Internal Symmetries in a Coupled-Channel Soluble Model with Inelasticity*

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(Received 25 February 1966)

The conjecture that an internal symmetry group may be selected by a bootstrap mechanism is studied within the framework of a closed, exactly soluble model. In order to have two-body unitarity and crossing without necessarily identically zero amplitudes, two types of particles of different mass (each possessing internal quantum numbers corresponding to an unspecified internal symmetry group and each assigned to irreducible representations of this group) are allowed to scatter in a two-dimensional world. The symmetry group makes its appearance explicitly only through the crossing matrix, characterized by a parameter which is to be determined self-consistently. All two-body channels, both elastic and inelastic, are treated exactly. Unitary, crossing-symmetric, analytic scattering amplitudes corresponding to the various processes are constructed for continuous ranges of the parameter of the crossing matrix. Even with the additional constraint of a self-consistency requirement in the form of Levinson's theorem, an internal symmetry group is not selected by the bootstrap mechanism in this model. Also, a technique is developed for converting a coupled set of singular, linear Cauchy integral equations into an equivalent uncoupled Fredholm set.

I. INTRODUCTION

BECAUSE of the reasonable success of some of the early bootstrap calculations,[1] numerical ratios of coupling constants produced by crude N/D calculations[2] have sometimes been interpreted as indicating that a particular internal symmetry group is favored, or possibly chosen uniquely, by the strong interactions.[3] The general philosophy is that a particular symmetry group may be the only one compatible with a unitary, crossing-symmetric, self-consistent set of scattering amplitudes. The hope is that the nonlinear, coupled integral equations may be sufficiently complicated to impose stringent constraints on any allowed internal symmetries. However, in practice, although elastic unitarity is ensured by use of the N/D method in bootstrap calculations, the crossing symmetry is badly violated in constructing approximate solutions, so that it is unclear to what extent a symmetry generated in this manner is indicative of a symmetry possessed by the actual scattering amplitudes. It is possible that the apparent symmetry is a result of the approximations made.

Therefore, it is of interest to examine an exactly soluble model possessing as many features of a realistic scattering situation as possible in order to see in detail whether or not the internal symmetry will indeed flow from the enunciated principles as hoped. Certainly the most serious obstacle to a soluble, realistic model is that two-body unitarity in all channels and exact crossing are incompatible.[4] That is, in a four-dimensional world the simultaneous requirements of crossing and of a truncated two-body unitarity condition everywhere above threshold in all channels imply that the resulting scattering amplitude must vanish identically. This result depends very essentially upon the fact that the amplitudes are functions of two independent complex variables and upon relativistic invariance. Therefore, since three-body unitarity would render a realistic problem intractable, the only hope of proving the bootstrap conjecture within the framework of a closed two-body model would appear to be in a two-dimensional world where Aks'[4] theorem does not apply.

Martin and McGlinn[5] have considered a static-limit Chew-Low-type model with two channels and constructed exact solutions for arbitrary values of the parameter in the 2×2 crossing matrix. In other words, their requirements of unitarity, crossing, and analyticity did not suffice for a unique selection of those discrete values of this parameter that correspond to $SU(2)$. More recently, Huang and Low[6] and Huang and Mueller[7] have studied the Chew-Low equation for various theories (i.e., neutral scalar, charged scalar, etc.) in which they impose a bootstrap requirement in the form of Levinson's theorem in addition to those of unitarity, crossing, and analyticity. The basic idea is that within the context of a bootstrap philosophy all particles are simply bound states of other particles, there being no elementary particles. In potential theory Levinson's theorem states that the difference between the phase shift at threshold and that at infinity in a particular channel is equal to π times the number of bound states in that channel. Therefore, since the number of bound states (i.e., poles corresponding to particles) in a given

* Work performed under the auspices of the U. S. Atomic Energy Commission.
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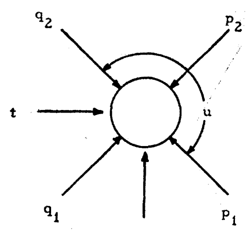

FIG. 1. Two-body scattering diagram.

channel is known for a particular model, the fulfillment of Levinson's theorem can simply be imposed as a bootstrap, or self-consistent, condition. This additional requirement places strong restrictions on the high-energy behavior allowed by the solutions, but practically none on the possible internal symmetry group.

It is the purpose of this paper to investigate whether or not the inclusion of inelastic effects may further constrain any allowed internal symmetry group. Therefore, we have considered a relativistic, two-dimensional model possessing unitary, crossing-symmetric, analytic scattering amplitudes in which all two-particle intermediate states, both elastic and inelastic, have been treated exactly. Finally, the effect of applying the bootstrap criterion, in the form of Levinson's theorem, to these amplitudes has been studied. Also, Appendix C presents what is believed to be a new method for converting a coupled set of singular linear integral equations of the Cauchy type into an equivalent uncoupled set of Fredholm integral equations.

II. DESCRIPTION OF THE MODEL

We begin by embedding our particles in a two-dimensional space-time background. In such a world there can be no space spin, either intrinsic or orbital, since there is only one space dimension. Physically, this is obvious since it is impossible to perform a spatial rotation if there exists only one space dimension, so that there are no generators for the rotations. Mathematically, this is simply a result of the fact that the Lorentz group becomes an Abelian one so that all of its irreducible representations are one-dimensional and, therefore, there exist only spin-zero particles. Lorentz invariance then requires that the scattering amplitudes be scalar functions of the scalar invariants of the system. Since we shall consider only two-body scattering, there are four two-vectors available. These are shown in Fig. 1. We may define the usual invariants

$$\begin{aligned} s&=(p_1+q_1)^2=(p_2+q_2)^2,\\ t&=(q_1+q_2)^2=(p_1+p_2)^2,\\ u&=(q_2+p_1)^2=(q_1+p_2)^2,\\ p_j^2&=M^2,\quad q_j^2=m^2, \end{aligned} \tag{1}$$

and obtain the relation

$$s+t+u=2(M^2+m^2).$$

However, in the center-of-mass system of our *one* space dimension we have

$$\mathbf{p}_1+\mathbf{q}_1=0=\mathbf{p}_2+\mathbf{q}_2,$$

so that $t\equiv 0$. Therefore,

$$s+u=2(M^2+m^2), \tag{2}$$

and there is just one independent variable left to describe the scattering.

For our two-body reactions we define an invariant T-matrix element in terms of the S matrix as

$$\langle \mathbf{p}_1\mathbf{q}_1 i|S|\mathbf{p}_2\mathbf{q}_2 j\rangle =\delta^{(2)}(P_i+P_j)\delta_{ij}+2i\delta^{(2)}(P_i+P_j)T_{ij}(s), \tag{3}$$

where

$$P_i=p_1+q_1,\quad P_j=p_2+q_2, \tag{4}$$

and where i and j are the channel indices for the internal degrees of freedom in the initial and final states, respectively. If we assume time-reversal invariance so that $T_{ij}(s)$ can be chosen symmetric in the indices i and j and integrate over the invariant volume element $d\mathbf{k}_j/k_j^0$, then the unitarity of S implies that

$$\begin{aligned} \delta^{(2)}(P_i+P_j)\,\mathrm{Im}T_{ij}(s) &=\sum_n \int_{s_n}^{\infty} T_{nj}^*(s;k_n^{(1)},k_n^{(2)})T_{in}(s;k_n^{(1)},k_n^{(2)})\\ &\times\delta(k_n^{(1)2}-M_n^2)\delta(k_n^{(2)2}-m_n^2)\\ &\times\theta[s_n-(k_n^{(1)}+k_n^{(2)})^2]\\ &\times\delta^{(2)}(P_i+k_n^{(1)}+k_n^{(2)})\delta^{(2)}(P_j+k_n^{(1)}+k_n^{(2)})\\ &\times d^2k_n^{(1)}d^2k_n^{(2)}, \end{aligned} \tag{5}$$

where the sum over n represents the sum over all possible two-body intermediate states. This can be reduced directly to

$$\begin{aligned} \mathrm{Im}T_{ij}&=\sum_n T_{in}^*(s)\rho_n(s)T_{nj}(s)\theta(s-s_n),\\ \rho_n(s)&=\tfrac{1}{2}\{[s-(M_n+m_n)^2][s-(M_n-m_n)^2]\}^{-1/2}. \end{aligned} \tag{6}$$

In the following we shall consider the scattering of two types of strongly interacting particles, denoted by a and b, having masses m and M, respectively, with $m<M$, and each carrying internal quantum numbers. That is, we assign these particles to multiplets of the internal symmetry group which is not specified further. We shall consider a model in which particle a belongs to a singlet representation and is its own antiparticle while particle b belongs to a doublet representation so that the antiparticle $\bar{b}$ is distinct from b. Graphically, the allowed two-particle scattering processes will be those shown in Fig. 2.

A few comments are in order about Fig. 2. First, all of the particles are drawn as formally ingoing. Lines corresponding to ingoing actual particles have positive timelike two-vectors, while the negative timelike vectors represent outgoing actual antiparticles.[8] Second, the t channel in all cases has been labeled as $t\equiv 0$ to indicate that this channel does not exist in a two-dimensional model. Also connected with this peculiarity of a two-dimensional model is the fact that the amplitudes $F(s,u)$ and $G(s,u)$, corresponding, respectively, to Figs. 2(b) and 2(c), are *distinct* amplitudes—as oppsoed to the situation in a four-dimensional world in which these two diagrams are the same and differ only in the labeling of the s, t, and u channels which all exist. Finally, the amplitude $B^{\alpha}(s,u)$, corresponding to Fig. 2(d), has an index α taking on values which correspond to the quantum numbers of the possible reactions. Since the particle b belongs to a doublet, we would *a priori* expect α to take on *four* values. However, we assume that there are just *two* independent amplitudes $\alpha=1$, 2, which are sufficient to describe the scattering of $b\bar{b}$. If we had specified our internal symmetry group to be, say, $SU(2)$, then this would follow from the charge independence of the strong interactions. It will become apparent later

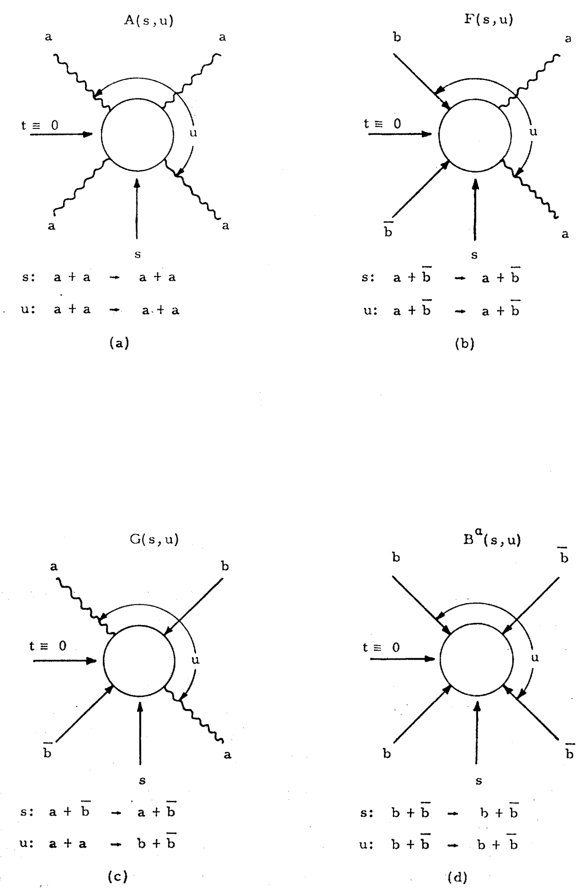

FIG. 2. Two-body processes considered in the model.

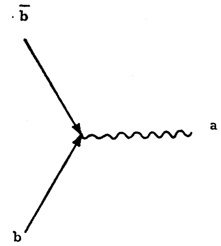

FIG. 3. Basic vertex of the model.

that this simplifying assumption does not invalidate our conclusions.

We shall also assume that the a particle communicates only with the $\alpha=1$ channel of $b\bar{b}$. This would correspond to the vertex of Fig. 3. Obviously, we have in mind the scattering of isotopic-spin-0 from isotopic-spin-$\frac{1}{2}$ particles, although it is precisely this interpretation of the model which we hope will follow necessarily from an analysis of the coupled-channel effects.

We must now apply the general requirements of unitarity, crossing, and analyticity to this model. The unitarity condition (6) locates the threshold singularities for the various amplitudes of Fig. 2. A qualitative sketch of the s plane for these amplitudes is given in Fig. 4. If we define the phase-space factors

$$\rho_1(s)=\tfrac{1}{2}[s(s-4m^2)]^{-1/2}, \tag{7a}$$

$$\rho_2(s)=\tfrac{1}{2}\{[s-(M+m)^2][s-(M-m)^2]\}^{-1/2}, \tag{7b}$$

$$\rho_3(s)=\tfrac{1}{2}[s(s-4M^2)]^{-1/2}, \tag{7c}$$

then the unitarity conditions (6) become the following:

(i) In the direct (i.e., s) channel for $A(s,u)$,

$$\begin{aligned}\operatorname{Im}A(s)=\rho_1(s)A^*(s)A(s)\theta(s-4m^2)&\\ +\rho_3(s)G^*(-s+2M^2+2m^2)&\\ \times G(-s+2M^2+2m^2)\theta(s-4M^2).&\end{aligned} \tag{8}$$

A similar relation holds in the physical region of the u channel since, by crossing,

$$A(4m^2-s)=A(s).$$

(ii) In the direct channel for $F(s,u)$,

$$\operatorname{Im}F(s)=\rho_2(s)[F^*(s)F(s)+G^*(s)G(s)]\times\theta[s-(M+m)^2]. \tag{9}$$

Again, the crossed channel is related simply by the relation

$$F[2(M^2+m)-s]=F(s).$$

(iii) In the direct channel for $G(s,u)$,

$$\operatorname{Im}G(s)=\rho_2(s)[G^*(s)F(s)+F^*(s)G(s)]\times\theta[s-(M+m)^2], \tag{10a}$$

while in the crossed channel

$$\operatorname{Im}G(u)=\rho_1(u)A^*(u)G(u)\theta(u-4m^2)+\rho_3(u)G^*(u)B^1(u)\theta(u-4M^2). \tag{10b}$$

[8] G. F. Chew, *S-Matrix Theory of Strong Interactions* (W. A. Benjamin, Inc., New York, 1961), p. 11.

(iv) In the direct channel for $B^1(s,u)$,

$$\begin{aligned}\mathrm{Im}B^1(s)=&\rho_3(s)B^{1*}(s)B^1(s)\theta(s-4M^2)\\&+\rho_1(s)G^*(-s+2M^2+2m^2)\\&\times G(-s+2M^2+2m^2)\theta(s-4m^2),\end{aligned}\tag{11a}$$

while in the direct channel for $B^2(s,u)$,

$$\mathrm{Im}B^2(s)=\rho_3(s)B^{2*}(s)B^2(s)\theta(s-4M^2).\tag{11b}$$

With the aid of crossing symmetry and an assumed analytic behavior for the various scattering amplitudes, we can write the coupled integral equations describing this model. We shall assume that there are no bound states in either channel of $A(s,u)$, but that there are bound-state poles at M^2 in the s and u channels of $F(s,u)$, a bound-state pole at M^2 in the s channel of $G(s,u)$ and a pole at m^2 in the u channel of $G(s,u)$, poles at m^2 in both channels for $B^1(s,u)$, and, finally, no bound-state poles in either channel of $B^2(s,u)$. Of course, $B^2(s,u)$ has the poles required by crossing. If we assume an unsubtracted dispersion relation sufficient for $A(s,u)$ and apply Cauchy's theorem in the s plane, we obtain

$$A(s)=\frac{1}{\pi}\int_{4m^2}^{\infty}ds'\rho_1(s')|A(s')|^2\left[\frac{1}{s'-s}+\frac{1}{s'+s-4m^2}\right]+\frac{1}{\pi}\int_{4M^2}^{\infty}ds'\rho_3(s')|G(-s'+2M^2+2m^2)|^2\left[\frac{1}{s'-s}+\frac{1}{s'+s-4m^2}\right].\tag{12}$$

If we take account of the bound-state poles at M^2 in each channel of $F(s,u)$, we find

$$F(s)=-\Gamma_1\left[\frac{1}{s-M^2}+\frac{1}{M^2+2m^2-s}\right]+\frac{1}{\pi}\int_{(M+m)^2}^{\infty}ds'\rho_2(s')[|F(s')|^2+|G(s')|^2]\left[\frac{1}{s'-s}+\frac{1}{s'+s-2M^2-2m^2}\right].\tag{13}$$

For $G(s,u)$ we have a pole at M^2 in the s channel and at m^2 in the u channel. Therefore,

$$G(s)=\frac{-\Gamma_2}{s-M^2}-\frac{\Gamma_3}{2M^2+m^2-s}+\frac{1}{\pi}\int_{(M+m)^2}^{\infty}\frac{ds'\rho_2(s')[F^*(s')G(s')+G^*(s')F(s')]}{(s'-s)}+\frac{1}{\pi}\int_{4m^2}^{\infty}\frac{ds'\rho_1(s')A^*(s')G(s')}{(s'+s-2M^2-2m^2)}+\frac{1}{\pi}\int_{4M^2}^{\infty}\frac{ds'\rho_3(s')G^*(s')B^1(s')}{(s'+s-2M^2-2m^2)}.\tag{14}$$

The amplitude $B^1(s,u)$ has poles at m^2 in both channels while $B^2(s,u)$ has only those poles required by crossing. Since the s and u channels are identical for each of these amplitudes, the crossing matrix[5] is

$$A=\begin{pmatrix}c & 1-c\\ 1+c & -c\end{pmatrix}.\tag{15}$$

Therefore, unitarity and crossing imply

$$B^1(s)=-\Gamma_4\left[\frac{1}{s-m^2}+\frac{c}{2M^2+m^2-s}\right]+\frac{1}{\pi}\int_{4M^2}^{\infty}ds'\rho_3(s')\left[\frac{|B^1(s')|^2}{s'-s}+\frac{c|B^1(s')|^2+(1-c)|B^2(s')|^2}{s'+s-4M^2}\right]+\frac{1}{\pi}\int_{4m^2}^{\infty}ds'\rho_1(s')|G(-s'+2M^2+2m^2)|^2\left[\frac{1}{s'-s}+\frac{c}{s'+s-4M^2}\right],\tag{16}$$

and

$$B^2(s)=\frac{-\Gamma_4(1+c)}{(2M^2+m^2-s)}+\frac{1}{\pi}\int_{4M^2}^{\infty}ds'\rho_3(s')\left[\frac{|B^2(s')|^2}{s'-s}+\frac{(1+c)|B^1(s')|^2-c|B^2(s')|^2}{s'+s-4M^2}\right]+\frac{(1+c)}{\pi}\int_{4m^2}^{\infty}\frac{ds'\rho_1(s')|G(-s'+2M^2+2m^2)|^2}{(s'+s-4M^2)}.\tag{17}$$

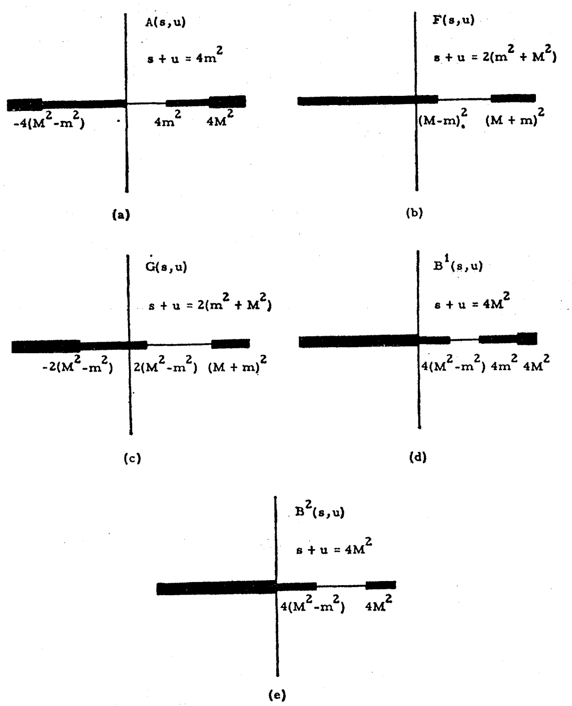

FIG. 4. The s planes for the amplitudes.

Although we have assumed that none of the amplitudes require any subtraction, it is a trivial matter to modify these equations if subtractions prove to be needed.

III. METHOD OF SOLUTION

We must now find solutions to the set of coupled equations (11)–(17). First, we consider the situation if $G(s,u)$ should vanish identically. Then the equations for the remaining amplitudes, $A_0(s,u)$, $F_0(s,u)$, and $B_0{}^\alpha(s,u)$ would uncouple. In fact, $A_0(s,u)$ and $F_0(s,u)$ could be found by a simple application of Cauchy's formula to the inverse amplitude. These solutions have been fully discussed in the literature.[9,10] Any arbitrariness in these solutions depends upon the number of Castillejo-Dalitz-Dyson (CDD) poles and the number of subtractions. In the spirit of the bootstrap philosophy, we shall assume that no CDD poles are present in any of the amplitudes. More particularly, we shall demand no poles in the amplitude $A_0(s,u)$ and bound-state poles at M^2 in both channels of $F_0(s,u)$. We would expect $F_0(s,u)$ to require one subtraction.[6,9]

The coupled-channel problem for $B_0{}^\alpha(s,u)$ can be solved by use of the techniques of Ref. 5 if we make the following observations. Define a new variable

$$z=(s-2M^2)/2M^2\,, \tag{18a}$$

so that

$$s=2M^2(1+z)\,, \tag{18b}$$

$$u=2M^2(1-z)\,. \tag{18c}$$

In terms of this new variable, the thresholds for the direct and crossed channels are at $z=+1$ and at $z=-1$, respectively. Also, crossing now consists of the statement

$$B_0{}^\alpha(-z)=\sum_{\beta=1}^{2} A_{\alpha\beta}B_0{}^\beta(z)\,. \tag{19}$$

This problem is solved for the corresponding S-matrix elements in Ref. 5 and these can be related to the above amplitudes from our Eq. (3) as

$$S_0{}^\alpha(s)=1+iB_0{}^\alpha(s)[s(s-4M^2)]^{-1/2}. \tag{20}$$

Explicit solutions can also be found in Ref. 6, where it is shown that one subtraction is required if there is a bound state and if Levinson's theorem is to be satisfied.

Of course, $G(s)\not\equiv 0$. However, there is a physically obvious parameter $\lambda\equiv m/M<1$ for which $\lambda=0$ (i.e., $M=\infty$) implies $G(s)\equiv 0$. That is, $M=\infty$ is the static limit in which particle b is always at rest. Also, when $\lambda=0$, $A(s)$, $F(s)$, and $B^\alpha(s)$ all reduce to the uncoupled forms we have just discussed. Therefore, we assume the following expansions as being physically reasonable:

$$A(s)=a_0(s)+\sum_{n=1}^{\infty}\lambda^n a_n(s)\,, \tag{21}$$

$$F(s)=f_0(s)+\sum_{n=1}^{\infty}\lambda^n f_n(s)\,, \tag{22}$$

$$G(s)=\sum_{n=1}^{\infty}\lambda^n g_n(s)\,, \tag{23}$$

$$B^\alpha(s)=b_0{}^\alpha(s)+\sum_{n=1}^{\infty}\lambda^n b_n{}^\alpha(s)\,. \tag{24}$$

The important observation here is that the unitarity conditions (8)–(11) involve $G(s)$ in such a fashion that the nth-order functions in the expansions (21)–(24) can always be solved for in terms of those of order $(n-1)$ and lower. In particular, $a_1(s)$, $f_1(s)$, and $b_1{}^\alpha(s)$ do *not* involve $g_1(s)$, so that one can begin with $a_0(s)$, $f_0(s)$, and $b_0{}^\alpha(s)$ alone and construct *all* the higher order functions from these by an iterative procedure. For example, in Eq. (8) consider the first-order correction to $A(s)$, namely,

$$\mathrm{Im}a_1(s)=\rho_1(s)[a_1{}^*(s)a_0(s)+a_0{}^*(s)a_1(s)]. \tag{25}$$

Since $a_0(s)$ itself satisfies unitarity, we can write

$$a_0(s)\equiv|a_0(s)|e^{i\eta(s)}=\frac{\sin\eta(s)e^{i\eta(s)}}{\rho_1(s)}\,, \tag{26}$$

where $\eta(s)$ is the phase shift at the static limit. Then

[9] R. W. Lardner, Nuovo Cimento 28, 1375 (1963).
[10] D. W. Schlitt, Nuovo Cimento 31, 858 (1964).

solving Eq. (25) for $\mathrm{Im}a_1(s)$ yields

$$\mathrm{Im}a_1(s)=\left[\frac{2\rho_1(s)\,\mathrm{Re}a_0(s)}{1-2\rho_1(s)\,\mathrm{Im}a_0(s)}\right]\mathrm{Re}a_1(s)$$
$$=\tan[2\eta(s)]\,\mathrm{Re}a_1(s). \quad (27)$$

We know from the unitarity condition (8) for the full $A(s)$ that $|A(s)|$ is bounded in the physical region $s>4m^2$. If the expansion (21) holds for any arbitrarily small but continuous range of values of the parameter λ, then all the $|a_n(s)|$ must also be finite in this physical region. In particular, then, the value of $\mathrm{Im}a_1(s)$ in Eq. (27) must be finite throughout the physical region in which (27) holds. Therefore, whenever $\tan[2\eta(s)]$ has a pole here, $\mathrm{Re}a_1(s)$ must vanish sufficiently rapidly to keep the product in (27) finite. We shall always assume this to be the case. Therefore, if we write an unsubtracted dispersion relation for $a_1(s)$ and use crossing symmetry for $a_1(s)$, we find

$$a_1(s)=\frac{1}{\pi}\int_{4m^2}^{\infty}ds'\,\mathrm{Im}a_1(s')\left[\frac{1}{s'-s}+\frac{1}{s'+s-4m^2}\right]. \quad (28)$$

From Eq. (27) this becomes

$$a_1(s)=\frac{1}{\pi}\int_{4m^2}^{\infty}ds'\,\tan[2\eta(s')]$$
$$\times\mathrm{Re}a_1(s')\left[\frac{1}{s'-s}+\frac{1}{s'+s-4m^2}\right]. \quad (29)$$

If we now let s approach the cut portion of the real axis from above, we obtain a linear, singular integral equation for $\mathrm{Re}a_1(s)$, namely,

$$\mathrm{Re}a_1(s)=\frac{1}{\pi}\mathcal{P}\int_{4m^2}^{\infty}ds'\,\tan[2\eta(s')]$$
$$\times\mathrm{Re}a_1(s')\left[\frac{1}{s'-s}+\frac{1}{s'+s-4m^2}\right], \quad (30)$$

where the $\mathcal{P}$ before the integral sign indicates that a principal-value integral is to be taken on the appropriate portion of the real s axis. On the other cut it simply reduces to an ordinary integral. Once we have $\mathrm{Re}a_1(s)$ on the cuts, we have $\mathrm{Im}a_1(s)$ there by Eq. (27) and then, by analytic continuation [e.g., via (28)], $a_1(s)$ everywhere.

The corresponding integral equation for $f_1(s)$ is

$$\mathrm{Re}f_1(s)=-\gamma_1^{(1)}\left[\frac{1}{s-M^2}+\frac{1}{M^2+2m^2-s}\right]$$
$$+\frac{1}{\pi}\mathcal{P}\int_{(M+m)^2}^{\infty}ds'\,\tan[2\tilde{\eta}(s')]$$
$$\times\mathrm{Re}f_1(s')\left[\frac{1}{s'-s}+\frac{1}{s'+s-2m^2-2M^2}\right], \quad (31)$$

where $\tilde{\eta}(s)$ is the phase of $f_0(s)$. We have made use of the fact that the residues Γ_j of the bound-state poles depend upon λ, as is evident from Eq. (14), and have written

$$\Gamma_j(\lambda)=\sum_{n=0}^{\infty}\lambda^n\gamma_n^{(j)}. \quad (32)$$

In fact, the integral equations for all the nth-order corrections to $A(s,u)$ and $F(s,u)$ have the form

$$\varphi(s)=\frac{-r_1}{s-\mu_1}-\frac{r_2}{\mu_2-s}+v(s)+\frac{1}{\pi}\mathcal{P}\int_{\mu_3}^{\infty}ds'\sigma(s')\varphi(s')$$
$$\times\left[\frac{1}{s'-s}+\frac{1}{s'+s-\mu_3}\right]. \quad (33)$$

The corresponding equation for $G(s,u)$ has two separate integrals since this amplitude is not crossing symmetric as are $A(s,u)$ and $F(s,u)$. However, we shall not write this down explicitly since Eq. (33) will be sufficient for our purposes. Here the locations of the poles (at μ_1, μ_2) are such that the poles do not lie on the cuts beginning at μ_3 and 0. The functions $\sigma(s)$ and $v(s)$ are known in terms of the previously determined lower-order functions in the expansions (21)–(24). The function $v(s)$ in general has a left- and a right-hand cut and satisfies a Hölder condition[11] on these cuts (as shown in Appendix B). Furthermore,

$$\sigma(\mu_3-s)=\sigma(s). \quad (34)$$

Under suitable conditions on $\delta(s)$, defined as

$$\sigma(s)\equiv\tan\delta(s), \quad (35)$$

at $s=\mu_3$, and at $s=\infty$, the solution of (33) is unique. As shown in Appendix B, in this case the solution of (33), extended to complex values of s, is

$$\psi(s)=\frac{-r_1}{s-\mu_1}-\frac{r_2}{\mu_2-s}+v(s)$$
$$+\frac{\Omega(s)}{\pi}\int_{\mu_3}^{\infty}ds'\left[\frac{-r_1}{s'-\mu_1}-\frac{r_2}{\mu_2-s'}+v(s')\right]\sin\delta(s')$$
$$\times\exp[-\rho(s')]\left[\frac{1}{s'-s}+\frac{1}{s'+s-\mu_3}\right], \quad (36)$$

where

$$\Omega(s)=\exp\left\{\frac{1}{\pi}\int_{\mu_3}^{\infty}ds'\delta(s')\left[\frac{1}{s'-s}+\frac{1}{s'+s-\mu_3}\right]\right\}, \quad (37)$$

$$\rho(s')=\frac{1}{\pi}\mathcal{P}\int_{\mu_3}^{\infty}d\zeta\,\delta(\zeta)\left[\frac{1}{\zeta-s'}+\frac{1}{\zeta+s'-\mu_3}\right]. \quad (38)$$

If these threshold and asymptotic behaviors on $\delta(s)$ are

[11] N. I. Muskhelishvili, *Singular Integral Equations* (P. Noordhoff, Gröningen, The Netherlands, 1953), p. 11.

not fulfilled, then there is some arbitrariness corresponding to subtractions. The connection between $\psi(s)$ in Eq. (36) and $\varphi(s)$ in (33) is given on the physical cuts $-\infty<s<0$ and $\mu_3<s<+\infty$ by

$$\mathrm{Re}\psi(s)=\varphi(s). \qquad (39)$$

Finally, the coupled-channel equations for the first-order corrections to $B_0{}^\alpha(s)$ are

$$\mathrm{Re}b_1{}^\alpha(s)=g_1{}^\alpha(s)+\sum_{\beta=1}^{2}\frac{1}{\pi}\mathcal{P}\int_{4M^2}^{\infty}ds'\tan[2\eta_\beta(s')]\times\mathrm{Re}b_1{}^\beta(s')\left[\frac{\delta_{\alpha\beta}}{s'-s}+\frac{A_{\alpha\beta}}{s'+s-4M^2}\right], \qquad (40)$$

where $g_1{}^\alpha(s)$ contains the pole terms of Eqs. (16) and (17). Similarly, the coupled equations for the real part of the nth-order correction to $B_0{}^\alpha(s)$ are given by an equation of the form

$$\chi^\alpha(s)=\gamma^\alpha(s)+\frac{1}{\pi}\mathcal{P}\int_{4M^2}^{\infty}ds'\sigma^\alpha(s')\chi^\alpha(s')\times\left[\frac{1}{s'-s}+\frac{A_{\alpha\alpha}}{s'+s-4M^2}\right]+\frac{A_{\alpha\beta}}{\pi}\mathcal{P}\int_{4M^2}^{\infty}\frac{ds'\sigma^\beta(s')\chi^\beta(s')}{(s'+s-4M^2)},$$

$$\alpha,\beta=1,2;\ \alpha\neq\beta. \qquad (41)$$

Here $\gamma^\alpha(s)$ contains the crossing-symmetric pole terms and the driving functions from lower order known terms, just as did $g(s)$ in Eq. (B7). The solution of this coupled set of equations is discussed in Appendix C. There, these coupled equations for $\chi^\alpha(s)$ are reduced to a pair of uncoupled Fredholm equations whose kernels have weak singularities.[12] Therefore, all the usual Fredholm theorems are applicable to these new equations for the $\chi^\alpha(s)$. Since this Fredholm kernel depends continuously upon the parameter c of the crossing matrix $A_{\alpha\beta}$ of Eq. (15), we are able to argue in Appendix C that these Fredholm equations for the nth-order functions of (24) will have solutions for any value of c, except possibly for some discrete set of values (i.e., in effect the characteristic values of the equation). Since there is at most a denumerable set of values of c in each order of λ for which no solution exists, the union of all such values of c must again be denumerable, so that only a *discrete* set of values of c can exist for which there is no $B^\alpha(s)$ satisfying the integral equations. There would still be many continuous ranges of c for which $B^\alpha(s)$ would exist.

Therefore, leaving aside the obviously difficult question of the convergence of the infinite series (21)–(24), we conclude that the requirements of unitarity, crossing, and analyticity are not sufficient to select an internal symmetry, even when coupling to two-body inelastic channels is taken into account in our model.

IV. THE BOOTSTRAP REQUIREMENT

We shall now impose the bootstrap requirement[6,7] via Levinson's theorem as discussed in Sec. I to see whether or not the parameter c is restricted in any way. Since this will be done by explicit calculation, we shall make only a first-order correction to the result for the static model. That is, we assume

$$B^\alpha(s)\approx b_0{}^\alpha(s)+\lambda b_1{}^\alpha(s)\equiv|B^\alpha(s)|e^{i\delta_\alpha(s)}. \qquad (42)$$

Stated in a form appropriate to a two-dimensional model,[9] Levinson's theorem is[12a]

$$\delta_\alpha(4M^2)-\delta_\alpha(\infty)=(n_\alpha-q_\alpha)\pi-\tfrac{1}{2}\pi, \qquad (43)$$

where n_α is the number of bound states in channel α and q_α is the number of CDD poles in channel α. The extra $\frac{1}{2}\pi$ is peculiar to a two-dimensional model and results from the fact that the phase-space factor $[s(s-4M^2)]^{-1/2}$ becomes infinite at threshold rather than vanishing there as does the corresponding factor $[(s-4M^2)/s]^{1/2}$ in the four-dimensional case. (Cf. Appendix A for an example of this behavior.)

Since we are going to rule out any CDD poles, we shall demand one bound state in the $\alpha=1$ channel and none in the $\alpha=2$ channel. If we let $\eta_\alpha(s)$ be the phase shift of $b_0{}^\alpha(s)$, then

$$b_0{}^\alpha(s)=|b_0{}^\alpha(s)|e^{i\eta_\alpha(s)}=2[s(s-4M^2)]^{1/2}\sin\eta_\alpha(s)e^{i\eta_\alpha(s)}. \qquad (44)$$

From unitarity we find, as previously for $a_1(s)$,

$$\mathrm{Im}b_1{}^\alpha(s)=\tan[2\eta_\alpha(s)]\,\mathrm{Re}b_1{}^\alpha(s). \qquad (45)$$

Therefore, the tangent of the phase shift of $B^\alpha(s)$ is

$$\tan\delta_\alpha(s)=\frac{\mathrm{Im}b_0{}^\alpha(s)+\lambda\,\mathrm{Im}b_1{}^\alpha(s)}{\mathrm{Re}b_0{}^\alpha(s)+\lambda\,\mathrm{Re}b_1{}^\alpha(s)}=\frac{2[s(s-4M^2)]^{1/2}\sin^2\eta_\alpha(s)\cos[2\eta_\alpha(s)]+\lambda\sin[2\eta_\alpha(s)]\,\mathrm{Re}b_1{}^\alpha(s)}{\cos[2\eta_\alpha(s)]\{[s(s-4M^2)]^{1/2}\sin[2\eta_\alpha(s)]+\lambda\,\mathrm{Re}b_1{}^\alpha(s)\}}. \qquad (46)$$

[12] S. G. Mikhlin, *Integral Equations* (The Macmillan Company, New York, 1954), pp. 59–66.

[12a] *Footnote added in proof.* Since the $\alpha=1$ channel of $B^\alpha(s)$ is coupled to $A(s)$ (cf. vertex of Fig. 3), Levinson's theorem for this channel is in general to be stated in terms of the *eigenphases* of this channel rather than in terms of $\delta_1(s)$, which is only the phase of $B^1(s)$ and not an eigenphase. The relevant quantity on the left side of Eq. (43) is $(1/2i)\ln(\det S_{\alpha\beta})$, where $S_{\alpha\beta}$ is a 2×2 matrix. However, the off-diagonal elements of $S_{\alpha\beta}$ are each proportional to λ [i.e., to $G(s)$], so that, to first order in λ, Eq. (43) is correct for $\alpha=1$, as well as for $\alpha=2$.

Since $\text{Im}b_1{}^\alpha(s)$ is bounded by unitarity as was $a_1(s)$, it follows that $\text{Re}b_1{}^\alpha(s)=0$ whenever $\cos[2\eta_\alpha(s)]=0$. Therefore, $\delta_\alpha(s)$ will pass through a resonance only for those values of $s=s_j$ for which

$$\text{Re}b_1{}^\alpha(s_j)=-\frac{[s_j(s_j-4M^2)]^{1/2}}{\lambda}\sin[2\eta_\alpha(s_j)]. \quad (47)$$

Furthermore, as $s\to 4M^2$ or as $s\to\infty$,

$$\begin{aligned}\tan\delta_\alpha(s)&\to\tan\eta_\alpha(s) \quad \text{if } \text{Re}b_1{}^\alpha(s)/\text{Re}b_0{}^\alpha(s)\to 0,\\ &\to\tan[2\eta_\alpha(s)] \text{ if } \text{Re}b_0{}^\alpha(s)/\text{Re}b_1{}^\alpha(s)\to 0. \quad (48)\end{aligned}$$

In order that the series (24) approach a limit uniformly as $\lambda\to 0$, we require that

$$\text{Re}b_1{}^\alpha(s)/\text{Re}b_0{}^\alpha(s)\to 0 \quad (49)$$

as $s\to 4M^2$ or as $s\to\infty$. This simply means, from (48), that the phase shift $\delta_\alpha(s)$ will approach its static-limit value $\eta_\alpha(s)$ continuously as $M\to\infty$. Condition (47) must be

$$\text{Re}b_1{}^1(s_1)=-\{[s_1(s_1-4M^2)]^{1/2}/\lambda\}\sin[2\eta_1(s_1)] \quad (50)$$

for only one value $s=s_1>4M^2$ in the $\alpha=1$ channel, while the equality (47) must never hold for any value of $s>4M^2$ in the $\alpha=2$ channel. Assume that there is no pole in (46) for $\alpha=2$ and that the expression (50) holds for one and only one value $s_1>4M^2$ for some value of c, the parameter in the crossing matrix. Since the functions in (50) are continuous functions of c and s, with $4M^2<s<\infty$, then if c is changed infinitesimally to $c+\delta c$, there will be a value of s, say $s_1+\delta s_1$, for which (50) will still hold.[12b] Therefore, the bootstrap condition in the form of Levinson's theorem will not restrict the allowed values of c to *a* discrete set, although it may limit c to finite continuous ranges.

V. CONCLUSIONS

As an introduction to this section, a few simple observations may be in order. This paper has been concerned with singlet-doublet scattering; that is, with events in which particle a, having only one possible value of its quantum number associated with an internal symmetry and assigned to a singlet representation of this group, is scattered from a particle b (or the antiparticle $\bar{b}$) having two possible values of this internal quantum number and assigned to a doublet representation of the symmetry group. This treatment can easily be extended to doublet-doublet scattering. In this case, all the equations for the iterated functions would be coupled ones and the methods of Appendix C would suffice. As is probably fairly evident, the conclusions remain in essence the same as above. There is still no unique solution for the parameter of the crossing matrix.

Also, we have not been concerned with the possible uniqueness of the solutions as functions of the variable s (the energy squared in the c.m. frame) as were Huang *et al.*[6,7] One reason for not investigating this question in any detail is that the asymptotic behavior of the scattering amplitude as a function of s does not depend critically upon the value of c. This has been discussed in detail for the case corresponding to $\lambda=0$ (i.e., all nonzero amplitudes uncoupled).[5–7] This feature also persists for the solutions found in Appendix C. Therefore, it does not appear that a detailed study of asymptotics will place stringent restrictions on allowed values for c.

Finally, it is desirable to evaluate the possible relevance of this oversimplified model to the question of bootstrapping internal symmetries in reality. It may certainly be true that the complications inherent in a four-dimensional world may be necessary to narrow the many *a priori* possible internal symmetries to one (or only a very few); but if this is the case, then the bootstrap mechanism would seem to have to remain an unproved conjecture. Although additional approximate calculations may generate an even greater number of correct values of ratios of coupling constants and may strengthen our belief in the mechanism, they cannot as such demonstrate that bootstrapping does indeed generate the only physical situation possible. Therefore, a tractable model that is mathematically similar to the actual physical situation appears to be the only reasonable means of demonstrating conclusively whether or not the bootstrap conjecture about the origin of internal symmetries is correct.

The model studied here is mathematically similar to what is generally believed to be the situation realized by nature in that unitarity, crossing, analyticity, nonlinearity, inelasticity, relativistic invariance, and a self-consistency condition are all present. A limitation of the work is that we have examined the possible bootstrap restrictions only to first order in λ (Sec. IV). Nevertheless, there is good reason to believe that the conclusions of that section are true in general since the continuity of the solutions as functions of c and s, which was essential to the argument, will persist to all orders in λ. Therefore, at the very least, the fact that for this model the bootstrap mechanism failed to select an internal symmetry group is not an encouraging result for the

[12b] *Footnote added in proof.* That Levinson's theorem be satisfied gives us an equation of the form

$$F(c,s)=0, \quad \text{(i)}$$

where F is a continuous function of the real variables c and s. The Eq. (i) above represents the intersection of a two-dimensional surface and the c-s plane. Now the argument based on continuity given in the text is valid unless the c-s plane intersects the surface only at a discrete set of points (i.e., only at extrema of the surface). However, the numerical value of F (and possibly the shape of the surface as well) depends upon the values of the coupling constants Γ_j [Eq. (32)] or upon λ. Therefore, if for a particular value of λ this extremum condition occurs accidently, we may simply vary λ and, correspondingly, the intersection defined by (i). Then the argument given in the text applies to these new solutions, so that a unique value of c is still not selected.

bootstrap conjecture about the origin of internal symmetries.

ACKNOWLEDGMENTS

The author wishes to thank Dr. Fritz Coester and Dr. William D. McGlinn for general discussions of the problem and for their criticisms of the manuscript. He has found the editorial comments of Dr. F. E. Throw of great assistance. He also acknowledges with gratitude the hospitality of Professor Abdus Salam at Imperial College, London, where this work was begun during the tenure of a National Science Foundation Postdoctoral Fellowship.

APPENDIX A: AN EXACT SCATTERING AMPLITUDE

The purpose of this Appendix is to exhibit briefly a scattering amplitude corresponding to $A_0(s)$ of Sec. III. With simple kinematical changes this will also serve as an example of $F_0(s)$. The claim made is that an unsubtracted $A_0(s)$ is

$$\frac{1}{A_0(s)}=\frac{1}{A_0(2m^2)}+\frac{1}{4m^2}-\frac{i}{2[s(s-4m^2)]^{1/2}}$$
$$\equiv\frac{1}{A_0(2m^2)}-\frac{(s-2m^2)}{2\pi}\int_{4m^2}^{\infty}\frac{ds'}{[s'(s'-4m^2)]^{1/2}(s'-2m^2)}\left[\frac{1}{s'-s}-\frac{1}{s'+s-4m^2}\right], \tag{A1}$$

and $1/A_0(2m^2)$ is real. This obviously satisfies the elastic unitarity version of Eq. (6) which implies

$$\mathrm{Im}[1/T(s)]=-\tfrac{1}{2}[s(s-4m^2)]^{-1/2}.$$

It is a straightforward matter to show that

$$\mathrm{Im}\left(\frac{1}{(s-2m^2)A(s)}\right)\Big/\mathrm{Im}s=\frac{-1}{|s-2m^2|^2A_0(2m^2)}$$
$$-\frac{1}{\pi}\int_{4m^2}^{\infty}\frac{ds'}{[s'(s'-4m^2)]^{1/2}(s'-2m^2)}\left[\frac{1}{(s'-\mathrm{Re}s)^2+(\mathrm{Im}s)^2}+\frac{1}{(s'+\mathrm{Re}s)^2+(\mathrm{Im}s)^2}\right]. \tag{A2}$$

If we require $1/[A_0(2m^2)]>0$, then $\mathrm{Im}[(s-2m^2)A_0(s)]^{-1}/\mathrm{Im}s<0$ everywhere in the finite s plane except possibly on the real axis (i.e., except where $\mathrm{Im}s=0$). Then $A_0(s)$ will have no poles in the complex s plane, except possibly on the real axis. This requirement on $A_0(2m^2)$ in turn implies

$$C\equiv\frac{1}{A_0(2m^2)}+\frac{1}{4m^2}>0.$$

Therefore,

$$1/A_0(s)=C-\tfrac{1}{2}i[s(s-4m^2)]^{-1/2} \tag{A3}$$

can vanish only on the strip $0<s<4m^2$ of the real axis. There are in fact two bound-state poles on the real axis, one on the strip $0<s<2m^2$ and one on $2m^2<s<4m^2$ (i.e., one in the s channel and one in the u channel). Since

$$\tan\eta(s)=\frac{1}{2C[s(s-4m^2)]^{1/2}}, \tag{A4}$$

we see that Levinson's theorem, Eq. (43), is satisfied.

APPENDIX B: THE UNCOUPLED EQUATIONS

In this Appendix we consider the solution to the singular integral equation

$$\varphi(s)=g(s)+\frac{1}{\pi}\mathcal{P}\int_{\mu_3}^{\infty}ds'\sigma(s')\varphi(s')\left[\frac{1}{s'-s}+\frac{1}{s'+s-\mu_3}\right], \tag{B1}$$

which is a generalization of the Omnès equation. Here $\mathcal{P}$ stands for the principal value since s is on one of the cuts $-\infty<s<0$ or $\mu_3<s<+\infty$. The real functions $g(s)$ and $\sigma(s)$ are given and satisfy Hölder conditions,[11] while solutions

$\varphi(s)$ are sought which also satisfy this condition. More specifically, $g(s)$ contains the bound-state pole terms as well as driving terms from lower order functions and is given by

$$g(s)=\frac{-r_1}{s-\mu_1}-\frac{r_2}{\mu_2-s}+v(s)\,, \tag{B2}$$

where $v(s)$ has both a right- and a left-hand cut. As usual, for complex values of $z(\mathrm{Re}z\equiv s)$ we define a function

$$F(z)=\frac{1}{2\pi i}\int_{\mu_3}^{\infty}ds'\sigma(s')\varphi(s')\left[\frac{1}{s'-z}+\frac{1}{s'+z-\mu_3}\right]. \tag{B3}$$

If we let $z^{\pm}=s\pm i\epsilon$, $\epsilon\to 0$, then

$$F(z^+)-F(z^-)=[\theta(s-\mu_3)+\theta(-s)]\sigma(s)\varphi(s)\,, \tag{B4}$$

$$F(z^+)+F(z^-)=\frac{1}{\pi i}\mathcal{P}\int_{\mu_3}^{\infty}ds'\sigma(s')\varphi(s')\left[\frac{1}{s'-s}+\frac{1}{s'+s-\mu_3}\right]. \tag{B5}$$

If we substitute these into (B1), we find

$$\{1-i\sigma(s)[\theta(s-\mu_3)+\theta(-s)]\}F(z^+)-\{1+i\sigma(s)[\theta(s-\mu_3)+\theta(-s)]\}F(z^-)=[\theta(s-\mu_3)+\theta(-s)]\sigma(s)g(s)\,. \tag{B6}$$

As usual, let

$$F(z)=\Phi(z)\Omega(z)\,, \tag{B7}$$

where $\Omega(z)$ satisfies the homogeneous version of (B6) [i.e., $g(s)\equiv 0$]. Then,

$$\ln\Omega(z)=\frac{1}{2\pi i}\int_{\mu_3}^{\infty}ds'\delta(s')\left[\frac{1}{s'-z}+\frac{1}{s'+z-\mu_3}\right], \tag{B8}$$

where

$$\sigma(s)\equiv\tan\delta(s)\,. \tag{B9}$$

Also, crossing symmetry has been used in the form

$$\sigma(\mu_3-s)=\sigma(s)\,.$$

If

$$\rho(s)\equiv\frac{1}{\pi}\mathcal{P}\int_{\mu_3}^{\infty}ds'\delta(s')\left[\frac{1}{s'-s}+\frac{1}{s'+s-\mu_3}\right], \tag{B10}$$

then

$$\Omega(z^{\pm})=\exp[\rho(s)\pm i\delta(s)]\,. \tag{B11}$$

The equation satisfied by $\Phi(z)$ of Eq. (B7) becomes

$$\Phi(z^+)-\Phi(z^-)=\frac{[\theta(s-\mu_3)+\theta(-s)]\sigma(s)g(s)}{\{1+i[\theta(s-\mu_3)+\theta(-s)]\sigma(s)\}\Omega(z^-)},$$

whose solution is

$$\Phi(z)=\frac{1}{2\pi i}\int_{\mu_3}^{\infty}\frac{ds'\sigma(s')g(s')\exp[-\rho(s')+i\delta(s')]}{[1+i\sigma(s')]}\left[\frac{1}{s'-z}+\frac{1}{s'+z-\mu_3}\right].$$

This can be simplified to

$$\Phi(z)=\frac{1}{2\pi i}\int_{\mu_3}^{\infty}ds'\sin\delta(s')g(s')\exp[-\rho(s')]\left[\frac{1}{s'-z}+\frac{1}{s'+z-\mu_3}\right]. \tag{B12}$$

We can now compute $\varphi(s)$ from (B4) as

$$\begin{aligned}\varphi(s)&=\frac{1}{\sigma(s)}[\Phi(z^+)\Omega(z^+)-\Phi(z^-)\Omega(z^-)]\\&=\cos\delta(s)\left\{g(s)\cos\delta(s)+\frac{\exp[\rho(s)]}{\pi}\mathcal{P}\int_{\mu_3}^{\infty}ds'\sin\delta(s')g(s')\exp[-\rho(s')]\left[\frac{1}{s'-s}+\frac{1}{s'+s-\mu_3}\right]\right\}.\end{aligned} \tag{B13}$$

Finally, the analytic function $\psi(s)$ for complex s of which $\varphi(s)$ is the real part for real s, i.e., the function for which

$$\mathrm{Re}\psi(s)=\varphi(s)\,,\quad s \text{ real}\,,$$

is

$$\psi(s)=g(s)+\frac{\Omega(s)}{\pi}\int_{\mu_3}^{\infty}ds'\,\sin\delta(s')g(s')\exp[-\rho(s')]\left[\frac{1}{s'-s}+\frac{1}{s'+s-\mu_3}\right]. \tag{B14}$$

Of course, to this solution of (B1) can be added any solution to the homogeneous equation

$$\varphi_0(s)=\frac{1}{\pi}\mathcal{P}\int_{\mu_3}^{\infty}ds'\sigma(s')\varphi_0(s')\left[\frac{1}{s'-s}+\frac{1}{s'+s-\mu_3}\right]. \tag{B15}$$

As before, we define

$$F_0(z)=\frac{1}{2\pi i}\int_{\mu_3}^{\infty}ds'\sigma(s')\varphi_0(s')\left[\frac{1}{s'-z}+\frac{1}{s'+z-\mu_3}\right],$$

so that

$$F_0(z^+)=e^{2i\delta(s)}F_0(z^-)\,.$$

If

$$F_0(z)=\Phi_0(z)\Omega(z)\,,$$

such that

$$\Phi_0(z^+)-\Phi_0(z^-)=0\,,$$

with $\Omega(z)$ still given by Eq. (B8), then

$$\Phi_0(z)=\frac{P_n[z(z-\mu_3)]}{2i[z(z-\mu_3)]^m}, \tag{B16}$$

where n and m are positive integers or zero, $P_n(x)$ is a real polynomial of degree n, and we have assumed that there are no essential singularities at μ_3 or ∞ and have demanded crossing symmetry. Then,

$$\varphi_0(s)=\frac{[F_0(z^+)-F_0(z^-)]}{\sigma(s)}=\frac{P_n[s(s-\mu_3)]}{[s(s-\mu_3)]^m}e^{\rho(s)}\cos\delta(s)\,. \tag{B17}$$

Therefore, the most general solution of (B1) for complex values of s is

$$\psi(s)=g(s)+\frac{\Omega(s)P_n[s(s-\mu_3)]}{[s(s-\mu_3)]^m}+\frac{\Omega(s)}{\pi}\int_{\mu_3}^{\infty}ds'\,\sin\delta(s')g(s')\exp[-\rho(s')]\left[\frac{1}{s'-s}+\frac{1}{s'+s-\mu_3}\right]. \tag{B18}$$

The values of n and m are fixed by requiring that $\psi(s)$ remain bounded at threshold and have a prescribed behavior as $s\to\infty$. The behavior of $\delta(s)$ near μ_3 and ∞ determines the behavior of the integral in (B18) in the neighborhoods of these points. One then chooses n and m in order to cancel any unwanted divergence at threshold and to give the prescribed behavior at infinity. For example, if $\delta(\mu_3)=0=\delta(\infty)$, then the solution of Eq. (B1) is unique and is given by (B13) or (B14). Since we shall not be concerned with the question of uniqueness, we shall not discuss the determination of n and m further here. This can, however, be found elsewhere.[13,14] We simply require a proof that there does exist a solution to (B1) and a knowledge of its form.

APPENDIX C: THE COUPLED EQUATIONS

In this Appendix we shall discuss a method for obtaining solutions to the singular coupled integral equations

$$\chi^\alpha(s)=\gamma^\alpha(s)+\frac{1}{\pi}\mathcal{P}\int_{4M^2}^{\infty}ds'\sigma^\alpha(s')\chi^\alpha(s')\left[\frac{1}{s'-s}+\frac{A_{\alpha\alpha}}{s'+s-4M^2}\right]+\frac{A_{\alpha\beta}}{\pi}\mathcal{P}\int_{4M^2}^{\infty}\frac{ds'\sigma^\beta(s')\chi^\beta(s')}{(s'+s-4M^2)},\quad \alpha,\beta=1,2;\ \alpha\neq\beta. \tag{C1}$$

As in Appendix B, we assume the given functions $\gamma^\alpha(s)$ and $\sigma^\alpha(s)$ to satisfy Hölder conditions[11] on the cuts $-\infty<s<0$ and $4M^2<s<+\infty$ on which Eq. (C1) is defined, and seek solutions $\chi^\alpha(s)$ satisfying such a

[13] R. Omnès, Nuovo Cimento 8, 316 (1958).
[14] N. I. Muskhelishvili, Ref. 11, p. 230 and p. 330.

condition. If we define

$$\tau^\alpha(s)=\gamma^\alpha(s)+\frac{A_{\alpha\beta}}{\pi}\mathcal{P}\int_{4M^2}^\infty \frac{ds'\sigma^\beta(s')\chi^\beta(s')}{(s'+s-4M^2)} \tag{C2}$$

and consider (C1) with α and β interchanged, we can use (B1) and (B13) to obtain

$$\chi^\beta(s)=\cos\delta_\beta(s)\left\{\tau^\beta(s)\cos\delta_\beta(s)+\frac{\exp[\rho_\beta(s)]}{\pi}\int_{4M^2}^\infty ds'\sin\delta_\beta(s')\tau_\beta(s')\exp[-\rho_\beta(s')]\left[\frac{1}{s'-s}+\frac{A_{\beta\beta}}{s'+s-4M^2}\right]\right\}, \tag{C3}$$

where

$$\sigma^\beta(s)=\tan\delta_\beta(s), \tag{C4}$$

$$\rho_\beta(s)=\frac{1}{\pi}\mathcal{P}\int_{4M^2}^\infty ds'\delta_\beta(s')\left[\frac{1}{s'-s}+\frac{A_{\beta\beta}}{s'+s-4M^2}\right]. \tag{C5}$$

If we now substitute Eq. (C3) for $\chi^\beta(s)$ back into (C1), we obtain an integral equation containing only $\chi^\alpha(s)$. However, this new equation involves double and triple principal-value integrals. Explicitly, we find

$$\chi^\alpha(s)=\tau^\alpha(s)+\frac{1}{\pi}\mathcal{P}\int_{4M^2}^\infty ds'\sigma^\alpha(s')\chi^\alpha(s')\left[\frac{1}{s'-s}+\frac{A_{\alpha\alpha}}{s'+s-4M^2}\right]+\frac{A_{\alpha\beta}}{\pi}\mathcal{P}\int_{4M^2}^\infty \frac{ds'\sin\delta_\beta(s')\cos\delta_\beta(s')\tau^\beta(s')}{(s'+s-4M^2)}$$
$$+\frac{A_{\alpha\beta}}{\pi}\mathcal{P}\int_{4M^2}^\infty \frac{ds'\sin\delta_\beta(s')\exp[-\rho_\beta(s')]}{(s'+s-4M^2)}\mathcal{P}\int_{4M^2}^\infty ds''\sin\delta_\beta(s'')\tau_\beta(s'')\exp[-\rho_\beta(s'')]\left[\frac{1}{s''-s'}+\frac{A_{\beta\beta}}{s''+s'-4M^2}\right]. \tag{C6}$$

We can now use the Bertrand-Poincaré formula[15,16]

$$\mathcal{P}\int_L \frac{dz}{(z-x)}\mathcal{P}\int_L \frac{F(x,y,z)dy}{(y-z)}=\mathcal{P}\int_L dy\,\mathcal{P}\int_L \frac{F(x,y,z)dz}{(z-x)(y-z)}-\pi^2F(x,x,x) \tag{C7}$$

to reduce these repeated integrals. Then, by the use of Eq. (C7) we can deduce the relation

$$\mathcal{P}\int_{4M^2}^\infty dt\left\{\mathcal{P}\int_{4M^2}^\infty dr\left[\frac{\varphi(r)}{r-t}+\frac{\chi(r)}{r+t-4M^2}\right]\right\}\left[\frac{f(t)}{t-s}+\frac{g(t)}{t+s-4M^2}\right]$$
$$=\int_{4M^2}^\infty \frac{dr[h(r)-h(s)]\varphi(r)}{(r-s)}+\int_{4M^2}^\infty \frac{dr[h(4M^2-r)-h(s)]\chi(r)}{(r+s-4M^2)}+\int_{4M^2}^\infty \frac{dr[k(r)-k(4M^2-s)]\varphi(r)}{(r+s-4M^2)}$$
$$+\int_{4M^2}^\infty \frac{dr[k(4M^2-s)-k(4M^2-r)]\chi(r)}{(r-s)}-\pi^2f(s)\varphi(s)\theta(s-4M^2)-\pi^2g(4M^2-s)\varphi(4M^2-s)\theta(-s), \tag{C8}$$

where

$$h(s)=\mathcal{P}\int_{4M^2}^\infty \frac{dtf(t)}{(t-s)},\qquad k(s)=\mathcal{P}\int_{4M^2}^\infty \frac{dtg(t)}{(t-s)}.$$

Notice that the final form of (C8) contains no principal-value integrals [except trivially in the definitions of $h(s)$ and $k(s)$]. We can now proceed to reduce the last two terms in (C6). The result is

$$\chi^\alpha(s)=\omega^\alpha(s)+\frac{1}{\pi}\mathcal{P}\int_{4M^2}^\infty ds'\sigma^\alpha(s')\chi^\alpha(s')\left[\frac{1}{s'-s}+\frac{A_{\alpha\alpha}}{s'+s-4M^2}\right]+\frac{A_{\alpha\beta}}{\pi}\int_{4M^2}^\infty K^\alpha(s,s')\chi^\alpha(s')ds', \tag{C9}$$

where

$$\omega^\alpha(s)=\gamma^\alpha(s)+\frac{A_{\alpha\beta}}{2\pi}\mathcal{P}\int_{4M^2}^\infty \frac{ds'\sin[2\delta_\beta(s')]\gamma^\beta(s')}{(s'+s-4M^2)}+\frac{A_{\alpha\beta}}{\pi}\mathcal{P}\int_{4M^2}^\infty \frac{ds'\sin\delta_\beta(s')\exp[-\rho_\beta(s')]}{(s'+s-4M^2)}$$
$$\times\mathcal{P}\int_{4M^2}^\infty ds''\sin\delta_\beta(s'')\gamma^\beta(s'')\exp[-\rho_\beta(s'')]\left[\frac{1}{s''-s'}+\frac{A_{\beta\beta}}{s''+s'-4M^2}\right], \tag{C10}$$

[15] N. I. Muskhelishvili, Ref. 11, p. 56.
[16] F. G. Tricomi, *Integral Equations* (Interscience Publishers, Inc.. New York, 1957), p. 171.

$$K^{\alpha}(s,s')=\frac{A_{\beta\alpha}}{\pi}\left\{\frac{[\varphi^{\beta}(4M^2-s)-\varphi^{\beta}(4M^2-s')]}{2(s'-s)}+\frac{[\zeta^{\beta}(4M^2-s',4M^2-s)-\zeta^{\beta}(4M^2-s',4M^2-s')]}{(s'-s)}\right.$$
$$\left.+A_{\beta\beta}\frac{[\kappa^{\beta}(4M^2-s',4M^2-s)-\kappa^{\beta}(4M^2-s',s')]}{(s'+s-4M^2)}\right\}\sigma^{\alpha}(s'),\quad \text{(C11)}$$

$$\zeta^{\beta}(s'',s)=\mathcal{P}\int_{4M^2}^{\infty}ds\frac{[\nu^{\beta}(s'')-\nu^{\beta}(s')]}{(s'-s)}\sin\delta_{\beta}(s')\exp[-\rho_{\beta}(s')],\quad \text{(C12)}$$

$$\kappa^{\beta}(s'',s)=\mathcal{P}\int_{4M^2}^{\infty}ds'\frac{[\nu^{\beta}(4M^2-s')-\nu^{\beta}(s'')]}{(s'-s)}\sin\delta_{\beta}(s')\exp[-\rho_{\beta}(s')],\quad \text{(C13)}$$

$$\nu^{\beta}(s)=\mathcal{P}\int_{4M^2}^{\infty}\frac{ds'\sin\delta_{\beta}(s')\exp[-\rho_{\beta}(s')]}{(s'-s)},\quad \text{(C14)}$$

$$\varphi^{\beta}(s)=\mathcal{P}\int_{4M^2}^{\infty}\frac{ds'\sin[2\delta_{\beta}(s')]}{(s'-s)}.\quad \text{(C15)}$$

We can now reduce Eq. (C9) to a Fredholm equation of the second kind as follows: If we again return to (B1) and (B13) and take

$$g^{\alpha}(s)=\omega^{\alpha}(s)+\frac{A_{\alpha\beta}}{\pi}\int_{4M^2}^{\infty}K^{\alpha}(s,s')\chi^{\alpha}(s')ds',\quad \text{(C16)}$$

and let $\chi_0{}^{\alpha}(s)$, which will be given by (B13), denote the solution of

$$\chi_0{}^{\alpha}(s)=\omega^{\alpha}(s)+\frac{1}{\pi}\mathcal{P}\int_{4M^2}^{\infty}ds'\sigma^{\alpha}(s')\chi_0{}^{\alpha}(s')\left[\frac{1}{s'-s}+\frac{A_{\alpha\alpha}}{s'+s-4M^2}\right],\quad \text{(C17)}$$

then we find

$$\chi^{\alpha}(s)=\chi_0{}^{\alpha}(s)+\frac{A_{\alpha\beta}}{\pi}\cos\delta_{\alpha}(s)\int_{4M^2}^{\infty}N^{\alpha}(s,s')\chi^{\alpha}(s')ds',\quad \text{(C18)}$$

where

$$N^{\alpha}(s,s')=\cos\delta_{\alpha}(s)K^{\alpha}(s,s')+\frac{\exp[\rho_{\alpha}(s)]}{\pi}\mathcal{P}\int_{4M^2}^{\infty}ds''\sin\delta_{\alpha}(s'')K^{\alpha}(s'',s')\exp[-\rho_{\alpha}(s'')]\left[\frac{1}{s''-s}+\frac{A_{\alpha\alpha}}{s''+s-4M^2}\right].\quad \text{(C19)}$$

The kernel in (C19) has a weak singularity,[12] as is evident from the definition of $K^{\alpha}(s,s')$ given in (C11). Therefore, (C18) is a Fredholm equation of the second kind with a weak singularity and, as such, is subject to all the familiar Fredholm theorems.[12]

Now recall that the functions $\chi_0{}^{\alpha}(s)$, $\delta_{\alpha}(s)$, and $N^{\alpha}(s,s')$ of (C18) depend upon the parameter c of the crossing matrix (15) in our model. It is known that a solution of (C18) will be given by the Fredholm resolvent unless, possibly, there is a nontrivial solution to the homogeneous, adjoint version of (C18). We wish to show that this can happen for at most a discrete set of values of c. To this end, let us study the modified kernel

$$M^{\alpha}(s,s')\equiv A_{\alpha\beta}\cos[\delta_{\alpha}(s)]N^{\alpha}(s,s').\quad \text{(C20)}$$

The first step is to show that $M^{\alpha}(c;s,s')$ of (C20) is a continuous function of c and s, s' on the cuts $-\infty<s<0$, and $4M^2<s<+\infty$. In fact, $M^{\alpha}(c;s,s')$ will be a holomorphic function of c in the complex c plane in a region $\mathfrak{D}$ that includes the real axis. First consider (C18) when $\chi^{\alpha}(s)=\mathrm{Re}b_1{}^{\alpha}(s)$, the first-order correction in Eq. (24). Then the functions $\chi_0{}^{\alpha}(s)$, $\cos\delta_{\alpha}(s)$, and $N^{\alpha}(s,s')$ of (C18) are defined in terms of analytic functions of and principal-value integrals of the zero-order functions [$b_0{}^{\alpha}(s)$ in our earlier notation] which are the solutions given by Martin and McGlinn.[5] As reference to their explicit solutions will show, these zero-order functions are continuous functions of s in the range $4M^2<s<+\infty$ and are analytic in the region $\mathfrak{D}$ of the c plane. Note that if a function $f(\lambda,t)$, which is continuous, bounded for $a<t<b$, and analytic for some domain of λ, is used to define a $g(\lambda,s)$ as

$$g(\lambda,s)=\mathcal{P}\int_a^b\frac{dtf(\lambda,t)}{(t-s)},\quad \text{(C21)}$$

then $g(\lambda,s)$ is also continuous for $a<s<b$ and is analytic in λ. The only thing requiring proof is the continuity in s, which is easily seen as follows. Let $\delta<\epsilon$, both sufficiently small. Then

$$|g(s+\delta)-g(s)|=\delta\left|\mathcal{P}\int_a^b \frac{dtf(t)}{(t-s)(t-s-\delta)}\right| \underset{\delta\to 0}{\longrightarrow}$$

$$\delta\left|\mathcal{P}\int_{s-\epsilon}^{s+\epsilon} \frac{dtf(t)}{(t-s)(t-s-\delta)}\right|$$

$$\simeq |f(s)|\left|\ln\left(\frac{1-\delta/\epsilon}{1+\delta/\epsilon}\right)\right| \simeq \frac{2\delta|f(s)|}{\epsilon} \equiv \epsilon' . \quad \text{(C22)}$$

Therefore, we see that $\chi_0{}^\alpha(c,s)$ and $\delta_\alpha(c,s)$ of Eq. (C18) are analytic functions of c in region $\mathfrak{D}$ of the c plane, while $N^\alpha(c;s,s')$, and hence $M^\alpha(c;s,s')$ of (C20), has a weak singularity on $4M^2<s$(or $s')<+\infty$ and also possesses the same analyticity in c. If we iterate (C18), the weak singularity will disappear. We conclude, then, that (C18) is equivalent to a Fredholm equation of the second kind whose kernel is continuous in s and s' on the cut from $4M^2$ to $+\infty$ and is an analytic function of the parameter c in region $\mathfrak{D}$ of the c plane. Now if we knew that the homogeneous adjoint equation associated with this Fredholm equation [in essence Eq. (C18)] could have nontrivial solutions for only a discrete set of values of c, then we could deduce that the Fredholm resolvent would produce solutions to (C18) for continuous ranges of c (that is, for all values of c except those belonging to this discrete spectrum of the kernel) and, therefore, that a unique value of the crossing parameter is not selected. The correctness of this assumption will now be proved.

In other words, we have an inhomogeneous Fredholm equation

$$\tilde{\psi}(\lambda,s)+\varphi(\lambda,s)=\int K(\lambda;s,s')\tilde{\psi}(\lambda,s')ds' , \quad \text{(C23)}$$

and we must show that the eigenvalues $\kappa(\lambda)$, given by

$$\kappa(\lambda)\psi(\lambda,s)=\int K(\lambda;s,s')\psi(\lambda,s')ds' , \quad \text{(C24)}$$

can take on the value $\kappa(\lambda)=+1$ for only a discrete set of values of λ. It is given that $\varphi(\lambda,s)$ and $K(\lambda;s,s')$ are continuous functions of s and s' on the range for which (C23) is defined and are analytic functions of the parameter λ in some domain of the λ plane and that

$$\int |\varphi(\lambda,s)|^2 ds<\infty , \quad \text{(C25)}$$

$$\int\int |K(\lambda;s,s')|^2 ds ds'<\infty . \quad \text{(C26)}$$

It is sufficient to prove this for (C24), which is actually the homogeneous equation, since the number of eigenvalues of the homogeneous adjoint equation is well known to be the same. As is usual for completely continuous operators such as (C26), we begin by considering a finite-rank kernel

$$K^N(\lambda;s,s')=\sum_{r=1}^N a_r(\lambda,s)b_r{}^*(\lambda,s') , \quad \text{(C27)}$$

where $a_r(\lambda,s)$ and $b_r(\lambda,s)$ are analytic functions of λ for the same domain of λ as is $K(\lambda;s,s')$ and

$$\langle a_m|a_n\rangle=\delta_{mn}=\langle b_m|b_n\rangle . \quad \text{(C28)}$$

Then,

$$\psi(\lambda,s)=\sum_{m=1}^N c_m(\lambda)b_m(\lambda,s) , \quad \text{(C29)}$$

so that (C24) becomes

$$\sum_{r=1}^N c_r(\lambda)a_r(\lambda,s)=\kappa(\lambda)\sum_{m=1}^N c_m(\lambda)b_m(\lambda,s) . \quad \text{(C30)}$$

This implies

$$\sum_{r=1}^N [f_{mr}(\lambda)-\delta_{mr}\kappa(\lambda)]c_r(\lambda)=0 , \quad \text{(C31)}$$

where the functions

$$f_{mr}(\lambda)\equiv\langle b_m(\lambda)|a_r(\lambda)\rangle \quad \text{(C32)}$$

are analytic in λ. As usual we must set

$$\det|f_{mr}(\lambda)-\delta_{mr}\kappa(\lambda)|=0 . \quad \text{(C33)}$$

We are interested in showing that there is only a discrete set of values of λ for which $\kappa=1$. Therefore, setting $\kappa=1$, we obtain

$$\det|f_{mr}(\lambda)-\delta_{mr}|=0 . \quad \text{(C34)}$$

This quantity is a polynomial of degree N in the $f_{mr}(\lambda)$, so that it is an analytic function of λ. Therefore, (C34) can hold for only a discrete set of values of λ, say $\{\lambda_j\}$, since the zeros of an analytic function are isolated.

We can now return to our exact equation (C24), since a completely continuous kernel satisfying (C26) can be approximated arbitrarily closely by a kernel of finite rank of the form (C27).[17] The solutions of (C24) may be approximated arbitrarily closely by those of

$$\kappa_0(\lambda)\psi_0(\lambda,s)=\int K^N(\lambda;s,s')\psi_0(\lambda,s')ds' , \quad \text{(C35)}$$

where

$$K(\lambda;s,s')=K^N(\lambda;s,s')+\epsilon R(\lambda;s,s') , \quad \text{(C36)}$$

in which the norm of $\epsilon R(\lambda;s,s')$ can be made as small as we please for N large enough.[18] Therefore, we shall write

$$\psi(\lambda,s)=\psi_0(\lambda,s)+\delta\chi(\lambda,s) . \quad \text{(C37)}$$

[17] B. Friedman, *Principles and Techniques of Applied Mathematics* (John Wiley & Sons, Inc., New York, 1956), p. 39.
[18] S. G. Mikhlin, Ref. 12, p. 22.

From Eqs. (C24) and (C35)–(C37) we find

$$|\kappa(\lambda)\psi(\lambda,s)-\kappa_0(\lambda)\psi_0(\lambda,s)| \leq \left|\int K^N(\lambda;\, s,s')[\psi(\lambda,s')-\psi_0(\lambda,s')]ds'\right| + \epsilon\left|\int R(\lambda;\, s,s')\psi(\lambda,s')ds'\right| < \epsilon', \quad \text{(C38)}$$

from which it follows that

$$|\kappa(\lambda)[\psi_0(\lambda,s)+\delta\chi(\lambda,s)]-\kappa_0(\lambda)\psi_0(\lambda,s)| < \epsilon'. \quad \text{(C39)}$$

Therefore, for sufficiently large N we can write

$$\kappa(\lambda)=\kappa_0(\lambda)+\eta\tilde{\kappa}(\lambda), \quad \text{(C40)}$$

where $|\eta\tilde{\kappa}(\lambda)|$ can be made arbitrarily small in the domain of λ being considered. We conclude that $\kappa(\lambda)$ can take on the value 1 for only a discrete set of values of λ.

We can also see easily that the solutions to (C24), or to (C23), are continuous functions of s on the range of s for which (C24) is defined. Since $K(s,s')$ is a continuous function of s and s', it follows that

$$|K(s+\delta,\, s')-K(s,s')| = \delta\tilde{K}(s,s')| < \epsilon, \quad \text{(C41)}$$

so that from Eq. (C24) one obtains

$$\kappa|\psi(s+\delta)-\psi(s)| = \left|\int [K(s+\delta,\, s')-K(s,s')]\psi(s')ds'\right| = \delta\left|\int \tilde{K}(s,s')\psi(s')ds'\right| < \epsilon', \quad \text{(C42)}$$

which states that $\psi(s)$ is continuous.

We began this argument for (C18) by letting $\chi^\alpha(s) = \mathrm{Re}b_1{}^\alpha(s)$. However, it is now evident that our conclusions hold for *any* of the $\mathrm{Re}b_n{}^\alpha(s)$ since the terms $\chi_0{}^\alpha(s)$, $\cos\delta_\alpha(s)$, and $N^\alpha(s,s')$ are given in terms of the lower-order functions, and ultimately in terms of the $b_0\alpha(s)$, which have the desired continuity and analyticity properties by induction. Therefore, we have the result quoted in Secs. III and IV, namely, that the $\mathrm{Re}b_n{}^\alpha(s)$ are continuous functions of c and of s in the range $4M^2 < s < +\infty$ and that they exist for all values of c, except possibly for a discrete set of values of c.

Finally, there is the important technical point of the equivalence of the Fredholm set, Eq. (C18), and of the original set of singular integral equations given by Eq. (C1). That is, we must show that for continuous ranges of the parameter c of the crossing matrix the solutions of Eq. (C18) are necessarily solutions of Eq. (C1) and that they satisfy a Hölder condition and have sufficiently well-behaved asymptotic behavior so that the integrals in Eq. (C1) all converge. We begin by demonstrating the equivalence of these sets of equations. Symbolically, Eq. (C1) may be written

$$\begin{aligned} K_1\varphi_1 &= L_1\varphi_2+f_1, \\ K_2\varphi_2 &= L_2\varphi_1+f_2, \end{aligned} \quad \text{(C43)}$$

where the K_j and L_j, $j=1, 2$, are integral operators with Cauchy singularities. Now Appendix B was concerned with the construction of operators $K_j{}^*$ and $L_j{}^*$ such that

$$\begin{aligned} K_jK_j{}^* &= I = K_j{}^*K_j, \\ L_jL_j{}^* &= I = L_j{}^*L_j, \end{aligned} \quad \text{(C44)}$$

where I is the identity and the index j is not summed. Therefore, Eq. (C43) is equivalent to the set

$$\begin{aligned} \varphi_1 &= K_1{}^*L_1\varphi_2+K_1{}^*f_1, \\ \varphi_2 &= K_2{}^*L_2\varphi_1+K_2{}^*f_2. \end{aligned} \quad \text{(C45)}$$

But Eqs. (C43) and (C45) together imply

$$\begin{aligned} K_1\varphi_1 &= L_1K_2{}^*L_2\varphi_1+g_1, \\ K_2\varphi_2 &= L_2K_1{}^*L_1\varphi_2+g_2, \end{aligned} \quad \text{(C46)}$$

where

$$\begin{aligned} g_1 &= L_1K_2{}^*f_2+f_1, \\ g_2 &= L_2K_1{}^*f_1+f_2. \end{aligned}$$

In turn, Eq. (C46) is equivalent to the Fredholm set

$$\begin{aligned} \varphi_1 &= \varphi_1{}^0+M_1\varphi_1, \\ \varphi_2 &= \varphi_2{}^0+M_2\varphi_2, \end{aligned} \quad \text{(C47)}$$

where the Fredholm operators are given as

$$\begin{aligned} M_1 &= K_1{}^*L_1K_2{}^*L_2, \\ M_2 &= K_2{}^*L_2K_1{}^*L_1, \end{aligned}$$

and

$$\begin{aligned} K_1\varphi_1{}^0 &= g_1, \\ K_2\varphi_2{}^0 &= g_2. \end{aligned}$$

In fact, Eq. (C47) is just Eq. (C18). Therefore, we need only show, for continuous ranges of the parameter c, that the solutions of Eq. (C46) are necessarily the solutions of Eq. (C43).

Direct use of Eq. (C46) shows that

$$\begin{aligned} (K_1\varphi_1-L_1\varphi_2-f_1) &\equiv N_1(K_1\varphi_1-L_1\varphi_2-f_1), \\ (K_2\varphi_2-L_2\varphi_1-f_2) &\equiv N_2(K_2\varphi_2-L_2\varphi_1-f_2), \end{aligned} \quad \text{(C48)}$$

where the Fredholm operators N_j are given as

$$\begin{aligned} N_1 &= L_1K_2{}^*L_2K_1{}^*, \\ N_2 &= L_2K_1{}^*L_1K_2{}^*. \end{aligned} \quad \text{(C49)}$$

Since N_j is not equal to the identity operator, Eq. (C48) implies

$$\begin{aligned} K_1\varphi_1-L_1\varphi_2-f_1 &= \psi_1, \\ K_2\varphi_2-L_2\varphi_1-f_2 &= \psi_2, \end{aligned} \quad \text{(C50)}$$

where the ψ_j are eigenvectors of N_j with unit eigenvalues; i.e.,

$$N_j\psi_j=\psi_j. \quad \text{(C51)}$$

However, as we have already seen in this Appendix, a Fredholm operator depending analytically upon a parameter c can have eigenvectors corresponding to unit eigenvalue for only a discrete set of values of c. Therefore, there are continuous ranges of c for which the only solutions to Eq. (C51) are $\psi_j\equiv 0$. For these continuous ranges of c, Eq. (C50) reduces to Eq. (C43), which establishes the equivalence of Eqs. (C18) and (C1).

The solutions of Eq. (C18) have been shown to be continuous functions of s, so that they obviously satisfy a Hölder condition by virtue of the mean-value theorem. The Hölder condition can be established under even weaker conditions than that of continuity.[19] Since the solutions of the Fredholm equations (C18) belong to the class of functions $\mathfrak{L}_2$, these functions must vanish for large values of s. All of the integrals in Eq. (C1) are weighted with the functions $\sigma^\alpha(s)$ which are assumed to vanish sufficiently rapidly to ensure convergence of the principal-value integrals. If this is not the case, then one simply makes enough subtractions to guarantee convergence.

[19] N. I. Muskhelishvili, Ref. 11, pp. 135–140.

PHYSICAL REVIEW VOLUME 148, NUMBER 4 26 AUGUST 1966

Ambiguity of the Meson-Baryon Couplings in a Bootstrap Static Model of $SU(6)$ Symmetry*

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(Received 8 April 1966)

In the present discussion we seek to clarify certain aspects of $SU(6)$ theory which have an important bearing on the problem of formulating a bootstrap dynamical description of symmetry breaking. We discuss here, in particular, a certain ambiguity of the meson-baryon coupling which exists even in the limit of exact $SU(6)$ symmetry, and which we call assignment mixing. It is possible to recover certain special theories, such as the so-called W-spin theory, by a particular choice of the assignment mixing angles. Bootstrap equations in the exact $SU(6)$ limit do not fix the angles, unless one also considers mesonic bootstrap equations corresponding to Fermi-Yang–type theories. It is also shown that in the exact $SU(6)$ limit, the normalization and vertex equations of the Cutkosky-Leon bootstrap method both yield the same equation, which relates the coupling constant f to the ratio of the meson mass m to the cutoff parameter k_Λ. Approximate solutions of the Bethe-Salpeter equation are obtained.

I. INTRODUCTION AND SUMMARY

It is commonly believed that $SU(6)$ theory is a closed subject in the static limit and that the only interest is in the formulation of a relativistic version of the $SU(6)$ group. However, our experience with the bootstrap version of $SU(6)$ theory has been that there are still some features of $SU(6)$ symmetry even in the nonrelativistic domain which, to the best of our knowledge, have not yet been thoroughly discussed. One of these features is the problem of assignment mixing, which we will discuss in the present paper. This mixing leads to the situation that even though there is only one $SU(6)$ Clebsch-Gordan coefficient[1] for coupling $35\otimes 56\supset 56$, there still remains an ambiguity in the meson-baryon couplings.

The present paper is in the first place an extension of the bootstrap version of $SU(6)$ symmetry of Capps[2] and of Belinfante and Cutkosky.[3] In addition, we intend to provide an elementary and rather explicit discussion of the model. Our emphasis therefore is not on the various successful features of the $SU(6)$ bootstrap theory, but on the conceptual problems involved in the formulation of the theory.

Since we wish to discuss low-energy meson-baryon scattering, it is reasonable to take advantage of the great simplifications which arise by making use of the static model, suitably extended to include vector mesons and spin-$\frac{3}{2}$ isobars. The simplifications include, first of all, the limitation to p-wave orbital angular momentum states. A second nice feature of the static model is that the baryon mass M disappears from the final bootstrap equations,[4] thus reducing the number of parameters in the theory. In the third place, the static model is very familiar and we have therefore the advantage of being able to build on previously acquired intuition. Finally, it is our belief that the close relation of the static model to relativistic dispersion theory[5] may help to provide a link between the present nonrelativistic theory and a relativistic $SU(6)$ theory, if such a theory exists at all. In a relativistic theory there are several vertices which

* Work supported in part by the U. S. Atomic Energy Commission.
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[2] R. H. Capps, Phys. Rev. Letters 14, 31 (1965).
[3] J. G. Belinfante and R. E. Cutkosky, Phys. Rev. Letters 14, 33 (1965).
[4] In the case of broken symmetry, only the mass differences between the various baryons will appear.
[5] G. F. Chew, M. L. Goldberger, F. E. Low, and Y. Nambu, Phys. Rev. 106, 1337 (1957).