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Resistance of a Plasma Slab between Juxtaposed Disk Electrodes

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The classical solution of the important (to plasma physics) problem of two opposite electrodes on an infinitely extensive slab of conductor is in principle seriously incorrect. An exact solution of the problem in two dimensions is given. The solution of the three-dimensional problem is presented in the form of a one-dimensional integral equation by a new method. Rapidly converging series solutions are then obtained. The method used has implications of usefulness which go well beyond the present problem.

INTRODUCTION

A FREQUENTLY and currently quoted[1] solution of the problem once known as Nobili's rings, given by Weber[2] in 1873, is incorrect. This problem envisions two infinitely conducting circular disks juxtaposed on the two faces of an infinitely extensive slab of moderately conducting material. The problem becomes of modern importance when one inquires what should be expected of electrodes or probes implanted in a plasma, leaving aside any complexities introduced by the anodic and cathodic behaviors of plasma electrodes.[3] Weber's result led to an approximation that when the slab thickness $2a$ is large, but not excessively large, compared with the electrode radius c, the resistance observed would be given by

$$R=\frac{1}{2\sigma c}\left(1-\frac{2c\ln 2}{\pi a}\right).$$

Weber's method of solution was to solve the problem of a single electrode on an infinite half-space of conductor, and then combine two such solutions, facing each other, to obtain the slab solution. Unfortunately, he assumed that the current distribution to the single electrodes was not altered in the two-electrode case, and so his result needs investigation in respect to the most interesting part, namely, that term, the second one, which involves the conductivity deep in the medium, since experimentally speaking, the first term is of little interest, being strongly affected by local anodic and cathodic processes. This investigation reveals many interesting facts, among them that the approximate Weber solution even though incorrect is in fact accurate to the second order in c/a.

Weber's Solution

The Nobili problem requires that over each disk the potential be constant (or the tangential component of E be zero) that is, at $z=\pm a$, $V=\pm V_0$ for $\rho<c$. Outside the disks, the component of E normal to the surface must be zero to fulfill Khirchof's first law, since the slab is assumed to be insulated on both faces everywhere except at the electrodes. Thus, at $z=\pm a$, $E_z=0$ for $\rho>c$. The problem is intrinsically difficult because it falls into the class of mixed boundary-value problems.

Taking as a basis for his approach the Weber-Schafheitlin discontinuous integral

$$V=\frac{2V_0}{\pi}\int_0^\infty(\sin\lambda c)J_0(\lambda\rho)\frac{d\lambda}{\lambda},$$

which has the value V_0 for $\rho<c$, and $(2V_0/\pi)\sin^{-1}(c/\rho)$ for $\rho>c$, Weber found the potential for an infinite half-space of conductor lying on the right-hand side of a single disk by adding the z dependence dictated by the Laplace equation. This yielded

$$V=\frac{2V_0}{\pi}\int_0^\infty e^{-\lambda z}(\sin\lambda c)J_0(\lambda\rho)\frac{d\lambda}{\lambda}.$$

Then $E_z=-\partial V/\partial z$ gives a second discontinuous integral at $z=0$,

$$E_z=\frac{2V_0}{\pi}\int_0^\infty(\sin\lambda c)J_0(\lambda\rho)d\lambda,$$

which has the values $E_z=0$ for $\rho>c$, and $E_z=(2V_0/\pi)\times(c^2-\rho^2)^{-1/2}$ for $\rho<c$.

Moving two such solutions to new origins at $z=\pm a$, Weber gave the final expression,

$$V=\frac{2V_0}{\pi}\int_0^\infty\frac{\sinh\lambda z}{\cosh\lambda a}(\sin\lambda c)J_0(\lambda\rho)\frac{d\lambda}{\lambda}$$

as the solution of Nobili's problem. Although this still fulfills the boundary condition $E_z=0$ for $\rho>c$ at both $z=\pm a$, it no longer fulfills $V=$constant for $\rho<c$ on either electrode.

[1] A. Gray, G. B. Mathews, and T. M. MacRobert, *Treatise on Bessel Functions* (Macmillan and Company Ltd., London, 1952).
[2] W. Weber, J. für Math. 75, 75 (1873).
[3] S. H. Lam, Phys. Fluids 8, 73 (1965).

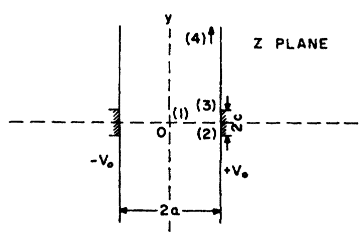
A Two-Dimensional Nobili Problem

An exact solution can be obtained in closed form for the two dimensional analog of the Nobili problem. It will be presented here because it has a result which demonstrates the pitfalls of simple intuition, and it can be used as an interpolative connection between limiting approximations for the three-dimensional case. In place of the opposite circular disks, we now imagine two opposite infinite strip electrodes of the same width. The slab is perpendicular to the x axis, extending between $y=\pm c$, as shown in Fig. 1. From the symmetry of the problem, one may restrict the investigation to the positive quadrant, and then find that around the edges of this quadrant the boundary conditions are

$$x=0,\ V=0,\ \ 0<y<\infty; \qquad x=a,\ E_x=0,\ \infty>y>c$$
$$x=a,\ V=V_0,\ c>y>0; \qquad y=0,\ E_y=0,\ \ 0<x<a.$$

If this quadrant is now regarded as a part of a complex z plane, it can be mapped on an upper half-plane by the transformation $t=-\cos(\pi z/a)$. This moves the turning points of the z plane to corresponding points along the real axis shown in Fig. 2(a).

FIG. 1. Geometry for a pair of two-dimensional strip electrodes in the two-dimensional case.



It will be noted that the segment where $V=V_0$, i.e., the segment between points (2) and (3), lies unsymmetrically arranged with respect to the origin. Therefore, a fractional linear transformation to an s plane is made by means of

$$s=(\alpha t+\beta)/(\gamma t+\delta)$$

with the correspondence of points shown in Fig. 2(b).

The new location of the outer boundary points is given in terms of a constant k:

$$\frac{1}{k}=\frac{1}{u-1}[u+3+(8u+8)^{1/2}]; \quad u=\cosh(\pi c/a)$$

which is between 1 and ∞. It is plotted in Fig. 3. The four constants of the fractional linear transformation are

$$\alpha=D/k, \quad \gamma=D,$$
$$\delta=(D/2)\{(1/k-1)u-(1/k+1)\},$$
$$\beta=(D/2)\{(1/k-1)u+(1/k+1)\}.$$

The constant D remains arbitrary.

Now we map the upper half of the s plane into the interior of a rectangle in a ζ plane with $s=sn(\zeta,k)$, for

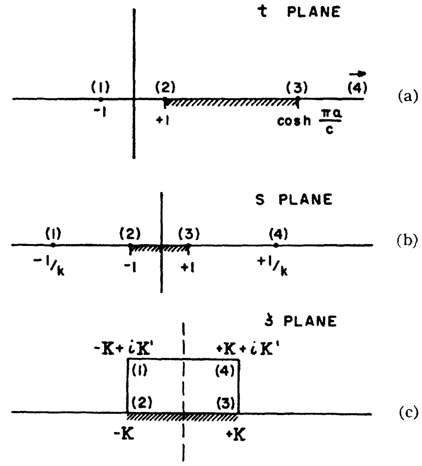


FIG. 2. (a) Complex t plane. (b) Complex s plane. (c) Complex ζ plane.

which the correspondence of points is shown in Fig. 2(c). The new constants K and K' are the two complete elliptic-integral periods regarded as functions of k.

A solution of the Laplace equation is now needed fulfilling the above boundary conditions. This is quite simple:

$$V(\xi,\eta)=V_0(1-\eta/K'),$$

and with this the problem is solved.

Since we desire only the resistance of the medium between the probes in this analysis, the answer falls out at once. Let $\varphi=V+iU$ be the complex potential for the problem. Then

$$I=-\sigma\int_{-c}^{c}\left(\frac{\partial V}{\partial x}\right)_{x=a}dy$$
$$=-\sigma\int_{-c}^{c}\left(\frac{\partial U}{\partial y}\right)_{x=a}dy$$
$$=-\sigma[U(a,c)-U(a,-c)].$$

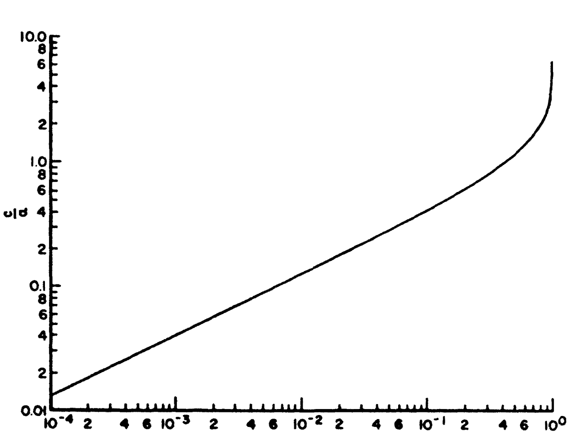


FIG. 3. The elliptic-integral-constant k as a function of c/a. Abscissa is $1/k$.

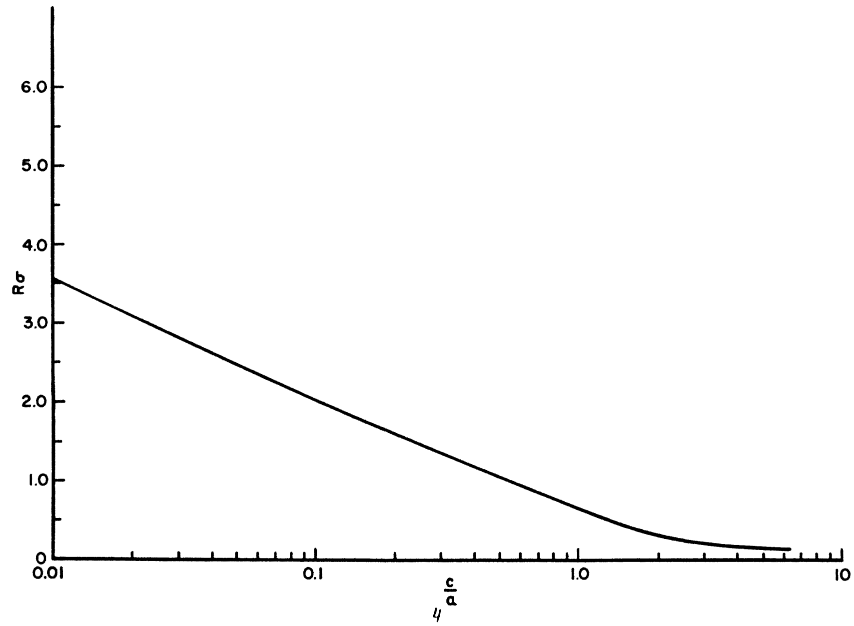


FIG. 4. The geometrical resistance behavior $R\sigma$ for two strip electrodes.

By the symmetry of the problem, this is

$$I=-2\sigma[U(a,c)-U(a,0)].$$

Our problem is therefore one of finding the imaginary part of $\varphi(z)$. The complex potential which satisfies the Laplace equation in the ζ plane is

$$\varphi=V_0(1+i\zeta/K').$$

Hence, our need is to evaluate the imaginary part of φ at the points $(a,c)\rightarrow(K,0)$ and $(a,0)\rightarrow(-K,0)$. So

$$\begin{aligned}I&=-2\sigma\,\mathrm{Im}[V_0(1+i\zeta/K')]_{(-K,0)}{}^{(K,0)}\\&=-2\sigma V_0(2K/K').\end{aligned}$$

Now the potential difference between two strips at a and $-a$ is $2V_0$, so the resistance per unit length electrode is

$$R=K'/2\sigma K.$$

The minus sign found for the current expression disappears when the current-voltage convention is observed. This expression is plotted as a function of c/a in Fig. 4.

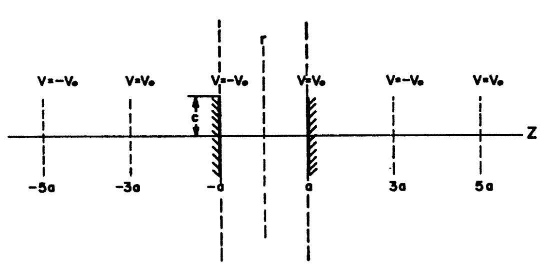


FIG. 5. Geometry and image electrode system for a pair of disk electrodes in the three-dimensional case.

The Three-Dimensional Nobili Problem

The solution of the three-dimensional problem can be obtained in the form of an integral equation using a technique related to one employed by Hafen[4] for the solution of certain electrostatic problems.

We consider the problem of infinitely many regularly spaced disk electrodes, with charges alternating in sign, as shown in Fig. 5. Then the expression

$$V=\frac{2}{\pi}\int_0^\infty d\lambda\,[\cdots+e^{-|z+3a|\lambda}-e^{-|z+a|\lambda}+e^{-|z-a|\lambda}-e^{-|z-3a|\lambda}\cdots]J_0(\lambda\rho)A(\lambda)$$

satisfies by symmetry the boundary condition $\partial V/\partial z=0$ for $\rho>c$ on every plane $z=(2n+1)a$. (That the final solution of the problem does indeed fulfill this boundary condition can be verified *a posteriori* also.) We propose that $A(\lambda)$ can be so chosen that the conditions $V=\pm V_0$ for $\rho<c$ can also be satisfied on the planes $z=(4n\pm1)a$, respectively, V_0 being a constant. Between the principal electrodes $z=\pm a$, the potential can be summed to

$$V=\frac{2}{\pi}\int_0^\infty d\lambda\,\frac{\sinh z\lambda}{\cosh a\lambda}J_0(\lambda\rho)A(\lambda),$$

which is reminiscent of Weber's expression, and on $z=\pm a$, this becomes

$$V_0=\frac{2}{\pi}\int_0^\infty d\lambda\,(\tanh a\lambda)J_0(\lambda\rho)A(\lambda).$$

Rather than determine $A(\lambda)$ from this, we shall

[4] M. Hafen, Math. Ann. **69**, 517 (1910).

attempt to find a function $f(\xi)$ connected with $A(\lambda)$ through

$$A(\lambda)=\int_0^c d\xi\,(\cos\lambda\xi)f(\xi).$$

Changing for the moment the old variable ρ into η, we act on the integral for V_0 with the operator

$$\frac{d}{d\rho}\int_0^\rho \frac{\eta d\eta}{(\rho^2-\eta^2)^{1/2}},$$

with the following result:

$$V_0=\frac{d}{d\rho}\int_0^\rho \frac{\eta d\eta}{(\rho^2-\eta^2)^{1/2}}\int_0^\infty d\lambda\, J_0(\eta\lambda)\left(1-\frac{2}{e^{2a\lambda}+1}\right)\int_0^c d\xi \times(\cos\lambda\xi)f(\xi).$$

The η integral can be performed, and by drawing the ξ integral to the front, one has

$$V_0=f(\xi)-\int_{-c}^{+c} d\eta\, f(\eta)K(\xi,\eta),$$

where

$$K(\xi,\eta)=\frac{2}{\pi}\int_0^\infty \frac{\cos\lambda\xi\,\cos\lambda\eta}{e^{2a\lambda}+1}d\lambda.$$

This is a standard inhomogeneous integral equation with the symmetric kernel $K(\xi,\eta)$.

In terms of new dimensionless variables as follows:

$F=f(\xi)/V_0$, $\xi/c=x$, $\eta/c=x_1$, $2a\lambda=\mu$, and $c/2a=\epsilon$,

a solution can be found in terms of Legendre polynomials letting

$$F(x)=\sum_0^\infty a_{2m}P_{2m}(x).$$

The integral equation then takes the form of an infinite set of algebraic equations

$$\frac{1}{4n+1}a_{2n}-\frac{\epsilon}{\pi}\sum_{m=0}^{\infty} a_{2m}M_{2m,2n}=\delta_{0,2n}$$

involving a matrix whose elements are given by

$$M_{2m,2n}=\int_{-1}^1 dx\int_{-1}^1 dx_1 \times P_{2m}(x)P_{2n}(x_1)\int_0^\infty \frac{d\mu\,\cos\epsilon\mu x\,\cos\epsilon\mu x_1}{e^\mu+1}.$$

These elements can be treated by expansion of the denominator of the kernel into a series in $e^{-l\mu}$, and performing the infinite integral term by term to give

$$M_{2m,2n}=\frac{2}{\pi}\sum_{l=1}^{\infty}(-1)^{l+1}\frac{1}{l}\int_{-1}^1 dx\int_{-1}^1 dx_1\frac{P_{2m}(x)P_{2n}(x_1)}{1+(\epsilon^2/l^2)(x+x_1)^2}.$$

If we define

$$M_{2m,2n}=\sum_{l=1}^{\infty}(-1)^{l+1}M^l{}_{2m,2n},$$

then the expressions $M^l{}_{2m,2n}$ can be found in a series by expanding the denominator of the integral for $M_{2m,2n}$, or in closed form by integration by parts. Both results (given in Table I) are useful in computation, because the circle of convergence of the series is determined by ϵ/l, denoted hereafter as θ, being $\leq\frac{1}{2}$. For $\epsilon=\frac{1}{2}$, the extreme case in which all the series representations are convergent, the explicit numerical values for a_{2m} are $a_0=1.626$, $a_2=-0.0819$, $a_4=0.00387$. For $\epsilon=\frac{1}{4}$ the explicit values were $a_0=1.269$, $a_2=-0.0127$, and $a_4=0.00027$. For larger values of ϵ, the $M^l{}_{2m,2n}$ can be evaluated by use of the exact expressions to establish those with the lower values of l, and the series can be used as soon as l is sufficiently large to ensure convergence. It has not been possible to derive a type form for the coefficients of either the series or the closed forms, except for small ϵ, when the summation over l can be accomplished in terms of Riemann's zeta function to yield

$$M_{2m,2n}=\epsilon^{2(m+n)}\sum_{\nu=0}^{\infty}C_{2m,2n;\,\nu-(m+n)}\epsilon^{2\nu}\left[1-\frac{1}{4(\nu-m-n)}\right]\times\zeta(2\nu-2m-2n+1),$$

where the coefficients C are the tabular coefficients of the various series expansions of Table I.

Let the entire coefficient of $\epsilon^{2\nu}$ of any term in the series for $M_{2m,2n}$ be labeled $\Gamma_{2m,2n;\,\nu}$. It is now possible to solve the infinite set of algebraic equations for the a_{2n}, obtaining

$$a_0=1+\frac{1}{\pi}\Gamma_{00,0}\epsilon+\frac{1}{\pi^2}\Gamma_{00,0}{}^2\epsilon^2+\left(\frac{1}{\pi^3}\Gamma_{00,0}{}^3+\frac{1}{\pi}\Gamma_{00,2}\right)\epsilon^3+\frac{1}{\pi^2}\Gamma_{00,0}\Gamma_{00,2}\epsilon^4+\left(\frac{1}{\pi^3}\Gamma_{00,0}{}^2\Gamma_{00,2}+\frac{5}{\pi^2}\Gamma_{02,0}+\frac{1}{\pi}\Gamma_{00,4}\right)\epsilon^5\cdots,$$

$$a_2=\frac{5}{\pi}\Gamma_{02,0}\epsilon^3+\frac{1}{\pi^2}\Gamma_{00,0}\Gamma_{02,0}\epsilon^4+\left(\frac{5}{\pi}\Gamma_{02,0}+\frac{5}{\pi^3}\Gamma_{02,0}\Gamma_{00,0}{}^2\right)\epsilon^5\cdots,$$

$$a_4=\frac{9}{\pi}\Gamma_{04,0}\epsilon^5\cdots.$$

TABLE I. Matrix elements for solution of the two-disk integral equation.

m	n	$M^{l}_{2m,2n}\pi l/2 \quad (\theta=\epsilon/l)$
0	0	$(4/\theta)\tan^{-1}2\theta-(1/\theta^2)\ln(1+4\theta^2)$
		$4\sum_{\nu=0}^{\infty}\frac{(-1)^{\nu}2^{2\nu}}{(\nu+1)(2\nu+1)}\theta^{2\nu}$
0	1	$-(3/\theta^3)\tan^{-1}2\theta+(1/2\theta^4-1/\theta^2)\ln(1+4\theta^2)+4/\theta^2$
		$4\sum_{\nu=1}^{\infty}\frac{(-1)^{\nu}\nu 2^{2\nu}}{(\nu+1)(\nu+2)(2\nu+3)}\theta^{2\nu}$
1	1	$\frac{4}{5\theta}\tan^{-1}2\theta-\left(\frac{3}{40\theta^6}+\frac{1}{2\theta^4}+\frac{1}{\theta^2}\right)\ln(1+4\theta^2)+\frac{3}{10\theta^4}+\frac{7}{5\theta^2}$
		$4\sum_{\nu=2}^{\infty}\frac{(-1)^{\nu}(\nu-1)\nu 2^{2\nu}}{(\nu+1)(\nu+2)(\nu+3)(2\nu+1)}\theta^{2\nu}$
0	2	$\left(\frac{35}{4\theta^5}-\frac{10}{\theta^3}\right)\tan^{-1}2\theta+\left(-\frac{7}{8\theta^6}+\frac{15}{2\theta^4}-\frac{1}{\theta^2}\right)\ln(1+4\theta^2)+\left(-\frac{14}{\theta^4}+\frac{19}{3\theta^2}\right)$
		$4\sum_{\nu=2}^{\infty}\frac{(-1)^{\nu}(\nu-1)\nu(2\nu-1)2^{2\nu}}{(\nu+1)(\nu+2)(\nu+3)(2\nu+3)(2\nu+5)}\theta^{2\nu}$
1	2	$-\frac{7}{\theta^3}\tan^{-1}2\theta+\left(-\frac{1}{16\theta^8}+\frac{5}{8\theta^6}+\frac{3}{\theta^4}-\frac{1}{\theta^2}\right)\ln(1+4\theta^2)+\left(-\frac{1}{4\theta^6}-\frac{2}{\theta^4}+\frac{17}{3\theta^2}\right)$
		$4\sum_{\nu=3}^{\infty}\frac{(-1)^{\nu}(\nu-2)(\nu-1)\nu 2^{2\nu}}{(\nu+1)(\nu+2)(\nu+3)(\nu+4)(2\nu+3)}\theta^{2\nu}$
2	2	$\frac{4}{9\theta}\tan^{-1}2\theta+\left(-\frac{35}{9\times 2^7\theta^{10}}-\frac{5}{16\theta^8}-\frac{9}{8\theta^6}-\frac{5}{3\theta^4}-\frac{1}{\theta^2}\right)\ln(1+4\theta^2)+\left(\frac{35}{288\theta^8}+\frac{145}{144\theta^6}+\frac{143}{54\theta^4}+\frac{43}{18\theta^2}\right)$
		$4\sum_{\nu=4}^{\infty}\frac{(-1)^{\nu}(\nu-3)(\nu-2)(\nu-1)\nu 2^{2\nu}}{(\nu+1)(\nu+2)(\nu+3)(\nu+4)(\nu+5)(2\nu+1)}\theta^{2\nu}$

The numerical values of the coefficients needed for these series have been calculated and are given in Table II.

Calculation of the Resistance

The potential in completely expanded form is

$$V=\frac{2V_0\epsilon}{\pi}\int_0^{\infty}d\mu\,\frac{\sinh(\mu z/2a)}{\cosh(\mu/2)}J_0\!\left(\frac{\rho\mu}{2a}\right)\int_0^1 dx_1\,(\cos\epsilon\mu x_1)F(x_1).$$

Then differentiating with respect to z

$$E_z=-\frac{2V_0}{\pi}\frac{\epsilon}{2a}\int_0^{\infty}\mu d\mu\,J_0\!\left(\frac{\rho\mu}{2a}\right)\int_0^1 dx_1\,(\cos\epsilon\mu x_1)F(x_1).$$

TABLE II. Numerical values of coefficients.

$\Gamma_{00,0}=2.7726$	$\Gamma_{00,2}=-2.0738$	$\Gamma_{00,4}=4.1476$
$\Gamma_{02,0}=0.4808$	$\Gamma_{02,2}=\ 1.481$	
$\Gamma_{04,0}=0.09875$		

The flux of σE over the disk is I, and so

$$I=\sigma\int_0^c 2\pi\rho d\rho\,E_z$$

$$=-\frac{2V_0}{c}\int_0^c\rho d\rho\int_0^{\infty}\mu d\mu\,J_0\!\left(\frac{\rho\mu}{2a}\right)\int_0^1 dx_1\,(\cos\epsilon\mu x_1)F(x_1).$$

Defining a resistance function $1/\mathcal{R}=-I/4\sigma V_0 c$, and new integrating variables $\rho/c=x$, $\epsilon\mu=\nu$, and introducing the Legendre expansion for $F(x_1)$,

$$\frac{1}{\mathcal{R}}=\sum_{m=0}^{\infty}a_2(\epsilon)\int_0^1 xdx\int_0^{\infty}\nu d\nu\,J_0(\nu x)\int_0^1 dx_1\,(\cos\nu x_1)P_{2m}(x_1).$$

We can now perform the indicated integrations in any order, with the result that

$$1/\mathcal{R}=a_0(\epsilon).$$

It is a remarkable physical fact that only the coefficient a_0 influences the resistance, and for small ϵ,

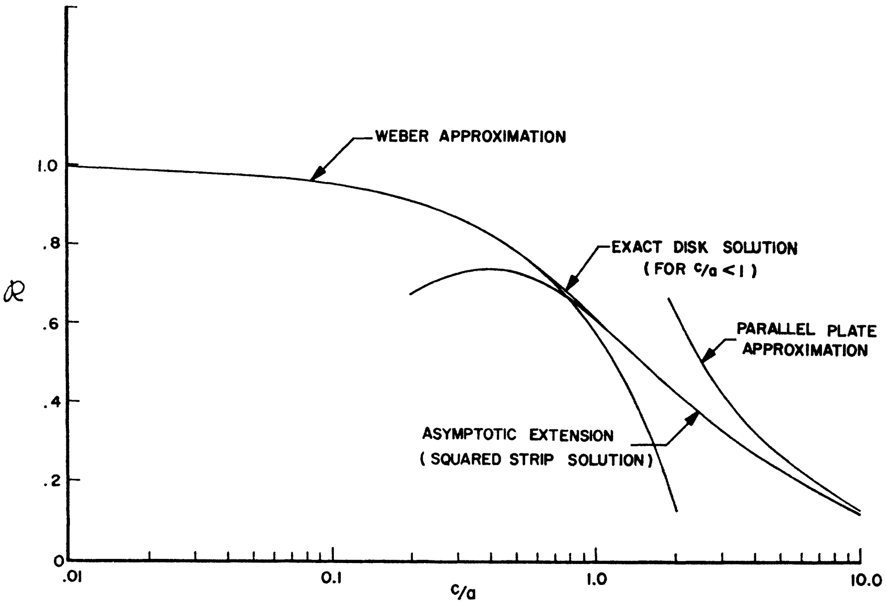


FIG. 6. The resistance function for disk electrodes, showing how the various approximations and the exact solution compare.

it deviates from the Weber solution only in the third order. Thus, the resistance proper as given by Weber was

$$R=(1/2\sigma c)[1-(2c/\pi a)\ln 2],$$

while for the above method, to second order it is

$$R=\frac{1}{2\sigma c}\left[1+\frac{2c}{\pi a}\ln 2+\left(\frac{4c}{\pi a}\ln 2\right)^2\right]^{-1}.$$

Comments on the Result

It becomes increasingly difficult to evaluate the disk solution as ϵ exceeds $\frac{1}{2}$, although it must certainly approach the simple solution for two parallel infinite plates. We believe, however, that by combining the disk solution for small ϵ with a suitable modification of the two-dimensional strip solution, this connection can be established adequately for practical applications. One notes that in the case of the latter, as $c/a\rightarrow\infty$, R varies as $(c/a)^{-1}$. For infinite parallel disks, R must be given by $a/\sigma\pi c^2$. Thus, for constant a, as c increases toward infinity the function $\mathcal{R}$ should approach

$$\mathcal{R}=(4c/a)(K'/2K)^2.$$

The exact function $\mathcal{R}=1/a_0$ is plotted in Fig. 6 over its known range, together with the asymptotic extension expression above, and the Weber approximation and parallel-plate approximation, these last being for comparison purposes only.

That intuition deriving from Ohm's law is a failure in extensive-medium problems like these is shown by the fact that the resistance between finite disk probes on opposite sides of an infinitely thick slab of infinite lateral extent is finite, while the resistance between infinitely long strips of the same width, under like placement, is infinite. The reason can be found in the logarithmic behavior of the potential in the two-dimensional case, and it is instructive to compute the resistance between two parallel cylinders, infinitely separated, which leads to

$$R=(1/2\pi\sigma)\ln[a/c+(a^2/c^2-\tfrac{1}{4})^{1/2}],$$

and displays this logarithmic singularity of the resistance clearly.

Note added in proof. C. J. Tranter, Quart. J. Mech. Appl. Math. **3**, 411 (1950), has presented a solution of this problem by a somewhat different technique, which leads also to the small-ϵ approximation we have given above. Our use of the Legendre polynomial expansion seems to provide a more rapid convergence for large values of ϵ.