Dispersion Relation for the Axial-Vector Vertex and a Sum Rule for the Axial-Vector Coupling-Constant Renormalization*

H. SUURA AND L. M. SIMMONS, JR.
School of Physics, University of Minnesota, Minneapolis, Minnesota
(Received 1 April 1966)

A dispersion relation for the nucleon axial-vector vertex, in which the subtraction term at infinite nucleon mass is given by the unrenormalized vertex, is derived from the axial current-field commutation relations. Using the hypothesis of partially conserved axial-vector current, a sum rule for the axial-vector coupling-constant renormalization g_A is obtained in terms of the π-N form factor. In the elastic-unitarity approximation, the sum rule involves the integral over the difference of the P_{11} and S_{11} phase shifts. Certain experimental phase shifts up to 1-GeV pion kinetic energy give $g_A \approx 1.1$. However, because of the lack of consistent experimental data beyond 500 MeV, and also because of the lack of reliable theory to estimate the important contribution from inelastic channels, all we can conclude at the moment is that the sum rule is not inconsistent with the experimental data.

I. INTRODUCTION

THE sum rule for the axial-vector coupling constant in β decay has been derived by Adler[1] and Weisberger[2] using Gell-Mann's current algebra[3] and the hypothesis of partially conserved axial-vector current[3] (PCAC). The success of the sum rule in giving an excellent numerical value has been a big stimulus to the opening of a new chapter in physics called "current algebra dynamics." Although the new technique is powerful and promising, one has yet to see how it fits with, or goes beyond the conventional formulation of field theory. In an attempt to understand the new technique in relation to field theory, we pointed out in our previous paper that the current-current commutator in the Adler-Weisberger sum rule can be interpreted as the bare Born term in a certain amplitude and that it serves as the subtraction constant at infinity in the dispersion relation for the amplitude. We also pointed out that in a similar way we could use the dispersion relation for the axial-vector vertex with the subtraction constant at infinity given by the bare vertex to derive a sum rule for the axial-vector coupling-constant renormalization. In this paper, we will give a somewhat detailed derivation of the sum rule, which can be expressed in a simple form in terms of the pion form factor using PCAC (Sec. II).

In Sec. III, we evaluate the pion form factor using the phase representation. In the elastic region, the phase of the pion form factor is given by the difference of the P_{11} and S_{11} phase shifts. The trouble lies in the inelastic region where the absorption in the P_{11} channel is enormous, so that the use of experimental phase shifts needs justification. A simple analysis of the inelastic channel contribution is given, and it is shown that the phase of the form factor is sensitive to the phase of the inelastic contribution.

Inasmuch as this latter phase is unknown, we were forced to neglect it, and to evaluate our sum rule purely on the basis of the rather poorly known elastic phase shifts.

If we use the elastic phase shifts from the Saclay analysis (see Sec. III) and integrate up to a pion laboratory kinetic energy of 1 GeV we obtain $g_A \approx 1.11$, a result not inconsistent with experiment.

II. DERIVATION OF THE SUM RULE

We begin with our definition of the vector and axial-vector β-decay currents: $V_\mu{}^a(x)$ and $A_\mu{}^a(x)$ $(a=1,2,3)$. We assume,[4] in place of current commutators,

$$\begin{aligned}\delta(x_0)[V_\mu{}^a(x),\bar{\psi}(0)]\gamma_0 &= G_0\bar{\psi}(0)\gamma_\mu(\tau_a/2)\delta^4(x)\,,\\ \delta(x_0)[A_\mu{}^a(x),\bar{\psi}(0)]\gamma_0 &= G_0\bar{\psi}(0)\gamma_\mu\gamma_5(\tau_a/2)\delta^4(x)\,,\end{aligned} \tag{1}$$

where G_0 is the unrenormalized universal coupling constant. The nucleon operator $\psi(x)$ (we neglect the neutron-proton mass difference) can be either a renormalized or an unrenormalized operator, as long as they differ only by the common wave function renormalization $Z_2{}^{1/2}$. (This situation no longer holds in case a refers to the unitary spin, as Z_2 has different values for different baryon masses.) For definiteness we will use the *unrenormalized* operator in the following. A model which gives Eq. (1) is, for instance,

$$V_\mu{}^a+A_\mu{}^a=G_0\bar{\psi}(x)\gamma_\mu(1+\gamma_5)(\tau_a/2)\psi(x) +(V'^a+A'^a)\,, \tag{2}$$

where $V'^a+A'^a$ is assumed to commute with $\psi(x)$ at equal times. The observed Fermi and Gamow-Teller coupling constants of β decay, G_V and G_A, are given by the one-nucleon matrix elements of $V_\mu{}^a$ and $A_\mu{}^a$.

$$\begin{aligned}\langle p|V_\mu{}^a(0)|p\rangle &= G_V\bar{u}(p)\gamma_\mu(\tau_a/2)u(p)\,,\\ \langle p|A_\mu{}^a(0)|p\rangle &= G_A\bar{u}(p)\gamma_\mu\gamma_5(\tau_a/2)u(p)\,.\end{aligned} \tag{3}$$

The vector-coupling constant is not renormalized because

* This work was supported in part by the U. S. Atomic Energy Commission.

[1] S. L. Adler, Phys. Rev. Letters **14**, 1051 (1965); Phys. Rev. **137**, B1022 (1965); **139**, B1638 (1965); **140**, B736 (1965).

[2] W. I. Weisberger, Phys. Rev. Letters **14**, 1047 (1965); Phys. Rev. **143**, 1302 (1966).

[3] For an extensive listing of the relevant literature see Refs. 1 and 2.

[4] Our conventions are: $a\cdot b=a^\mu b_\mu=a^0b^0-\mathbf{a}\cdot\mathbf{b}$, $\{\gamma^\mu,\gamma^\nu\}_+=2g^{\mu\nu}$, $g^{00}=-g^{ii}=1$, $\gamma_\mu{}^\dagger=\gamma_0\gamma_\mu\gamma_0$, $\gamma_5=-i\gamma_0\gamma_1\gamma_2\gamma_3$. Plane-wave normalizations are such that the density of states is $(2\pi)^{-3}(m/p_0)d^3p$ for fermions and $(2\pi)^{-3}(2q_0)^{-1}d^3q$ for bosons.

of the strict current conservation, and we have

$$G_V = G_0. \tag{4}$$

Now we consider the axial-vector vertex part, as described in Fig. 1. The nucleon of momentum p is on the mass shell, $p^2=m^2$, and the momentum transfer q will be held at $q^2=0$ throughout. Our variable is then the off-shell nucleon mass

$$W = [(p+q)^2]^{1/2}. \tag{5}$$

This is analogous to the vector vertex considered by Bincer,[5] and by Drell and Pagels.[6] We define the axial-vector vertex $\Gamma_\mu{}^a(p, p+q)$ by

$$\bar{u}(p)\Gamma_\mu{}^a(p, p+q) = iZ_2^{-1/2}\int \langle p|(A_\mu{}^a(0),\bar{\psi}(x))_+|0\rangle \overleftarrow{D}_x e^{-i(p+q)x}dx, \tag{6}$$

where $\overleftarrow{D}_x = (i\gamma\cdot\overleftarrow{\partial}+m)$. In the limit $W\to m$, we have

$$\bar{u}(p)\Gamma_\mu{}^a(p,p)u(p) = \langle p|A_\mu{}^a(0)|p\rangle. \tag{7}$$

To express $\Gamma_\mu{}^a$ in terms of invariant amplitudes, we can use either of the two following projection operators,

$$P_\pm(p) \equiv (\pm\gamma\cdot p+m)/2m, \tag{8}$$

or

$$\Lambda_\pm(p) \equiv (\pm\gamma\cdot p+\sqrt{p^2})/2\sqrt{p^2}. \tag{9}$$

Correspondingly, we write

$$\begin{aligned}\bar{u}(p)\Gamma_\mu{}^a(p, p+q) = \bar{u}(p)\gamma_\mu\gamma_5(\tau_a/2) &\times[F_+(W^2)P_+(p+q) \\ &+F_-(W^2)P_-(p+q)]+\cdots,\end{aligned} \tag{10}$$

$$\begin{aligned}\bar{u}(p)\Gamma_\mu{}^a(p, p+q) = \bar{u}(p)\gamma_\mu\gamma_5(\tau_a/2) &\times[G_+(W)\Lambda_+(p+q) \\ &+G_-(W)\Lambda_-(p+q)]+\cdots,\end{aligned} \tag{11}$$

where the dotted parts contain $q_\mu\gamma_5$ and $\gamma_5\sigma_{\mu\nu}q^\nu$ terms, which do not contribute in the following derivation. Note that $F_\pm(W^2)$ are the boundary values of functions analytic in the complex W^2 plane with a branch cut along the positive real axis, while $G_\pm$ do not have this property because of W in $\Lambda_\pm(p+q)$. They are related by

$$\begin{aligned}G_\pm(W) = (1/2m)[&(m\pm W)F_+(W^2+i\epsilon) \\ &+(m\mp W)F_-(W^2+i\epsilon)].\end{aligned} \tag{12}$$

The following derivation is somewhat neater if we work in terms of $G_\pm$ rather than $F_\pm$ which we used previously.[7]

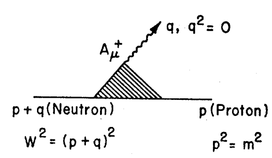

FIG. 1. Axial-vector vertex.

[5] A. M. Bincer, Phys. Rev. **118**, 855 (1960).
[6] S. D. Drell and H. R. Pagels, Phys. Rev. **140**, B397 (1965).
[7] H. Suura and L. M. Simmons, Jr., Phys. Rev. Letters **16**, p. 598 (1966).

We can continue $G_+(W)$ defined for $W>0$ into the whole complex W plane by

$$\begin{aligned}G(W) = (1/2m)[&(W+m)F_+(W^2) \\ &+(m-W)F_-(W^2)],\end{aligned} \tag{13}$$

which now has singularities along the positive and negative real axis, where

$$\begin{aligned}G(W+i\epsilon) &\equiv G_+(W), \quad W>0 \\ &\equiv G_-{}^*(|W|), \quad W<0.\end{aligned} \tag{14}$$

From Eqs. (3), (7), and (13), we have

$$G(m) = G_+(m) = F_+(m^2) = G_A. \tag{15}$$

To project out $G(W)$, we multiply Eq. (11) by q_μ and a projection operator

$$\begin{aligned}\Lambda(W) = [\gamma\cdot(p+q)+W]/2W &= \Lambda_+(p+q), \quad W>0 \\ &= \Lambda_-(p+q), \quad W<0,\end{aligned} \tag{16}$$

and obtain

$$\begin{aligned}\Gamma^a(W) &\equiv -q^\mu\bar{u}(p)\Gamma_\mu{}^a(p, p+q)\Lambda(W) \\ &= (W+m)G(W)\bar{u}(p)\gamma_5(\tau_a/2)\Lambda(W),\end{aligned} \tag{17}$$

from which we have a condition

$$\Gamma^a(-m) = 0. \tag{18}$$

On the other hand, performing $\overleftarrow{D}_x$ in Eq. (10) and also converting q_μ into ∂_μ, we reduce $\Gamma^a(W)$ to

$$\begin{aligned}\Gamma^a(W) = -Z_2^{-1/2}\int e^{iq\cdot x}\{&q^\mu\delta(x_0)\langle p|[A_\mu{}^a(x),\bar{\psi}(0)]|0\rangle \\ &+\delta(x_0)\langle p|[A_0{}^a(x),\bar{f}(0)]|0\rangle\}\Lambda(W)+\tilde{\Gamma}^a(W),\end{aligned} \tag{19}$$

$$\tilde{\Gamma}^a(W) = Z_2^{-1/2}\int e^{iq\cdot x} \times\langle p|(\partial^\mu A_\mu{}^a(x),\bar{f}(0))_+|0\rangle\Lambda(W), \tag{20}$$

where $\bar{f}(x) = \bar{\psi}(x)\overleftarrow{D}_x$. The first commutator gives, from (1) and $\langle p|\bar{\psi}(0)|0\rangle = Z_2{}^{1/2}\bar{u}(p)$,

$$\Gamma_0{}^a(W) = G_0(W+m)\bar{u}(p)\gamma_5(\tau_a/2)\Lambda(W). \tag{21}$$

The second commutator gives a constant independent of W, which we must cancel by the subtraction constant for $\tilde{\Gamma}^a(W)$ at $W=-m$ to satisfy (18). Thus, denoting by $A(\tilde{\Gamma}^a)$ the absorptive part of $\tilde{\Gamma}^a$,

$$\Gamma^a(W) = \Gamma_0{}^a(W)+\frac{W+m}{\pi}\times\int_{-\infty}^{\infty}\frac{A(\tilde{\Gamma}^a(W'))}{(W'+m)(W'-W)}dW'. \tag{22}$$

The absorptive part of $\tilde{\Gamma}^a$ is related by PCAC to that of the π-N vertex, defined by

$$\bar{u}(p)\Gamma_5{}^a(p,p+q)=iZ_2{}^{-1/2}\int\langle p|(j_\pi{}^a(0),\bar{\psi}(x))_+|0\rangle\overleftarrow{D}_x e^{-i(p+q)\cdot x}dx\,, \quad (23)$$

where

$$(\Box+\mu^2)\phi_\pi{}^a=j_\pi{}^a. \quad (24)$$

Corresponding to (11), we write

$$\bar{u}(p)\Gamma_5{}^a(p,p+q)=i\bar{u}(p)\gamma_5\tau_a[K_+(q^2,W)\Lambda_+(p+q)+K_-(q^2,W)\Lambda_-(p+q)].$$

Again we can define a real analytic function $K(q^2,W)$, in the whole complex W plane, such that

$$\begin{aligned}K(q^2,W+i\epsilon)&=K_+(q^2,W)\quad W>0\\&=K_-{}^*(q^2,|W|)\quad W<0.\end{aligned} \quad (25)$$

In terms of $\Lambda(W)$ defined by (16), we can write, as in (17),

$$\begin{aligned}\Gamma_5{}^a(W)&\equiv\bar{u}(p)\Gamma_5{}^a(p,p+q)\Lambda(W)\\&=K(q^2,W)i\bar{u}(p)\gamma_5\tau_a\Lambda(W).\end{aligned} \quad (26)$$

By PCAC[8]

$$\partial^\mu A_\mu{}^a(x)=(G_Am\mu^2/K(0,m))\phi_\pi{}^a(x)\,, \quad (27)$$

we can relate $\tilde{\Gamma}^a$ of (20) to $\Gamma_5{}^a$ of (23) and (26). Thus,

$$\begin{aligned}A(\tilde{\Gamma}^a(W))&=(G_Am/K(0,m))A(-i\Gamma_5{}^a(0,W))\\&=(G_Am/K(0,m))\,\mathrm{Im}K(0,W)\bar{u}(p)\gamma_5\tau_a\Lambda(W).\end{aligned} \quad (28)$$

Inserting (28), (17), and (21) into (22), we obtain

$$G(W)=G_0+\frac{G_A}{K(0,m)}\frac{2m}{\pi}\times\int_{-\infty}^{\infty}\frac{\mathrm{Im}K(0,W')}{(W'+m)(W'-W)}dW'. \quad (29)$$

Taking $W=m$, using (15), (4), and the conventional notation $g_A=G_A/G_V$, we have the sum rule

$$\frac{1}{g_A}=1-\frac{1}{K(0,m)}\frac{2m}{\pi}\int_{-\infty}^{\infty}\frac{\mathrm{Im}K(0,W')}{(W'^2-m^2)}dW'. \quad (30)$$

If $K(0,W)$ satisfies a dispersion relation with at most one subtraction, the last term is simply

$$-[K(0,m)-K(0,-m)]/K(0,m)\,,$$

so that[9]

$$g_A=K(0,m)/K(0,-m). \quad (31)$$

If we assume that $K(0,W)$ has no zero, and satisfies a phase representation

$$K(0,W)=K(0,0)\exp\left[\frac{W}{\pi}\int_{-\infty}^{\infty}\frac{\varphi(W')}{W'(W'-W)}dW'\right], \quad (32)$$

then

$$g_A=\exp\left[\frac{2m}{\pi}\int_{m+\mu}^{\infty}\frac{\varphi(W')+\varphi(-W')}{W'^2-m^2}dW'\right]. \quad (33)$$

III. EVALUATION OF THE SUM RULE

In order to evaluate the sum rule (31), we first replace $K(0,W)$ by $K(\mu^2,W)$ [for simplicity denoted as $K(W)$], so that we can use experimental π-N scattering phase shifts. The phase representation for $K(W)$ has been worked out by Bincer[5] and Ida[10] in the elastic approximation, but we will give a brief derivation considering the effect of the inelastic channels. From (23) and (24)

$$\begin{aligned}\mathrm{Im}K_\pm(W)i\bar{u}(p)\gamma_5\tau_\alpha\Lambda_\pm(p+q)&=\tfrac{1}{2}(2\pi)^4\textstyle\sum_\alpha\langle p|j_\pi{}^a(0)|\alpha_{\rm in}\rangle\delta(p+q-p_\alpha)\\&\quad\times Z_2{}^{-1/2}\langle\alpha_{\rm in}|\bar{f}(0)|0\rangle\Lambda_\pm(p+q).\end{aligned} \quad (34)$$

Taking π-N intermediate states $|p'q_b'\rangle$ we find

$$\begin{aligned}Z_2{}^{-1/2}\langle p'q_b'{}^{\rm out}|\bar{f}(0)|0\rangle&=iZ_2{}^{-1/2}Z_3{}^{-1/2}(\pi)\\&\quad\times\int\langle p'|(j_\pi{}^a(0),\bar{\psi}(x))_+|0\rangle\overleftarrow{D}_x e^{-i(p+q)\cdot x}dx\,,\end{aligned}$$

where $Z_3(\pi)$ is the pion renormalization constant. Hence, from (23) and the usual substitution law we have

$$\begin{aligned}Z_2{}^{-1/2}\langle p'q_b'{}^{\rm in}|\bar{f}(0)|0\rangle\Lambda_\pm(p+q)\\=Z_3{}^{-1/2}(\pi)K_\pm{}^*(W)i\bar{u}(p)\gamma_5\tau_b\Lambda_\pm(p+q).\end{aligned} \quad (35)$$

$Z_3{}^{-1/2}(\pi)\langle p|j_\pi{}^a(0)|p'q_b'{}^{\rm in}\rangle$ is the π-N scattering amplitude, which we expand into partial waves after Pauli reduction

$$\begin{aligned}Z_3{}^{-1/2}(\pi)\langle p|j_\pi{}^a(0)|p'q_b'{}^{\rm in}\rangle&=\sum_{l=0}^{\infty}(2l+1)\langle X|(T_{l+}{}^{ab}P_{l+}+T_{l-}{}^{ab}P_{l-})|X'\rangle\\&\quad\times P_l(\cos\theta_{pp'}).\end{aligned} \quad (36)$$

Here $|X\rangle$ and $|X'\rangle$ are spin functions, and

$$\begin{aligned}P_{l\pm}&=(l+1+\sigma\cdot L)/(2l+1)\\&=(l-\sigma\cdot L)/(2l+1)\end{aligned}$$

---

[8] From (13), $\langle p|\partial^\mu A_\mu(0)|p\rangle=imG_A\bar{u}(p)\gamma_5\tau_a u(p)$. On the other hand, $\langle p|\phi^a(0)|p\rangle=\mu^{-2}\langle p_a|j_\pi{}^a(0)|p\rangle$, which by (22) and (25) is equal to $\bar{u}(p)\Gamma_5(p,p)u(p)=iK(0,m)\bar{u}(p)\gamma_5\tau_a u(p)$. Comparing these two expressions we obtain the coefficient of (26). With our definition of $\phi_\pi{}^a$ as the unrenormalized operator, we have $K(0,m)=Z_3{}^{1/2}(\pi)g_\pi$, but the normalization of K does not matter in our sum rule.

[9] This result has also been obtained by S. L. Adler (private communication), who has proved its equivalence to the equation derived by J. Bernstein, M. Gell-Mann, and L. Michel, Nuovo Cimento **16**, 560 (1960).

[10] M. Ida, Phys. Rev. **136**, B1767 (1965).

are projection operators into $j=l\pm\frac{1}{2}$. Introducing (35) and (36) into (34), and reducing $\bar{u}(p')\gamma_5\Lambda_\pm(p+q)$ into nonrelativistic form, we can easily perform the angular integration over the direction of p', and the summation over spin states X' and over the isospin. The isospin sum projects the $I=\frac{1}{2}$ amplitudes $T_{l\pm}^{1/2}$ out of $T_{l\pm}^{ab}$. The angular integration projects out $T_{1-}^{1/2}$ for Λ_+ and $T_{0+}^{1/2}$ for Λ_-. We obtain

$$\begin{aligned}\mathrm{Im}K_+(W)&=(mp/4\pi W)T_{1-}^{1/2}(W)K_+^*(W)+R_+(W)\,,\\ \mathrm{Im}K_-(W)&=(mp/4\pi W)T_{0+}^{1/2}(W)K_-^*(W)+R_-(W)\,,\end{aligned} \tag{37}$$

where $R_\pm(W)$ represents the contribution from the inelastic channels.[11] With our definition of $T_{l\pm}$ by (36), it is normalized to

$$T_{l\pm}(W)=(4\pi W/mp)(\eta_{l\pm}e^{2i\delta_{l\pm}}-1)/2i\,, \tag{38}$$

where $\eta_{l\pm}$ is the usual absorption parameter and $\delta_{l\pm}$ is the phase shift. We write

$$R_\pm(W)=\gamma_\pm(W)e^{i\alpha_\pm(W)}\,, \tag{39}$$

where $\gamma_\pm=|R_\pm|$. Equation (37) imposes a rather strong condition on $R_\pm(W)$, as the right-hand sides must be real. Using (25), (34), and (37) we obtain

$$\begin{aligned}e^{2i\varphi(W)}&=K_+^*/K_+\\ &=[\eta_{P_{11}}e^{2i\delta_{P_{11}}}-e^{2i\alpha_+}]/[1-\eta_{P_{11}}e^{-2i(\delta_{P_{11}}-\alpha_+)}]\,,\\ e^{2i\varphi(-W)}&=K_-^*(|W|)/K_-(|W|)\\ &=[1-\eta_{S}e_{11}^{-2i(\delta_{S_{11}}-\alpha_-)}]/[\eta_{S_{11}}e^{2i\delta_{S_{11}}}-e^{2i\alpha_-}].\end{aligned} \tag{40}$$

We have employed the usual notation S_{11}, P_{11} for the $l=0,1$ states with $J=\frac{1}{2}$, $T=\frac{1}{2}$. In the elastic region, $\eta=1$, and we have

$$\begin{aligned}\varphi(W)&=\delta_{P_{11}}(W)\,, && W>0\\ &=-\delta_{S_{11}}(|W|)\,, && W<0.\end{aligned} \tag{41}$$

The sum rule (33) can be written as

$$g_A=\exp\left[\frac{2m}{\pi}\int_{m+\mu}^{\infty}dW'\frac{\delta_{P_{11}}(W')-\delta_{S_{11}}(W')}{W'^2-m^2}\right]\times\exp\left[\frac{2m}{\pi}\int_{W_i}^{\infty}dW'\frac{\tilde{\varphi}(W')+\tilde{\varphi}(-W')}{W'^2-m^2}\right], \tag{42}$$

where W_i is the inelastic threshold and

$$\begin{aligned}\tilde{\varphi}(W')&=\varphi(W')-\delta_{P_{11}}(W')\,, && W'>0\\ &=\varphi(W')-\delta_{S_{11}}(W')\,, && W'<0.\end{aligned} \tag{43}$$

If the absorption in the P_{11} and S_{11} states were small ($\eta\approx1$), the correction $\tilde{\varphi}(W)$ would be small, as we see from (38). As discussed below, however, both $\eta_{P_{11}}$ and $\eta_{S_{11}}$ become rather small at intermediate energies. This does not necessarily exclude the possibility that $\tilde{\varphi}$ is small. This would be the case, for example, if $\alpha_+\approx\delta_P-\frac{1}{2}\pi$ and $\alpha_-\approx\delta_S-\frac{1}{2}\pi$. This requires a strong coherence between elastic and inelastic contributions. In any case, a great amount of theoretical work, beyond the scope of this paper, will be required to draw a definite conclusion about the inelastic contribution in Eq. (42). For simplicity, we shall neglect the correction factor involving $\tilde{\varphi}$ and use

$$g_A\approx\exp\left[\frac{2m}{\pi}\int_{M+\mu}^{W^L}dW'\frac{\delta_{P_{11}}(W')-\delta_{S_{11}}(W')}{W'^2-M^2}\right]. \tag{44}$$

The upper limit W^L is imposed by certain practical considerations. Several large-scale π-N phase-shift analyses[12–15] have recently appeared. Although they agree with one another on many points, they do not agree in all details for the states of interest here. This is partly due to the fact that for both states η becomes small, making the cross section insensitive to δ. We may summarize the situation as follows.[16]

(i) For pion laboratory kinetic energy $T^\pi\lesssim250$ MeV, $\delta_{P_{11}}-\delta_{S_{11}}<0$.

(ii) $\delta_{P_{11}}-\delta_{S_{11}}>0$ from about 250 MeV up to at least 1 GeV.

(iii) Both phase shifts become "large" (possibly exceeding $\frac{1}{2}\pi$) in this energy range.

(iv) All four of these analyses give values of $\delta_{P_{11}}-\delta_{S_{11}}$ which are in rough agreement up to about 500 MeV. Beyond this point they do not agree.

If we use the results of the Livermore analysis[11] in (42) with W^L given by $T_\pi^L\approx500$ MeV, the integral is approximately zero and we obtain $g_A\approx1$. (For the Saclay data[14] we need $T_\pi^L\approx550$ MeV to get the same results). Thus, all of the renormalization comes effectively from the region $T_\pi\gtrsim500$ MeV and we cannot hope for a reliable determination of g_A. If we employ the Saclay results[14] and integrate up to their limit, $T_\pi\approx1$ GeV, Eq. (42) yields $g_A\approx1.11$. At $T_\pi=1$ GeV the integrand of (42) is positive and non-negligible so it appears that a larger value of g_A would be obtained if it were possible to extend the integration to higher energy. In view of the theoretical and experimental uncertainties involved, we can only say that our sum rule is not in disagreement with the known facts.

[11] The first named author acknowledges a useful conversation on this subject with Professor D. A. Geffen.

[12] L. D. Roper, R. M. Wright, and B. T. Feld, Phys. Rev. 138, B190 (1965).

[13] P. Auvil, A. Donnachie, A. T. Lea, and C. Lovelace, Phys. Letters 12, 76 (1964).

[14] B. H. Bransden, P. J. O'Donnell, and R. H. Moorhouse, Phys. Rev. 139, B1566 (1965).

[15] P. Bareyre, C. Bricman, A. V. Stirling, and G. Villet, Phys. Letters 18, 342 (1965).

[16] The current states of π-N phase shift analyses has recently been summarized by L. D. Roper at the Willamsburg Conference on Intermediate Energy Physics, 1966 (unpublished). We are grateful to Dr. Roper for a copy of his address.