gamma_k+\Gamma_{k''}-z)(\Gamma_q-\Gamma_k+z)} - \frac{\Gamma_q\Gamma_{k''}+z^2}{(\Gamma_q-z)(\Gamma_{k''}-z)}\right]$$
$$-3\zeta(2)N(0)\left(\frac{i\beta z}{2\pi}\right)\int \frac{d\Omega_p}{4\pi}\left[\frac{1}{(\Gamma_k+\Gamma_{k''}-z)(\Gamma_q-\Gamma_k+z)} - \frac{1}{(\Gamma_q-z)(\Gamma_{k''}-z)}\right] . \tag{A2}$$

Equation (A2) can be written:

$$R(z) = \tfrac{7}{2}\zeta(3)N(0)z(\beta/2\pi)^2[J_1(\mathbf{k}+\mathbf{k}'')-J_2(\mathbf{k})-J_1(\mathbf{q})-J_1(\mathbf{k}'')-2zI_1(\mathbf{q},\mathbf{k}'')]$$
$$+3\zeta(2)N(0)(i\beta z/2\pi)[I_1(\mathbf{k}+\mathbf{k}'', \mathbf{k}-\mathbf{q})+I_1(\mathbf{q},\mathbf{k}'')] , \tag{A3}$$

where

$$J_1(\mathbf{s}) = \int \frac{d\Omega_p}{4\pi} \frac{1}{(\Gamma_s - z)}\bigg|_{z=\omega+i\delta} , \tag{A4}$$

$$J_2(\mathbf{s}) = \int \frac{d\Omega_p}{4\pi} \frac{1}{\Gamma_s+z}\bigg|_{z=\omega+i\delta} = -J_1(s) , \tag{A5}$$

and

$$I_1(\mathbf{s},\mathbf{s}') = \int \frac{d\Omega_p}{4\pi} \frac{1}{(\Gamma_s-z)(\Gamma_{s'}-z)} . \tag{A6}$$

In what follows, as we are interested in the power absorption, only terms in R which are odd functions of z will give a contribution.

From Eq. (A4)

$$J_1(\mathbf{s})=\frac{\Theta(1-\omega/v_Fs)}{2v_Fs}\left[i\pi+\ln\left|\frac{1-\omega/v_Fs}{1+\omega/v_Fs}\right|\right]+\frac{\Theta(1-v_Fs/\omega)}{2v_Fs}\ln\left|\frac{1-v_Fs/\omega}{1+v_Fs/\omega}\right|. \tag{A7}$$

Thus as far as the power absorption is concerned

$$J_1(\mathbf{s})=i\pi\Theta(1-\omega/v_Fs)/2v_Fs, \tag{A8}$$

where $\Theta(1-x)$ is the Heavyside step function.

The terms involving the integral I_1 are complicated to evaluate in all generality: we will just pick up those points which are significant for our computation.

Using a trick due to Feynman, $I(\mathbf{s},\mathbf{s}')$ can be written:

$$I_1(\mathbf{s},\mathbf{s}')=\int_0^1 du\int\frac{d\Omega_p}{4\pi}\frac{1}{(\Gamma_Q-z)^2}, \tag{A9}$$

where

$$\Gamma_Q=v_F|\mathbf{s}+(\mathbf{s}-\mathbf{s}')u|x=v_F|Q|x. \tag{A10}$$

Thus

$$I_1=\frac{1}{2}\int_0^1 du\left[\frac{\partial}{\partial\omega}P\int_{-1}^1\frac{dx}{(\Gamma_Q-\omega)}+i\pi\int_{-1}^1\frac{\partial}{\partial\omega}\delta(\Gamma_Q-\omega)dx\right]\quad\text{if}\quad \omega/v_F|Q|<1, \tag{A11}$$

$$=\frac{1}{2}\int_0^1 du\frac{\partial}{\partial\omega}P\int_{-1}^1\frac{dx}{\Gamma_Q-\omega}\qquad\text{if}\quad \omega/v_F|Q|>1.$$

A. The Principal Part

$$I_{1p}=-\int_0^1\frac{du}{v_F^2|Q|^2-\omega^2}. \tag{A12}$$

We have then an integral of the type

$$I_{1p}=\int_0^1\frac{du}{Au^2+Bu+C},\quad\text{where}\quad \begin{aligned}A&=v_F^2|\mathbf{s}-\mathbf{s}'|^2,\\ B&=v_F^2 2\mathbf{s}'\cdot(\mathbf{s}-\mathbf{s}'),\\ C&=v_F^2s'^2-\omega^2,\end{aligned} \tag{A13}$$

and

$$L=4AC-B^2=4v_F^4\{(\mathbf{s}'-\mathbf{s})^2s'^2[1-\omega^2/(v_Fs')^2]-[\mathbf{s}'\cdot(\mathbf{s}-\mathbf{s}')]^2\}$$
$$=4v_F^4\{(s'-s)^2s^2[1-\omega^2/(v_Fs)^2]-[\mathbf{s}\cdot(\mathbf{s}-\mathbf{s}')]^2\}. \tag{A14}$$

Thus

$$I_{1p}=-\frac{2}{(\sqrt{L})}\left[\tan^{-1}\frac{2v_F^2\mathbf{s}\cdot(\mathbf{s}-\mathbf{s}')}{(\sqrt{L})}-\tan^{-1}\frac{2v_F^2\mathbf{s}'\cdot(\mathbf{s}-\mathbf{s}')}{(\sqrt{L})}\right]. \tag{A15}$$

Properties of I_{1p}. I_{1p} can only have singularities when (1)$L=0$, (2)$L<0$.

(1)

$$\lim_{L\to0}I_{1p}=\frac{1}{v_F^2}\left[\frac{1}{\mathbf{s}\cdot(\mathbf{s}-\mathbf{s}')}-\frac{1}{\mathbf{s}'\cdot(\mathbf{s}-\mathbf{s}')}\right]. \tag{A16}$$

The type-I limit which corresponds to $\mathbf{s}=\mathbf{s}'=\mathbf{q}$ can be extracted from Eq. (A16) as a special case. We see in fact in this case there is no singularity other than the trivial $s=s'=0$ which is not allowed by the theory.

$$\lim_{s'\to s}I_{1p}{}^{L=0}=-(v_F^2s^2)^{-1}[1-\omega^2/(v_Fs)^2]^{-1}. \tag{A17}$$

The important singularities arise when in the first term of Eq. (A16) $\mathbf{s}\|(\mathbf{s}-\mathbf{s}')$ is consistent with $L=0$, namely when

$$1-\omega^2/(v_Fs)^2=0. \tag{A18}$$

Similarly the second term is singular when $\mathbf{s}'\|(\mathbf{s}-\mathbf{s}')$ and

$$1-\omega^2/(v_F s')^2=0. \tag{A19}$$

(2) $L<0$. Inspection of I_{1p} when $L<0$ shows that again the first term is singular when $1-\omega^2/(v_F s)^2=0$, and the second term when $1-\omega^2/(v_F s')^2=0$.

Finally we note that I_{1p} is an even function of ω and therefore will only contribute to the $\zeta(2)$ terms of $R(z)$.

B. The δ Function

$$I_\delta=\frac{i\pi}{2}\int_{-1}^{1}\frac{\partial}{\partial\omega}\delta(\Gamma_Q-\omega)dx\,,$$

i.e.,

$$I_\delta=-\frac{i\pi}{2V_F|\mathbf{Q}|}\int_0^1[\delta(v_F|\mathbf{Q}|-\omega)-\delta(v_F|\mathbf{Q}|+\omega)]du\,. \tag{A20}$$

We see by inspection that I_δ is an odd function of ω and therefore will contribute only to the $\zeta(3)$ terms of $R(z)$. We may also easily check that $I_\delta(\mathbf{q},\mathbf{q})$, the only term arising in the power absorption, is identically zero.

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Mössbauer Hyperfine Spectra of Fe^{3+} in Corundum: Magnetic- and Crystal-Field Effects

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The Mössbauer technique was employed to study the paramagnetic hfs of Fe^{57}-doped corundum (α-Al_2O_3) in the presence of an external magnetic field. Experiments were performed with an oriented single crystal and with external field strengths to 41 kOe parallel to the trigonal axis of the crystal field at the iron site. The results showed noticeable field-dependent features. Several of these effects were analyzed by detailed calculations of theoretical hfs from the ground-term electronic levels of Fe^{3+}. Effects of small local fields and of mixing of electronic states by crystal-field terms and by the hyperfine interaction are shown to produce gross changes in the Mössbauer spectra. Some of these features were observed; others which require a more homogeneous magnetic field were not observed because major changes in the Mössbauer spectra occur over relatively small ranges of external field strengths.

I. INTRODUCTION

In previous work, the well-resolved, low-temperature paramagnetic hyperfine spectra of trivalent Fe^{57} in corundum (Al_2O_3) have been investigated by Mössbauer effect, primarily in the absence of an external magnetic field.[1–3] The major features of the Mössbauer absorption spectrum in zero field were explained on the basis of the "superposition" of three essentially independent hfs from the three Kramers doublets of the ground term of Fe^{3+}:$3d^5$, ${}^6S_{5/2}$. However, the observed and predicted spectra for the ground doublet were not in complete agreement. Since the spin-Hamiltonian parameters of Fe^{3+} in Al_2O_3 are known to high accuracy[4–6] so that theoretical hfs are readily calculated, we have extended these measurements to study magnetic-field-dependent features of the hfs and to obtain further comparisons of experiment with theory.

It is known from ESR work that a relatively small cubic term in the spin Hamiltonian causes a large mixing of electronic wave functions for certain critical-field strengths. It is shown below that this leads to dramatic changes in the hfs for small changes of the externally applied magnetic field. The magnetic field also causes certain levels to cross. At these points equally

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