where now the prime means that interior indices cannot be equal to i. Summing this equation first over even n and then over odd n, we get two coupled equations, as before, which are

$$\sum_{n=2,4,\cdots}(K^n)_{ii}=K_{ii}\sum_{n=1,2,\cdots}(K^n)_{ii}+R\beta\Big(1+\sum_{n=2,4,\cdots}(K^n)_{ii}\Big)$$

and

$$\sum_{n=1,3,\cdots}(K^n)_{ii}=K_{ii}\Big(1+\sum_{n=2,4,\cdots}(K^n)_{ii}\Big)+R\beta\sum_{n=1,3,\cdots}(K^n)_{ii},$$

from which we can get

$$A_{ii}=\sum_{n=1,3,\cdots}(K^n)_{ii}=\frac{K_{ii}}{(1-R\beta)^2-K_{ii}{}^2}. \tag{B23}$$

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Applications of the Chiral $U(6)\otimes U(6)$ Algebra of Current Densities*

J. D. BJORKEN
Stanford Linear Accelerator Center, Stanford University, Stanford, California
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Consequences of the local commutation relations of vector and axial currents proposed by Gell-Mann are explored: (1) A recipe for detecting and isolating Schwinger terms in the commutators, proportional to derivatives of the δ function, is discussed. (2) Under assumptions of smooth asymptotic behavior of form factors for forward scattering of the isovector current from a proton, we show that the $U(3)\otimes U(3)$ algebra for the time components of the currents implies the $U(6)\otimes U(6)$ algebra for space components, at least for spin-averaged diagonal single-particle states. (3) The derivation of the Adler-Weisberger formula for G_A/G_V is sharpened by giving arguments that, at fixed energy, the forward π-p Green's function satisfies an unsubtracted dispersion relation in the pion mass. (4) A lower bound for inelastic electron-nucleon scattering at high momentum transfer is derived on the basis of $U(6)\otimes U(6)$. (5) The contribution of very virtual photons to the hyperfine anomaly in hydrogen is shown to be related to an equal-time commutator of currents; this contribution is crudely estimated to be <4 parts per million (ppm). (6) The logarithmically divergent part of electromagnetic mass differences of hadrons is shown to be proportional to matrix elements of the equal-time commutator of the electromagnetic current with its time derivative. It is suggested that this "divergent" part be identified with the Coleman-Glashow "tadpoles"; this suggestion is discussed in the framework of a simple quark model. (7) The logarithmically divergent part of the electromagnetic correction to the process $\pi^-\to\pi^0+e^-+\bar{\nu}$ is, on the basis of the $U(6)\otimes U(6)$ current algebra, shown to be nonvanishing, and is computed. (8) A speculative argument is presented that the rate $e^++e^-\to$ hadrons is comparable to the rate $e^++e^-\to\mu^++\mu^-$ in the limit of large energies.

I. INTRODUCTION

IN this paper we apply the chiral $U(6)\otimes U(6)$ algebra of current densities proposed by Gell-Mann[1] and by Feynman, Gell-Mann, and Zweig[2] to various processes. We propose a criterion for detecting and isolating singular terms proportional to gradients of delta functions. These Schwinger terms[3] have inhibited the use of the full information contained in the algebra of current densities. In particular, the behavior of matrix elements of currents as the momentum q carried by the currents approaches infinity can be determined in terms of the current algebra. Some applications involving electromagnetic corrections to hadron processes have been found. The program of the paper is as follows:

Section II: We propose a criterion for identification of Schwinger terms. The crux of the matter is that the T product of currents used in making sum rules is in general not covariant. This was recognized and discussed[4] by Johnson in 1961. We give a rule for constructing the T product from the corresponding covariant amplitude. The difference of the two objects is the Schwinger term.

Section III: The claims of Sec. II are illustrated for the vacuum expectation value of the T product of two currents. This section is essentially a summary of Johnson's paper.

Section IV: We next take the T product of two isovector currents between protons at rest and show that the only Schwinger terms are in the disconnected graphs, provided certain form factors behave reasonably at infinity. If this is the case, we can furthermore show that if the time components of the current densities satisfy a $U(3)\otimes U(3)$ algebra, the space components satisfy the $U(6)\otimes U(6)$ algebra, at least for diagonal matrix elements between single-particle states, spin averaged.

* Work supported by the U. S. Atomic Energy Commission.
[1] M. Gell-Mann, Phys. Rev. **125**, 1062 (1962); Physics **1**, 63 (1964).
[2] R. Feynman, M. Gell-Mann, and G. Zweig, Phys. Rev. Letters **13**, 678 (1964).
[3] J. Schwinger, Phys. Rev. Letters **3**, 296 (1959).
[4] K. Johnson, Nucl. Phys. **25**, 431 (1961).

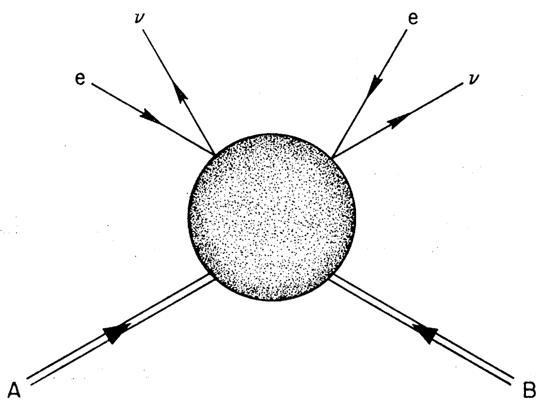

FIG. 1. Second-order weak-interaction S-matrix element.

Section V: In this section we show that the forward π-p scattering amplitude for a virtual pion [whose interpolating field is $\partial_\mu j_\mu(x)^{\text{axial}}$] satisfies an unsubtracted dispersion relation in the mass, for fixed laboratory energy. This allows one to sharpen the derivation of the Adler-Weisberger formula[5] for $|G_A/G_V|$ by giving some justification for the analytic continuations needed in that calculation.

Section VI: We look at the spin-dependent part of forward Compton scattering of a virtual photon and using the $U(6)\otimes U(6)$ algebra derive an inequality for inelastic electron-nucleon scattering:

$$\lim_{q^2\to-\infty}\lim_{E_{\text{inc}}\to\infty} q^4 E_{\text{inc}}\int_0^\infty \frac{d\nu}{\nu}\frac{d}{dq^2 d\nu}(\sigma_p+\sigma_n) > \frac{8\pi\alpha^2}{3}\left|\frac{G_A}{G_V}\right|, \quad (1.1)$$

where $\nu=E_{\text{inc}}-E_f$. We conjecture that

$$\frac{d\sigma_p}{dq^2}\gtrsim\frac{\pi\alpha^2}{q^4} \quad (1.2)$$

in the same limit.

Section VII: The results of Sec. VI are applied to the hyperfine structure in hydrogen; it is concluded that the contribution of very virtual photons ($q^2\ll -m_p^2$) is bounded by a few ($\sim$4) parts per million (ppm) and probably cannot explain the 20-ppm anomaly.

Section VIII: We show that the logarithmically divergent part of electromagnetic mass differences is proportional to matrix elements of the equal-time commutators of the currents with their time derivatives. On the basis of a simple quark model, we argue (but cannot prove) that these matrix elements are finite, nonvanishing, and have $SU(3)$ octet transformation properties. If the quark mass term in H is dominant, many of the Coleman-Glashow "tadpole" theory results emerge.

Section IX: We examine the radiative corrections to β decay of a pion, and show, on the basis of chiral $U(6)\otimes U(6)$, that to all orders of strong interaction the radiative correction diverges logarithmically; in particular

$$\mathfrak{M}\approx\mathfrak{M}_0\left\{1+\frac{3\alpha}{4\pi}\ln\frac{\Lambda^2}{M^2}\right\}, \quad (1.3)$$

where $\mathfrak{M}_0$ is the lowest order amplitude.

Section X: Finally we look at the process $e^++e^-\to$ hadrons, and show that the total cross section satisfies the relation

$$\int dq^2 q^4\sigma_{\text{tot}}(q^2) = 16\pi^2\alpha^2\int\langle 0|[j_z(0,\mathbf{x}),[H,j_z(0)]]|0\rangle d^3x, \quad (1.4)$$

where q^2 is the square of the total center-of-mass energy.

Using the toy Hamiltonian of Sec. VIII, we find a quartic divergence in the right-hand side, suggesting that within logarithmic factors

$$\sigma_{\text{tot}}(q^2)\sim\alpha^2/q^2 \quad \text{as} \quad q^2\to\infty. \quad (1.5)$$

II. THE SCHWINGER TERMS

The Schwinger terms[3,4] are singular terms in the commutator of current densities. Specifically, Schwinger showed that

$$[j_0(0,\mathbf{x}),\mathbf{j}(0,\mathbf{x}')]=C\nabla\delta(\mathbf{x}-\mathbf{x}'), \quad (2.1)$$

where $j_\mu(x)$ is, say, the electromagnetic current density. That such a term is present can be demonstrated by manipulation of the vacuum expectation value. In constructing sum rules such terms get in the way; what we shall endeavor to do is to give a recipe which identifies and isolates these contributions.

We argue that the existence of Schwinger terms is demanded by locality and Lorentz kinematics alone, and indeed may be isolated by using only this information. We illustrate what we mean by considering the isovector $\Delta S=0$ currents $j_\mu^\pm$ defined by the β-decay interaction satisfying

$$[Q^+,Q^-]=2Q_3,$$
$$[Q^3,Q^\pm]=\pm Q^\pm,$$
$$Q^\pm=\int d^3x\, j_0^\pm(\mathbf{x},0). \quad (2.2)$$

We center our considerations on the time-ordered product[6]

$$M_{\mu\nu}(q,\cdots)=-i\int d^4x\, e^{iq\cdot x}\langle A|T(j_\mu^+(x)j_\nu^-(0))|B\rangle \quad (2.3)$$

[5] S. Adler, Phys. Rev. Letters 14, 1051 (1965); W. Weisberger, *ibid.* 14, 1057 (1965).

[6] Throughout this paper, we normalize single-particle states such that $\langle p'|p\rangle=(E/M)(2\pi)^3\delta^3(\mathbf{p}'-\mathbf{p}')$.

and the absorptive parts

$$\rho_{\mu\nu}(q,\cdots)=\int d^4x\, e^{iq\cdot x}\langle A|j_\mu^+(x)j_\nu^-(0)|B\rangle,$$

$$\bar{\rho}_{\mu\nu}(q,\cdots)=\int d^4x\, e^{-iq\cdot x}\langle A|j_\mu^-(0)j_\nu^+(x)|B\rangle,$$

$$\rho_{\mu\nu}=\sum_n (2\pi)^4\delta^4(q+P_A-P_n)\times\langle A|j_\mu^+(0)|n\rangle\langle n|j_\nu^-(0)|B\rangle,$$

$$\bar{\rho}_{\mu\nu}=\sum_n (2\pi)^4\delta^4(q+P_B-P_n)\times\langle A|j_\mu^-(0)|n\rangle\langle n|j_\nu^+(0)|B\rangle. \quad (2.4)$$

The main point of this section is that in general, for kinematical reasons alone, *the time-ordered product is not a covariant amplitude;* i.e., it does not transform as a second-rank tensor. This will be shown below by considering the vacuum expectation value; however, before delving into the arithmetic, we note that there always does exist a covariant amplitude, defined by the second-order weak matrix element shown in Fig. 1. The covariance of the S matrix demands that the amplitude $\tilde{M}_{\mu\nu}$ which multiplies the lepton currents does transform as a second-rank tensor.

We now rephrase the problem of Schwinger terms in the following way: Given (phenomenologically) the covariant amplitude $\tilde{M}_{\mu\nu}$, how do we construct the time-ordered product? We want the T product, because we can then use the Fubini-Furlan technique to construct sum rules.

We propose that $M_{\mu\nu}$ and $\tilde{M}_{\mu\nu}$, considered as analytic functions of q_0, have the same absorptive parts $\rho_{\mu\nu}$ and $\bar{\rho}_{\mu\nu}$ [considered in coordinate space this means $M_{\mu\nu}(x)=\tilde{M}_{\mu\nu}(x)$ for $x_0\neq 0$]. Therefore, $M_{\mu\nu}$ and $\tilde{M}_{\mu\nu}$ differ at most by a polynomial in q_0 [in coordinate space, this means terms $\delta(x_0)$, $\delta'(x_0)$, $\cdots$]. Finally, the time-ordered product vanishes as $q_0\to\infty$, as seen from expanding (2.3) and truncating the intermediate-state sum (as we will eventually do in one way or another before confronting the theory with experimental numbers):

$$M_{\mu\nu}(q,\cdots)=\int_0^\infty\frac{dq_0'}{(2\pi)}\times\left[\frac{\rho_{\mu\nu}(q_0',\mathbf{q},\cdots)}{q_0-q_0'}-\frac{\bar{\rho}_{\nu\mu}(q_0',-\mathbf{q},\cdots)}{q_0+q_0'}\right]. \quad (2.5)$$

Thus, we construct the time-ordered product $M_{\mu\nu}$ from the covariant amplitude $\tilde{M}_{\mu\nu}$ by letting $q_0\to\infty$ at fixed $\mathbf{q}$, identifying any polynomial in q_0, and subtracting it off.

It is useful also to observe that the term $O(1/q_0)$ as $q_0\to\infty$ is proportional to the equal-time commutator of the currents

$$M_{\mu\nu}\xrightarrow[q_0\to\infty]{}\frac{1}{q_0}\int\frac{dq_0'}{(2\pi)}[\rho_{\mu\nu}(q_0',\mathbf{q},\cdots)-\bar{\rho}_{\nu\mu}(q_0',-\mathbf{q},\cdots)]$$
$$=\frac{1}{q_0}\int d^3x\langle A|[j_\mu^+(0,\mathbf{x}),j_\nu^-(0,0)]|B\rangle e^{-i\mathbf{q}\cdot\mathbf{x}}. \quad (2.6)$$

The higher terms involve multiple commutators of the currents with H; e.g., the next term is

$$\frac{1}{q_0^2}\int\frac{dq_0'}{2\pi}q_0'[\rho_{\mu\nu}(q_0',\mathbf{q},\cdots)+\bar{\rho}_{\nu\mu}(q_0',-\mathbf{q},\cdots)]$$
$$=\frac{1}{q_0^2}\int d^3x\, e^{-i\mathbf{q}\cdot\mathbf{x}}\langle A|[[j_\mu^+(0,\mathbf{x}),H]j_\nu^-(0)]|B\rangle. \quad (2.7)$$

III. VACUUM EXPECTATION VALUE

For completeness, we consider briefly the previous case with $|A\rangle=|B\rangle=|0\rangle$, although this is treated in Johnson's paper.[4] The absorptive part $\rho_{\nu\mu}$ has the form

$$\rho_{\mu\nu}(q)=\bar{\rho}_{\nu\mu}(q)=(q_\mu q_\nu-g_{\mu\nu}q^2)\rho(q^2) \quad (3.1)$$

and we take the covariant amplitude[7] as obtainable by a dispersion integral[8] over ρ:

$$\tilde{M}_{\mu\nu}=\frac{(q_\mu q_\nu-g_{\mu\nu}q^2)}{2\pi}\int\frac{d\sigma^2\rho(\sigma^2)}{q^2-\sigma^2}. \quad (3.2)$$

To obtain $M_{\mu\nu}$ we let $q_\mu=q_0\eta_\mu$ and let $q_0\to\infty$ [$\eta_\mu=(1,0,0,0)$]:

$$\tilde{M}_{\mu\nu}\xrightarrow[q_0\to\infty]{}(\eta_\mu\eta_\nu-g_{\mu\nu})\int\frac{d\sigma^2\rho(\sigma^2)}{2\pi}. \quad (3.3)$$

This is the Schwinger term; the time-ordered product is

$$M_{\mu\nu}=\tilde{M}_{\mu\nu}-(\eta_\mu\eta_\nu-g_{\mu\nu})\int\frac{d\sigma^2\rho(\sigma^2)}{2\pi}. \quad (3.4)$$

To evaluate the equal-time commutator, we may take $q_\mu M_\nu{}^\mu$ and integrate (2.3) by parts.

$$q^\mu M_{\mu\nu}=\int d^3x\langle A|[j_0^+(0,\mathbf{x}),j_\nu^-(0)]|B\rangle e^{-i\mathbf{q}\cdot\mathbf{x}}$$
$$=[-(\eta\cdot q)\eta_\nu+q_\nu]\int\frac{d\sigma^2\rho(\sigma^2)}{2\pi}$$
$$=(0,\mathbf{q})\int\frac{d\sigma^2\rho(\sigma^2)}{2\pi}. \quad (3.5)$$

This is the result of Schwinger. If we use (2.6) and

[7] We assume conserved currents, although the existence of the Schwinger terms does not depend upon this.

[8] Any extra constant "mass" term $\sim g_{\mu\nu}$ is removed in forming $M_{\mu\nu}$, cf. Eq. (3.4).

evaluate the term $O(1/q_0)$ from (3.2) and (3.4), we arrive at the same conclusion. It may also be obtained by directly evaluating (2.5); here the Schwinger term arises because the polynomial projection operator $(q_\mu q_\nu - g_{\mu\nu}q^2)$ depends upon the integration variable q_0'.

IV. EXPECTATION VALUE BETWEEN PROTONS

We now consider the same commutator between proton states of the same spin and momentum, and averaged over spins.[9] (The spin-dependent terms will be considered later.) This is the case considered by Adler, who has derived fixed-momentum-transfer sum rules.[10] The general form of the covariant amplitude $\tilde{M}_{\mu\nu}$ is then

$$\begin{aligned}\tilde{M}_{\mu\nu} &= \frac{1}{2}\sum_s \int d^4x\, e^{iq\cdot x}\langle Ps|T(j_\mu^+(x)j_\nu^-(0))|Ps\rangle \\ &\qquad + (\text{Schwinger terms}) \\ &= P_\mu P_\nu \mathfrak{F}_1(q^2,\nu) + (P_\mu q_\nu + P_\nu q_\mu)\mathfrak{F}_2(q^2,\nu) \\ &\qquad + q_\mu q_\nu \mathfrak{F}_3(q^2,\nu) + g_{\mu\nu}\mathfrak{F}_4(q^2,\nu), \quad \nu = q\cdot P/M, \end{aligned} \tag{4.1}$$

which we rewrite as

$$\begin{aligned}\tilde{M}_{\mu\nu} &= P_\mu P_\nu M_0(\nu) \\ &+ [q^2 P_\mu P_\nu - (q\cdot P)(q_\mu P_\nu + q_\nu P_\mu) + (q\cdot P)^2 g_{\mu\nu}]M_1(q^2,\nu) \\ &+ (q_\mu q_\nu - g_{\mu\nu}q^2)M_2(q^2,\nu) \\ &+ (q_\mu P_\nu + q_\nu P_\mu - g_{\mu\nu} q\cdot P)M_3(q^2,\nu) \\ &\qquad + g_{\mu\nu}M_4(q^2,\nu) + B_{\mu\nu} + \tilde{D}_{\mu\nu}. \end{aligned} \tag{4.2}$$

$B_{\mu\nu}$ is the Born term and $\tilde{D}_{\mu\nu}$ the disconnected covariant amplitude, identical to (3.2).[11]

$$\begin{aligned}B_{\mu\nu} &= \frac{2(P_\mu + \frac{1}{2}q_\mu)(P_\nu + \frac{1}{2}q_\nu)}{M(q^2+2M\nu)}\left[F_{1v}^2 - \frac{q^2}{4M^2}F_{2v}^2\right] \\ &\qquad - \frac{(q_\mu q_\nu - g_{\mu\nu}q^2)}{2M(q^2+2M\nu)}(F_{1v}+F_{2v})^2, \end{aligned} \tag{4.3}$$

where F_{1v} and F_{2v} are the Dirac isovector form factors normalized to 1 and $(\kappa_P - \kappa_N)$, respectively.

For $q^2<0$ (spacelike), the absorptive parts of $\tilde{M}_{\mu\nu}$ are confined to the conserved pieces M_1 and M_2 which satisfy fixed q^2 dispersion relations in ν. Thus M_0, M_3, and M_4 are polynomials in ν for fixed q^2.

In constructing $\tilde{M}_{\mu\nu}$ from $M_{\mu\nu}$, we see that we will obtain a Schwinger term from $\tilde{D}_{\mu\nu}$, as well as lose M_0 completely. However, provided

$$\begin{aligned} q^2 M_1 &\to 0, \\ q^2 M_2 &\to 0, \\ q\cdot P M_3 &\to 0, \\ M_4 &\to 0, \end{aligned} \tag{4.4}$$

as $q_0 \to i\infty$ $(\nu \to i\infty;\, q^2 \to -\infty)$, no other Schwinger terms will be induced. These are reasonable assumptions, which we hereafter accept. Then $M_{\mu\nu}$ is given by (4.2), with M_0 omitted and $\tilde{D}_{\mu\nu} \to D_{\mu\nu}$. Taking the divergence, we find

$$\begin{aligned} q^\mu M_{\mu\nu} &= q^2 P_\nu M_3 + q_\nu M_4 \\ &\qquad + \frac{(P_\nu + \frac{1}{2}q_\nu)}{M}\left[F_{1v}^2 - \frac{q^2}{4M^2}F_{2v}^2\right] + q^\mu D_{\mu\nu}. \end{aligned} \tag{4.5}$$

From local commutation relations,[1] we expect, from (2.2) and (2.3),

$$\begin{aligned} q^\mu M_{\mu\nu} &= \frac{1}{2}\sum_s \int d^3x \langle Ps|[j_0^+(0,\mathbf{x}), j_\nu^-(0)]|Ps\rangle e^{-i\mathbf{q}\cdot\mathbf{x}} \\ &= \sum_s \langle Ps|j_\nu^3(0)|Ps\rangle + (\text{disconnected piece}) \\ &= (P_\nu/M) + (\text{disconnected piece}). \end{aligned} \tag{4.6}$$

Equating (4.6) and (4.5), we have

$$\begin{aligned} M_3 &= \frac{1}{Mq^2}\left[1 - \left\{F_{1v}^2 - \frac{q^2}{4M^2}F_{2v}^2\right\}\right], \\ M_4 &= -\frac{1}{2M}\left[F_{1v}^2 - \frac{q^2}{4M^2}F_{2v}^2\right]. \end{aligned} \tag{4.7}$$

We find an interesting result in the case that

$$\begin{aligned} q_0^3 M_1 &\to 0, \\ q_0^3 M_2 &\to 0, \\ q^2 F_i &< \infty. \end{aligned} \tag{4.8}$$

Then as $q_0 \to i\infty$, everything comes from M_3:

$$M_{\mu\nu} \to (\eta_\mu P_\nu + \eta_\nu P_\mu - g_{\mu\nu}\eta\cdot P)/Mq_0, \tag{4.9}$$

and aside from the Schwinger term in the disconnected part[12] we find

$$\frac{1}{2}\sum_s \int d^3x\, e^{-i\mathbf{q}\cdot\mathbf{x}}\langle Ps|[j_\mu^+(0,\mathbf{x}), j_\nu^-(0)]|Ps\rangle = \frac{\eta_\mu P_\nu + \eta_\nu P_\mu - g_{\mu\nu}\eta\cdot P}{M}. \tag{4.10}$$

This is what is expected from quark currents, e.g.,

$$[j_i(0,\mathbf{x}), j_i(0)] = 2j_0^3(0)\delta^3(\mathbf{x}). \tag{4.11}$$

Thus it appears that the theory which is "as smooth as possible" is that for which the current algebra is chiral $U(6)\otimes U(6)$.

The Adler sum rule[10] is obtained by demanding that the coefficient of $(P_\mu q_\nu + P_\nu q_\mu)$ in $\tilde{M}_{\mu\nu}$ satisfy an unsubtracted dispersion relation; we need not go into

[9] The same calculation goes through for any single-particle state.

[10] S. Adler, Phys. Rev. **143**, 1144 (1966).

[11] Within a factor $(2\pi)^3(E/M)\delta^3(0)$.

[12] In fact, we may choose $\mathbf{q}=0$; there is, in that case, no Schwinger term at all.

detail here, as it has been discussed considerably elsewhere.[13]

V. THE ADLER-WEISBERGER FORMULA

As an application of some of these ideas, we sharpen the derivation of the Adler-Weisberger sum rule[5] for the axial-vector renormalization in β decay. We consider

$$M(q^2,\nu)=-i\int d^4x\, e^{iq\cdot x}\langle P|T(D^+(x)D^-(0))|P\rangle. \tag{5.1}$$

Here $|P\rangle$ is a proton state of momentum P and

$$D^{\pm}(x)=\partial j_\mu(x)_{\text{axial}}{}^{\pm}/\partial x_\mu. \tag{5.2}$$

For spacelike q^2, M satisfies a dispersion relation in ν; the even part we subtract once, and the odd part we leave unsubtracted in accordance with the Pomeranchuk theorem.

$$M(q^2,\nu)=B(q^2,\nu)+\frac{\nu}{2\pi}\int_0^\infty \frac{d\nu'\rho^{\text{odd}}(q^2,\nu')}{(\nu'^2-\nu^2)}+M^{\text{even}}(q^2,\nu). \tag{5.3}$$

B is the Born term. The rigorous formula, from the current algebra, is that

$$(\partial A/\partial\nu)|_{q_2=\nu=0}\equiv(\partial/\partial\nu)(M-B)_{q_2=\nu=0}=1-G_A{}^2. \tag{5.4}$$

To relate this to pion scattering, one argues that the continuum amplitude A is dominated by the double-pion pole for small values of q^2.

$$A(q^2,\nu)\approx[a^2\mu^4/(q^2-\mu^2)^2]A_{\pi p}(\nu), \tag{5.5}$$

where a is a constant related to the pion-decay amplitude. The pole dominance (5.5) is plausible, if A satisfies an unsubtracted dispersion relation in q^2. One knows[14] that for fixed ν, A is analytic in the cut q^2 plane, with branch point at $\sim 8.5\,\mu^2$ for $\nu\sim 0$. Now, on the basis of reasonable commutation relations, we shall show that A indeed satisfies an unsubtracted dispersion relation in q^2, strengthening the argument (5.5). Although this is a fine point in the $\Delta S=0$ sum rule, it may be of some significance in understanding why the $\Delta S=1$ sum rule[15] works at all.

We return to (5.1) and let $q_0\to i\infty$. The term of order $1/q_0$ is odd in ν, and has the form [see (2.5) and (2.6)]

$$M\xrightarrow[q_0\to i\infty]{}\frac{1}{q_0}\int d^3x\, e^{-i\mathbf{q}\cdot\mathbf{x}}\langle P|[D^+(0,\mathbf{x}),D^-(0)]|P\rangle. \tag{5.6}$$

From the dispersion relation, valid for spacelike q^2 as is the case here, we find, assuming the Born terms vanish rapidly for large q^2,

$$M^{\text{odd}}\to\frac{\nu}{2\pi}\int_{\nu_0}^\infty\frac{d\nu'\rho^{\text{odd}}(q^2,\nu')}{(\nu'^2-\nu^2)}. \tag{5.7}$$

But the threshold ν_0 in the dispersion integral is $\sim|q_0|^2/2M$ for large q_0; therefore

$$\left|\frac{\nu'}{\nu}\right|\gtrsim\frac{|q_0|}{2M}\gg 1. \tag{5.8}$$

Thus, as $q_0\to i\infty$

$$M\to\frac{\nu}{2\pi}\int_0^\infty\frac{d\nu'\rho^{\text{odd}}(q^2,\nu')}{\nu'^2}=\nu\left.\frac{\partial A(q^2,\nu)}{\partial\nu}\right|_{\nu=0}. \tag{5.9}$$

We conclude that

$$\frac{E_P}{M}\left.\frac{\partial A(q^2,\nu)}{\partial\nu}\right|_{\nu=0}\xrightarrow[q^2\to-\infty]{}\frac{1}{q^2}\int d^3x\, e^{-i\mathbf{q}\cdot\mathbf{x}}\times\langle P|[D^+(0,\mathbf{x}),D^-(0)]|P\rangle. \tag{5.10}$$

Thus, $A'(q^2,0)$ satisfies an unsubtracted dispersion relation provided the commutator exists.[16] If $D^{\pm}(x)$ is proportional to a canonical pion field, the commutator vanishes. If it is bilinear in Fermi fields, e.g.,

$$D^+(x)=C\bar\psi\gamma_5\tau^+\psi, \tag{5.11}$$

then

$$\left.\frac{\partial A}{\partial\nu}\right|_{\nu=0}\to\frac{|C|^2}{q^2}\quad\text{as}\quad q^2\to-\infty. \tag{5.12}$$

VI. SPIN-DEPENDENT VIRTUAL COMPTON SCATTERING

We apply these ideas to the antisymmetric part of the virtual Compton amplitude from a proton, assuming quark structure for the electromagnetic currents j_μ.

$$[j_\mu(0,\mathbf{x}),j_\nu(0)]=-2i\epsilon_{\mu\nu\lambda\sigma}\eta^\lambda j_5{}^\sigma(0)\delta^3(\mathbf{x})+(\text{gradient terms}), \tag{6.1}$$

where

$$j_\mu=\bar\psi\gamma_\mu Q\psi,\qquad j_\mu{}^5=\bar\psi\gamma_5\gamma_\mu Q^2\psi=(2/9)\bar\psi\gamma_5\gamma_\mu\psi+\tfrac{1}{3}\bar\psi\gamma_5\gamma_\mu Q\psi, \tag{6.2}$$

and

$$Q=\begin{pmatrix}\frac{2}{3}&0&0\\0&-\frac{1}{3}&0\\0&0&-\frac{1}{3}\end{pmatrix},\quad\epsilon_{0123}=1. \tag{6.3}$$

The general structure of the antisymmetric part $M_{\mu\nu}{}^{(a)}$

[13] N. Cabibbo and L. Radicati, Phys. Letters **12**, 697 (1965); S. Adler, Ref. 10.

[14] To all orders of perturbation theory.

[15] C. Levinson and I. Muzinich, Phys. Rev. Letters **15**, 715 (1965); D. Amati, C. Bouchiat, and J. Nuyts, Phys. Letters **19**, 59 (1965); L. Pandit and J. Schechter, *ibid.* **19**, 56 (1965); W. Weisberger, Phys. Rev. **143**, 1302 (1966).

[16] This is required on experimental grounds; the success of the Adler sum rule for the even pion-nucleon amplitude [S. Adler, Phys. Rev. **137**, B1022 (1965)] demands that this commutator be small. See also K. Kawarabayashi and W. Wada, Phys. Rev. **146**, 1209 (1966).

of $M_{\mu\nu}$, defined as in (2.3) with j_μ replacing $j_\mu^\pm$, is given by

$$M_{\mu\nu}^{(a)}=\tfrac{1}{2}\bar{u}\{[\gamma_\nu,\boldsymbol{q}]P_\mu-[\gamma_\mu,\boldsymbol{q}]P_\nu+[\gamma_\mu,\gamma_\nu]P\cdot q\}\,uG_1(q^2,\nu)$$
$$+\tfrac{1}{2}\bar{u}\{[\gamma_\nu,\boldsymbol{q}]q_\mu-[\gamma_\mu,\boldsymbol{q}]q_\nu+[\gamma_\mu,\gamma_\nu]q^2\}uG_2(q^2,\nu)\,. \quad (6.4)$$

From crossing symmetry,

$$M_{\mu\nu}^{(a)}(q,\cdots)=+M_{\nu\mu}^{(a)}(-q,\cdots)=-M_{\mu\nu}^{(a)}(-q,\cdots)$$

and

$$G_1(q^2,-\nu)=G_1(q^2,\nu)\,,\qquad G_2(q^2,-\nu)=-G_2(q^2,\nu)\,. \quad (6.5)$$

We assume unsubtracted dispersion relations for both G_1 and G_2.

For $q^2=0$, $M_{\mu\nu}^{(a)}$ is related to the spin-dependent part of the forward Compton-scattering amplitude; for $q^2<0$, the absorptive part of $M_{\mu\nu}^{(a)}$ is related to the spin-dependent part of inelastic electron-proton scattering. Specifically,

$$\frac{d\sigma^{\uparrow\uparrow}}{dq^2dE'}-\frac{d\sigma^{\downarrow\uparrow}}{dq^2dE'}=\frac{4\alpha^2}{q^2E^2}\times[M(E+E'\cos\theta)\,\mathrm{Im}G_1+q^2\,\mathrm{Im}G_2]\,, \quad (6.6)$$

where $d\sigma^{\uparrow\uparrow}$ is the cross section when the spins of electron and proton are parallel and along the direction of motion of incident electron, and $d\sigma^{\downarrow\uparrow}$ is the cross section for antiparallel spins. E, E', and θ are energies and scattering angle of the electron, $q^2=-4EE'\sin^2(\theta/2)$, and $\nu=E-E'$. We have set m_e equal to 0.

The photoproduction cross section for $q^2=0$ is given by the optical theorem:

$$e^2\,\mathrm{Im}G_1=(1/2M)[\sigma^{\uparrow\uparrow}-\sigma^{\downarrow\uparrow}]\,, \quad (6.7)$$

where $\sigma^{\uparrow\uparrow}$ is the cross section for photon and proton spins aligned. The Born terms are given by

$$G_1^{\mathrm{Born}}=\frac{-2q^2F_1(F_1+F_2)}{M(q^4-4M^2\nu^2)}\,,\qquad G_2^{\mathrm{Born}}=\frac{-2\nu F_2(F_1+F_2)}{q^4-4M^2\nu^2}\,. \quad (6.8)$$

We extract a useful result by considering the limit $q_0\to i\infty$ of $M_{\mu\nu}^{(a)}$. Using (2.6) and (6.1) we find

$$\lim_{q_0\to i\infty} M_{\mu\nu}^{(a)}=\frac{-2i\epsilon_{\mu\nu0\sigma}}{q_0}\langle P|j_5^\sigma(0)|P\rangle$$
$$=\frac{-2i\epsilon_{\mu\nu\lambda\sigma}q^\lambda}{q^2}\langle P|j_5^\sigma(0)|P\rangle. \quad (6.9)$$

We define

$$\langle ps|j_5^\sigma(0)|ps\rangle=Z\bar{u}\gamma_5\gamma^\sigma u=-Zs^\sigma. \quad (6.10)$$

Upon identification of this with the asymptotic behavior of G_1 and G_2 we find that

$$G_1\to -2Z/Mq^2\,,\qquad G_2\to O(1/q_0^4)\quad (q_0\to i\infty)\,. \quad (6.11)$$

On the other hand, we may evaluate G_1 by using an unsubtracted dispersion relation

$$G_1(q^2,\nu)=\frac{2}{\pi}\int_0^\infty\frac{d\nu'\nu'\,\mathrm{Im}G_1(q^2,\nu')}{\nu'^2-\nu^2} \quad (6.12)$$

and[17] as $q_0\to i\infty$

$$G_1(q^2,\nu)\approx\frac{2}{\pi}\int_0^\infty\frac{d\nu'}{\nu'}\,\mathrm{Im}G_1(q^2,\nu')=G_1(q^2,0)\,, \quad (6.13)$$

or

$$\int_0^\infty\frac{d\nu'}{\nu'}\,\mathrm{Im}G_1(q^2,\nu')\xrightarrow[q^2\to-\infty]{}\frac{-\pi Z}{Mq^2}. \quad (6.14)$$

Using Eq. (6.6) as $E\to\infty$ at fixed q^2

$$\mathrm{Im}G_1(q^2,\nu)\to\frac{q^2E}{8\alpha^2M}\left[\frac{d\sigma^{\uparrow\uparrow}}{dq^2d\nu}-\frac{d\sigma^{\downarrow\uparrow}}{dq^2d\nu}\right] \quad (6.15)$$

and we find

$$\lim_{q^2\to-\infty}\lim_{E\to\infty}\int_0^\infty\frac{d\nu'}{\nu'}\left[\frac{d\sigma^{\uparrow\uparrow}}{dq^2d\nu'}-\frac{d\sigma^{\downarrow\uparrow}}{dq^2d\nu'}\right]=\frac{-8\pi\alpha^2Z}{q^4E}. \quad (6.16)$$

It will be a long time before these cross sections are measured. Furthermore, we do not know the value of Z, although $SU(6)$ predicts $Z_p=5/9$ and $Z_n=0$. However, if we take the difference between proton and neutron we know from (6.2), assuming always the $U(6)\otimes U(6)$ current algebra, that

$$Z_p-Z_n=\frac{1}{3}\left(\frac{G_A}{G_V}\right). \quad (6.17)$$

Thus,

$$\int_0^\infty\frac{d\nu'}{\nu'}\frac{d}{dq^2d\nu'}\times[\sigma_p{}^{\uparrow\uparrow}-\sigma_p{}^{\downarrow\uparrow}-\sigma_n{}^{\uparrow\uparrow}+\sigma_n{}^{\downarrow\uparrow}]\to\frac{-8\pi\alpha^2}{3q^4E}\left(\frac{G_A}{G_V}\right). \quad (6.18)$$

Something may be salvaged from this worthless equation by constructing an inequality[18]:

$$\lim_{q^2\to-\infty}\lim_{E\to\infty}q^4E\int_0^\infty\frac{d\nu}{\nu'}\left[\frac{d\sigma_p}{dq^2d\nu'}+\frac{d\sigma_n}{dq^2d\nu'}\right]>\frac{8\pi\alpha^2}{3}\left|\frac{G_A}{G_V}\right|. \quad (6.19)$$

---

[17] Compare Eqs. (5.7)–(5.9).

[18] $\sigma_p=\sigma_p{}^{\uparrow\uparrow}+\sigma_p{}^{\uparrow\downarrow}$.

Aside from the factor $1/\nu'$ in the dispersion integral, this is similar in form to the Adler sum rule[10] for neutrino processes

$$\lim_{q^2\to-\infty}\lim_{E\to\infty}\left(\frac{d\bar\sigma}{dq^2}-\frac{d\sigma}{dq^2}\right)\cong\frac{G^2}{\pi},\tag{6.20}$$

where e^4/q^4 replaces G^2. We suspect the factor $1/\nu'$ is due to our inefficiency in using only the spin-dependent amplitude, and conjecture that a "practical" inequality for electron scattering is

$$\lim_{q^2\to-\infty}\lim_{E\to\infty}\frac{d\sigma_p}{dq^2}\gtrsim\frac{\pi\alpha^2}{q^4}\tag{6.21}$$

in direct analogy to the result for neutrinos.

VII. HYPERFINE INTERACTION

The asymptotic part of the spin-dependent Compton amplitude (6.4) will also contribute to the hyperfine interaction and has not been included in previous analyses,[19] for which the asymptotic behavior (e.g., Born terms) is more rapid than $1/k_0$. The second-order matrix element for the spin-dependent part of forward electron-proton scattering asymptotically ($k_0\to i\infty$) approaches

$$\mathfrak{M}^{(2)}=-ie^4\int\frac{d^4k}{(2\pi)^4k^6}\bar u\gamma^\mu k\gamma^\nu u M_{\mu\nu}{}^{(a)}(P,k).\tag{7.1}$$

Inserting (6.9), and doing the spin algebra, we find

$$\begin{aligned}\mathfrak{M}^{(2)}&=\frac{+\alpha^2 Z}{2\pi^2}\int\frac{d^4k}{k^6}\bar u\gamma^\mu\gamma^\lambda\gamma^\nu u\epsilon_{\mu\nu\lambda\sigma}s^\sigma\\&=+3\alpha^2 Z\bar u\gamma_5 s u\int_{\bar m^2}^\infty\frac{dk^2}{k^4},\end{aligned}\tag{7.2}$$

where $\bar m^2$ is some effective lower cutoff. Comparing with the first-order term, we find a correction

$$\left|\frac{\Delta\nu}{\nu}\right|=\frac{9\alpha}{2\pi}\frac{|Z_p|}{\mu_p}\left[\frac{m_eM_p}{\bar m^2}\right].\tag{7.3}$$

Choosing $Z_p\sim1$ and $\bar m^2=m_p{}^2$, we find an answer

$$\left|\frac{\Delta\nu}{\nu}\right|\sim3.5\times10^{-6}\equiv3.5\text{ ppm}.\tag{7.4}$$

This appears to be too small by nearly an order of magnitude to account for the anomaly[19] of $\sim$20 ppm, and we conclude that within the general picture we have taken [convergent dispersion integrals and chiral $U(6)\otimes U(6)$ current algebra] that the large k^2 region is probably not a major contributor to the hyperfine anomaly.

[19] C. K. Iddings, Phys. Rev. **138**, B446 (1965); this contains references to earlier work.

VIII. ELECTROMAGNETIC MASS SHIFTS

The same methods may be applied to any process where high-momentum (spacelike) virtual photons are involved, in particular, to radiative corrections to processes involving hadrons. The most interesting are the electromagnetic mass shifts of hadrons and the radiative corrections to weak interactions. We survey first the mass shifts. We consider the expression (4.1) for $\bar M_{\mu\nu}$, with electromagnetic currents replacing isospin currents, and spins averaged. To be explicit, we consider the proton, although our results will be general. Then the analog to (4.2) is

$$\begin{aligned}M_{\mu\nu}=&[q^2P_\mu P_\nu-(q\cdot P)(q_\mu P_\nu+q_\nu P_\mu)+(q\cdot P)^2g_{\mu\nu}]\\&\times M_1(q^2,q\cdot P)+(q_\mu q_\nu-g_{\mu\nu}q^2)M_2(q^2,q.P)\\&+B_{\mu\nu}-\frac{g_{\mu\nu}}{M}\left(F_{1p}{}^2-\frac{q^2}{4M^2}F_{2p}{}^2\right)+D_{\mu\nu}.\end{aligned}\tag{8.1}$$

In order to be consistent with the absence of Schwinger terms and with a chiral $U(6)\otimes U(6)$ current algebra, we demand, aside from the disconnected graphs $D_{\mu\nu}$, that $M_{\mu\nu}\to O(1/q_0{}^2)$ as $q_0\to i\infty$. This means

$$\begin{aligned}M_1&\to O(1/q_0{}^4),\\M_2&\to O(1/q_0{}^4),\end{aligned}\tag{8.2}$$

as $q_0\to i\infty$. We assume the Born terms may be ignored in this limit, which is satisfied if

$$\begin{aligned}F_{1p}&\to O(1/q^2),\\F_{2p}&\to O(1/q^2)\\&\text{as }q^2\to-\infty.\end{aligned}\tag{8.3}$$

We concentrate our attention on the divergent part of the electromagnetic mass shift which we calculate according to Cottingham.[20] This comes from the terms associated with M_1 and M_2, since the Born contributions have been evaluated[21] in terms of measured electromagnetic form factors and found to be convergent.

$$\begin{aligned}\delta M&=\frac{-ie^2}{2(2\pi)^4}\int\frac{d^4q}{q^2}[M_\mu{}^\mu(q,P)-D_\mu{}^\mu]\\&=\frac{-i\alpha}{8\pi^3}\int\frac{d^4q}{q^2}[(q^2+2\nu^2)M^2M_1(q^2,\nu)-3q^2M_2(q^2,\nu)]\\&\qquad+(\text{Born terms}),\quad M\nu=q\cdot P.\end{aligned}\tag{8.4}$$

Following Cottingham,[20] we rotate the q_0 integration contour to iq_0 and then express M_1 and M_2 in terms of dispersion integrals over ν, which we leave unsubtracted.[22]

[20] W. Cottingham, Ann. Phys. **25**, 424 (1963).

[21] M. Cini, E. Ferrari, and R. Gatto, Phys. Rev. Letters **2**, 7 (1959).

[22] The case where subtractions are necessary is interesting and deserves study; it, in particular, has implications for inelastic electron scattering at high q^2.

$$M_i(q^2,\nu)=\frac{2}{\pi}\int_{\nu_{\min}}^{\infty}\frac{d\nu'\nu'\,\mathrm{Im}M_i(q^2,\nu')}{\nu'^2-\nu^2}. \quad (8.5)$$

From this dispersion relation, we know (because the threshold $\nu_{\min}\to\infty$ like $-q^2/M$) that, for $q^2=-k^2$, $\nu=ik\cos\theta$, $\cos\theta\leq 1$,

$$M_i(q^2,\nu)\to M_i(q^2,0) \quad \text{as} \quad k\to\infty. \quad (8.6)$$

The divergent part of δM is therefore

$$\delta M^{\mathrm{div}}=\frac{\alpha}{8\pi}\int^{\infty}k^2dk^2 \times[\tfrac{3}{2}M^2M_1(-k^2,0)-3M_2(-k^2,0)]. \quad (8.7)$$

We see that δM^{div} depends upon the term $O(1/k^4)$ in $M_i(k^2,0)$. It is precisely this term which is determined by the equal-time commutation relations of the currents with H. According to (2.7), we have[23] as $q_0\to i\infty$, $\mathbf{q}=0$:

$$M_{\mu\nu}\to\frac{1}{q^2}\frac{1}{2}\sum_s\int d^3x\langle ps|[[j_\mu(0,\mathbf{x}),H],j_\nu(0)]|ps\rangle \quad (8.8)$$

while from (8.1) and (8.6)

$$M_{\mu\nu}\to[P_\mu P_\nu-\eta\cdot P(\eta_\mu P_\nu+\eta_\nu P_\mu)+(\eta\cdot P)^2g_{\mu\nu}]q^2M_1(q^2,0)+(\eta_\mu\eta_\nu-g_{\mu\nu})q^2M_2(q^2,0), \quad (8.9)$$

where $q_\mu=\eta_\mu q_0$.

To proceed, we need a model for the strong Hamiltonian H in order to evaluate the double commutator. The results appear to be quite model-dependent. Within the framework of a quark model, however, we can plausibly argue that the double commutator will not vanish. To illustrate—and only to illustrate—the situation we consider a simple quark model for which

$$H=\sum_{i=1}^{3}\int d^3x\,\psi_i^\dagger(x)\times[-i\boldsymbol{\alpha}\cdot\boldsymbol{\nabla}+\beta m_i+g\beta\gamma_\mu B^\mu(x)]\psi_i(x)+H_B$$
$$=H_0+H_M+H_I+H_B. \quad (8.10)$$

The ψ_i are quark fields and $B^\mu(x)$ is a neutral-vector, $SU(3)$-singlet field. H_B is the Hamiltonian of the B meson, including possible self-interaction terms. The only virtue of this H is that it has a simple algebraic structure and a chiral $U(6)\otimes U(6)$ current algebra.

The commutator (8.8) can now be computed. Only space components of j_μ and j_ν need be considered, because $\eta_\mu M^{\mu\nu}=O(1/q_0^3)$, as follows from either (8.9) or (8.8). Then

$$\int d^3x[[j_i(0,\mathbf{x}),H_0],j_j(0)]=-4i\psi^\dagger[\alpha_i\nabla_j-\delta_{ij}\boldsymbol{\alpha}\cdot\boldsymbol{\nabla}]Q^2\psi,$$

$$\int d^3x[[j_i(0,\mathbf{x}),H_M],j_j(0)]=-4\bar{\psi}MQ^2\psi\delta_{ij}, \quad (8.11)$$

$$\int d^3x[[j_i(0,\mathbf{x}),H_I],j_j(0)]=4g\psi^\dagger[\alpha_iB_j-\delta_{ij}\boldsymbol{\alpha}\cdot\mathbf{B}]Q^2\psi.$$

In this model, we see that the "divergent" part of the mass splittings transforms, in the $SU(3)$ limit, as a unitary octet, since the matrices Q^2 and Q^2M can be reduced to linear combinations of 1, λ_3, and λ_8. As an instant generalization we have the theorem:

Theorem: If the part of H which depends upon quark fields ψ is bilinear in ψ and $\psi^\dagger$ and contains no off-diagonal $SU(3)$ matrices[24] (i.e., only 1, λ_3, λ_8), the divergent part of the electromagnetic mass splittings transforms as an octet [in the $SU(3)$ limit].

We have not shown that the matrix element is nonvanishing, and cannot, in fact, do so. However, something can be said about the mass term H_M and hereafter, in the spirit[25] of static $SU(6)$, we ignore the other two. With this reservation, we find from (8.11) and (8.8)

$$M_{\mu\nu}\to(4/q^2)\langle ps|\bar{\psi}_k(0)Q_k^2m_k\psi_k(0)|ps\rangle(g_{\mu\nu}-\eta_\mu\eta_\nu)\equiv(4m_0^{(p)}/q^2)(g_{\mu\nu}-\eta_\mu\eta_\nu). \quad (8.12)$$

Therefore, from (8.9)

$$q^4M_1\to 0,\quad M_2\to -4m_0^{(p)}/q^4 \quad \text{as} \quad q^2\to-\infty. \quad (8.13)$$

Thus, within these simple-minded assumptions, the electromagnetic mass of the proton diverges if the bare quark masses and $m_0^{(p)}$ are nonvanishing, and is given by[26]

$$\delta M_p\approx\frac{3\alpha}{2\pi}m_0^{(p)}\int\frac{dk^2}{k^2}=\frac{3\alpha}{2\pi}m_0^{(p)}\ln\frac{\Lambda^2}{k_{\min}^2}. \quad (8.14)$$

If H commutes with isospin, then $m_1=m_2$ in (8.10), and the isospin-dependent part of the mass splittings is given by

$$\delta M_3=\frac{\alpha}{2\pi}m_1\langle p|\bar{\psi}(0)T_3\psi(0)|p\rangle\int\frac{dk^2}{k^2} \quad (8.15)$$

[23] Notice that, were chiral $U(6)\otimes U(6)$ an exact symmetry, the commutator would vanish.

[24] This assumption is scarcely needed; were there such matrices, they would lead to other charged fields to which the quarks are coupled. But these fields would contribute to the currents, in contradiction with our original assumptions.

[25] We mean $\langle\psi^\dagger\boldsymbol{\alpha}\psi\rangle\ll\langle\psi^\dagger\beta\psi\rangle$.

[26] For free quarks, the contribution of the (here neglected) kinetic energy term reduces this by a factor 2.

with

$$T_3=\begin{bmatrix}\frac{1}{2} & 0 & 0\\ 0 & -\frac{1}{2} & 0\\ 0 & 0 & 0\end{bmatrix}$$

and matrix indices suppressed. The same calculation may be done for any particle state and the same factors

$$\frac{\alpha m_1}{2\pi}\int\frac{dk^2}{k^2} \tag{8.16}$$

will appear; only the reduced matrix element

$$\langle p|\bar{\psi}(0)T_3\psi(0)|p\rangle$$

will vary from particle to particle. Of course, for mesons δM_3 is replaced by $(\delta\mu^2)_3$.

We recognize from (8.15) a strong similarity to the Coleman-Glashow[27] "tadpole" picture of electromagnetic splittings. In particular, in the $SU(3)$ limit for the matrix elements $\langle p|\bar{\psi}T_3\psi|p\rangle$ we find that the splittings transform as an octet. Furthermore, the electromagnetic splittings can be related to the octet splittings of the $SU(3)$ multiplet in question[28]:

$$\delta M_8=(m_1-m_3)\langle p|\bar{\psi}Y\psi|p\rangle \tag{8.17}$$

and

$$Y=\begin{bmatrix}\frac{1}{3} & 0 & 0\\ 0 & \frac{1}{3} & 0\\ 0 & 0 & -\frac{2}{3}\end{bmatrix}. \tag{8.18}$$

Thus

$$\frac{\delta M_3}{\delta M_8}=\frac{\alpha}{2\pi}\left(\frac{m_1}{m_1-m_3}\right)\frac{\langle p|\bar{\psi}(0)T_3\psi(0)|p\rangle}{\langle p|\bar{\psi}(0)Y\psi(0)|p\rangle}\int\frac{dk^2}{k^2}. \tag{8.19}$$

For meson octets and the decuplet, the ratio of the matrix elements is simply a Clebsch-Gordan coefficient. For the baryon octet, the ratio depends only upon the f/d ratio in the octet mass formula.

From the form of (8.19), we find the general results of Coleman and Glashow that:

(1) The electromagnetic splittings are octet.

(2) The f/d ratio of the electromagnetic splittings is the same as for the octet splittings.

(3) The ratio of electromagnetic splittings to octet splittings is universal, i.e., independent, within Clebsch-Gordan coefficients, of the particle in question (to the extent that the logarithm

$$\int\frac{dk^2}{k^2}$$

is independent of the particle in question).

[27] S. Coleman and S. Glashow, Phys. Rev. **134**, B671 (1964); S. Coleman and H. Schnitzer, *ibid.* **136**, B223 (1964).

[28] Although this looks like perturbation theory, we may obtain the same result by using the Fubini-Furlan technique, keeping only the pole contributions, and ignoring the dispersion integrals. S. Fubini, G. Furlan, and C. Rossetti, Nuovo Cimento **40A**, 1171 (1965).

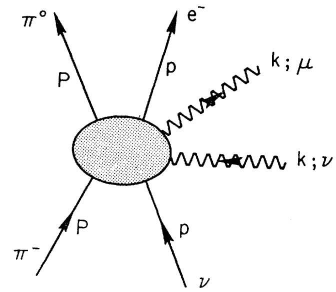

FIG. 2. Radiative amplitude for π-β decay.

In connection with this last result, we find in the Coleman-Glashow notation [their Eqs. (9) and (10)]

$$\frac{K^0-K^+}{K-\pi}=\frac{\Sigma^--\Sigma^+}{\Xi-N}=\frac{n-p}{\Xi-\Sigma}=\frac{\Xi^--\Xi^0}{\Sigma-N}$$
$$=\frac{\alpha}{2\pi}\left(\frac{m_1}{m_1-m_3}\right)\int\frac{dk^2}{k^2}. \tag{8.20}$$

Numerically the left-hand side of (8.20) varies from +0.017 to +0.038, and when nontadpole contributions are removed, a best value of about +0.035 results. Therefore,

$$\frac{m_1}{m_1-m_3}\int\frac{dk^2}{k^2}\approx 30.$$

It is curious that, contrary to the naive picture, the isosinglet quark here has the *smallest* bare mass.

All this is highly speculative, but we draw from this calculation the following conclusions:

(1) The contribution to the electromagnetic mass splittings from a quark mass term in the Hamiltonian is divergent.

(2) It is unlikely that such a contribution would be cancelled by others.

(3) Under fairly general assumptions on the structure of H, the divergent part of the mass splitting transforms as an $SU(3)$ octet.

(4) Assuming that the quark mass term is the dominant source of this splitting, many results of the Coleman-Glashow "tadpole" theory follow.

IX. RADIATIVE CORRECTIONS TO WEAK INTERACTIONS

Next we consider the divergent part of the radiative corrections to π^+ β decay.[29] We shall be able to show that, for a $U(6)\otimes U(6)$ current algebra, the first-order radiative correction diverges, to all orders of the strong interactions, and we compute the coefficient of the divergent logarithm.

We begin the calculation by considering the invariant amplitude, illustrated in Fig. 2, for the process

[29] N. P. Chang, Phys. Rev. **131**, 1272 (1963). G. DaPrato and G. Putzulu, Nuovo Cimento **21**, 541 (1961).

$\pi^- \to \pi^0 + e^- + \bar\nu + \gamma + \gamma'$, where γ and γ' are virtual photons of momenta k.

We shall write down the asymptotic part of this amplitude, all terms $O(1/k^2)$, and then tie together the photons and integrate to obtain the radiative correction. By first considering this amplitude, we can check that the result is gauge-invariant.

The amplitude of Fig. 2 is composed of three terms, illustrated in Fig. 3. We let the neutrino be virtual.

The amplitude $\mathfrak{M}_{\mu\nu}^{(a)}$ is perturbation theory and can be written down instantly. (We ignore the form-factor dependence in the pion vertex.)

$$\mathfrak{M}_{\mu\nu}^{(a)} = Ge^2 P_\alpha \bar u \left\{ \gamma_\mu \frac{1}{p+k-m} \gamma_\nu + \gamma_\nu \frac{1}{p-k-m} \gamma_\mu \right\} \frac{1}{p-m} \gamma^\alpha (1-\gamma_5)\,. \quad (9.1)$$

We record also the divergence of $\mathfrak{M}_{\mu\nu}^{(a)}$:

$$k^\mu \mathfrak{M}_{\mu\nu}^{(a)} = Ge^2 P_\alpha \bar u \gamma_\nu \frac{1}{p-k-m} \gamma^\alpha (1-\gamma_5)\,. \quad (9.2)$$

The hadronic piece of $\mathfrak{M}_{\mu\nu}^{(b)}$ is proportional to

$$\Gamma_{\mu\alpha}{}^+ = -i \int d^4x\, e^{ik\cdot x} \langle \pi^0 | T(j_\mu(x) j_\alpha{}^+(0)) | \pi^- \rangle\,, \quad (9.3)$$

where $j_\alpha{}^+$ is the total weak current $(V-A)$ and[30]

$$k^\mu \Gamma_{\mu\alpha}{}^+ = \langle \pi^0 | j_\alpha{}^+(0) | \pi^- \rangle = P_\alpha \sqrt2\,. \quad (9.4)$$

The general form of $\Gamma_{\mu\alpha}{}^+$ is, consistent with (9.4), Lorentz covariance, and isovector current conservation,[31]

$$\begin{aligned}\Gamma_{\mu\alpha}{}^+ = &[k^2 P_\mu P_\alpha - (k\cdot P)(P_\mu k_\alpha + P_\alpha k_\mu) \\ &+ (k\cdot P)^2 g_{\mu\alpha}] \Phi_1(k^2, k\cdot P) \\ &+ (k_\mu k_\alpha - g_{\mu\alpha} k^2) \Phi_2(k^2, k\cdot P) \\ &+ \epsilon_{\mu\alpha\beta\gamma} P_\beta k_\gamma \Phi_3(k^2, k\cdot P) \\ &- 2\sqrt2 \frac{(P_\mu - \frac12 k_\mu)(P_\alpha - \frac12 k_\alpha)}{k^2 - 2k\cdot P} F^2(k^2) \\ &+ \sqrt2 (P_\alpha k_\mu + P_\mu k_\alpha - g_{\mu\alpha} P\cdot k) \left[\frac{1-F^2(k^2)}{k^2}\right] \\ &+ \frac{1}{\sqrt2} g_{\mu\alpha} F^2(k^2)\,. \end{aligned} \quad (9.5)$$

We need the term $O(1/k)$ in $\Gamma_{\mu\alpha}{}^+$; this is determined by the equal-time commutation relations, as in (4.6).

[30] See Footnote 6; the pion propagator is $2(q^2-\mu^2)^{-1}$, and occasionally we set $\mu = 1$.

[31] See Sec. IV.

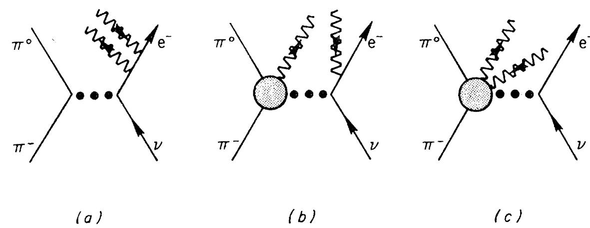

FIG. 3. Decomposition of the amplitude of Fig. 2.

Assuming unsubtracted dispersion relations for the Φ_i and $F(k^2)$ and carrying out an argument similar to (6.13), consistency with the $U(6)\otimes U(6)$ current algebra as $k_0 \to i\infty$ demands, as in Sec. IV,

$$\begin{aligned} \Phi_1(k^2,0) &\to O(1/k^4)\,, \\ \Phi_2(k^2,0) &\to O(1/k^4)\,, \\ \Phi_3(k^2,k\cdot P) &\to k\cdot P\, O(1/k^4)\,, \\ F(k^2) &\to 0\,. \end{aligned} \quad (9.6)$$

Then all the contributions of $\Gamma_{\mu\alpha}{}^+$ to the radiative correction $\mathfrak{M}_{\mu\nu}^{(b)}$ will be finite except the term

$$\sqrt2 \frac{P_\alpha k_\mu + P_\mu k_\alpha - g_{\mu\alpha} P\cdot k}{k^2} \equiv \Gamma_{\mu\alpha(\infty)}{}^+\,, \quad (9.7)$$

which follows from the $U(6)\otimes U(6)$ current algebra. So we take for $\mathfrak{M}_{\mu\nu}^{(b)}$ the approximate expression

$$\mathfrak{M}_{\mu\nu}^{(b)} \cong -Ge^2 \left[\frac{P_\alpha k_\mu + P_\mu k_\alpha - g_{\mu\alpha}(P\cdot k)}{k^2}\right] \bar u \gamma_\nu \times \frac{1}{p-k-m} \gamma^\alpha (1-\gamma_5) + \begin{pmatrix} \mu \leftrightarrow \nu \\ k \leftrightarrow -k \end{pmatrix}. \quad (9.8)$$

Its divergence is given by

$$k^\mu \mathfrak{M}_{\mu\nu}^{(b)} = -Ge^2 P^\alpha \bar u \gamma_\nu \frac{1}{p-k-m} \gamma_\alpha (1-\gamma_5) + Ge^2 \left[\frac{P_\alpha k_\nu + P_\nu k_\alpha - g_{\nu\alpha}(k\cdot P)}{k^2}\right] \bar u \gamma^\alpha (1-\gamma_5)\,. \quad (9.9)$$

Finally, we come to $\mathfrak{M}_{\mu\nu}^{(c)}$. Here the wave-function renormalization is a little more delicate than that of the electron line, which is mere perturbation theory. To cope with this we go back to the Fubini-Furlan method[32] and consider

$$T_{\mu\nu}(P,q) = -i \int d^4x \times e^{iq\cdot x} \langle \pi^- | T(j_\mu{}^-(x) j_\nu{}^+(0)) | \pi^- \rangle\,, \quad (9.10)$$

[32] S. Fubini, G. Furlan, and C. Rossetti, IAEA, Vienna (unpublished). Similar calculations have been carried out by R. Norton (private communication). See also Footnote 28.

considered to all orders of electromagnetism. Taking

$$\langle\pi^-|j_\mu^-(0)|\pi^0\rangle=(1/\sqrt{2})Z(P_{\pi^-}+P_{\pi^0})_\mu$$

we find as $q\to 0$

$$O(q^2)=q_\mu q_\nu T^{\mu\nu}=2q\cdot P+i\langle\pi^-|[\partial^\mu j_\mu^-(0),Q^+]|\pi^-\rangle$$
$$-i\int d^4x\, e^{iq\cdot x}\langle\pi^-|T(\partial^\mu j_\mu^-(x)\partial^\nu j_\nu^+(0))|\pi^-\rangle. \quad (9.11)$$

We extract the Born terms from the last term, using

$$\begin{aligned}-i\langle\pi^-|\partial_\mu j_\mu^-(0)|\pi^0\rangle&=(Z/\sqrt{2})(q^2+2q\cdot P)\\&=(Z/\sqrt{2})(\mu_0{}^2-\mu_-{}^2)\end{aligned} \quad (9.12)$$

while the continuum terms are obtained by replacing $\partial_\mu j_\mu^\pm$ by $\pm ieA_\mu j_\mu^\pm$ and contracting out the electromagnetic field. We obtain, to lowest order in e^2,

$$O(q^2)=2q\cdot P+\text{constant}+\frac{Z^2(\mu_-{}^2-\mu_0{}^2)^2}{(q^2+2q\cdot P+\mu_-{}^2-\mu_0{}^2)}$$
$$+e^2\int d^4x\, e^{iq\cdot x}D^{\mu\nu}(x)\langle\pi^-|T(j_\mu^-(x)j_\nu^+(0))|\pi^-\rangle, \quad (9.13)$$

where $D_{\mu\nu}$ is the photon propagator. Keeping the term linear in $q\cdot P$, we get a Fubini-Furlan (Weisberger-Adler) formula

$$(1-Z^2)P_\alpha=-\frac{e^2}{2}\frac{\partial}{\partial q^\alpha}\int\frac{d^4k}{(2\pi)^4}D^{\mu\nu}(k)T_{\mu\nu}(P,k-q)\bigg|_{q=0}. \quad (9.14)$$

By displacing the origin in k space, we can put the differentiation onto $D_{\mu\nu}(k)$; this makes $D_{\mu\nu}$ of order $1/k^3$ and allows us to keep only terms of order $1/k$ in $T_{\mu\nu}$. These terms, however, are known from the work in Sec. III, within the same assumptions about asymptotic behavior of inelastic form factors.

$$T_{\mu\nu}(P,k)=2\left(\frac{P_\mu k_\nu+P_\nu k_\mu-g_{\mu\nu}P\cdot k}{k^2}\right)+O(1/k^2).$$

Putting all this together, we find that the contribution of the renormalization terms to $\mathfrak{M}_{\mu\nu}{}^{(c)}$ is

$$\mathfrak{M}_{\mu\nu}{}^{(c)}\cong Ge^2\frac{\partial}{\partial k^\alpha}\left[\frac{P_\mu k_\nu+P_\nu k_\mu-g_{\mu\nu}P\cdot k}{k^2}\right]\bar{u}\gamma^\alpha(1-\gamma_5),$$
$$k^\mu\mathfrak{M}_{\mu\nu}{}^{(c)}\cong -Ge^2\left[\frac{P_\alpha k_\nu+P_\nu k_\alpha-g_{\nu\alpha}(P\cdot k)}{k^2}\right]\bar{u}\gamma^\alpha(1-\gamma_5). \quad (9.15)$$

Putting (9.2), (9.9), and (9.15) together, we find the consistency check

$$k^\mu\{\mathfrak{M}_{\mu\nu}{}^{(a)}+\mathfrak{M}_{\mu\nu}{}^{(b)}+\mathfrak{M}_{\mu\nu}{}^{(c)}\}=0. \quad (9.16)$$

We are now prepared to evaluate the radiative correction. We multiply by $\frac{1}{2}D_{\mu\nu}(k)d^4k/(2\pi)^4$, with $D_{\mu\nu}$ proportional to $(1/k^2)(g_{\mu\nu}+\lambda k_\mu k_\nu/k^2)$, and integrate. Diagrams (a) must be handled with care to account for the $\sqrt{Z_2}$ multiplying the amplitude from the reduction formula. The contribution is

$$\mathfrak{M}^{(a)}\cong GP_\alpha\bar{u}\gamma^\alpha(1-\gamma_5)\left[-\frac{\alpha}{8\pi}(1+\lambda)\int\frac{dk^2}{k^2}\right]$$
$$\equiv -\frac{\alpha}{8\pi}(1+\lambda)\ln\frac{\Lambda^2}{m^2}\mathfrak{M}_0. \quad (9.17)$$

From (9.8) and (9.15) we obtain the contributions of diagrams (b) and (c)

$$\mathfrak{M}^{(b)}\cong -ie^2\int\frac{d^4k}{(2\pi)^4}\frac{1}{k^2}\left(g^{\mu\nu}+\frac{\lambda k^\mu k^\nu}{k^2}\right)$$
$$\times\left[\frac{P_\alpha k_\mu+P_\mu k_\alpha-g_{\mu\alpha}P\cdot k}{k^2}\right]\bar{u}\gamma_\nu\frac{k}{k^2}\gamma^\alpha(1-\gamma_5)G$$
$$=\frac{\alpha}{\pi}\left(\frac{5}{8}+\frac{\lambda}{4}\right)\ln\frac{\Lambda^2}{m^2}\mathfrak{M}_0, \quad (9.18)$$

$$\mathfrak{M}^{(c)}\cong -\frac{ie^2}{2}\int\frac{d^4k}{(2\pi)^4}\frac{1}{k^2}\left(g^{\mu\nu}+\frac{\lambda k^\mu k^\nu}{k^2}\right)\frac{\partial}{\partial k^\alpha}$$
$$\times\left[\frac{P_\mu k_\nu+P_\nu k_\mu-g_{\mu\nu}(P\cdot k)}{k^2}\right]\bar{u}\gamma^\alpha(1-\gamma_5)G$$
$$=-\frac{\alpha}{8\pi}(1+\lambda)\ln\frac{\Lambda^2}{m^2}\mathfrak{M}_0. \quad (9.19)$$

Therefore, to leading order in α, the pion β-decay amplitude has the structure

$$\mathfrak{M}\cong GP_\alpha\bar{u}\gamma^\alpha(1-\gamma_5)u\left\{1+\frac{3\alpha}{8\pi}\ln\frac{\Lambda^2}{m^2}\right\}. \quad (9.20)$$

We conclude that a chiral $U(6)\otimes U(6)$ current algebra implies serious difficulties in making a consistent theory of radiative corrections to weak semileptonic processes, difficulties which cannot be blamed upon our ignorance of strong-interaction form factors. However, even for a cutoff $\Lambda^2\sim 1/G$, the correction (9.20) is only about 1%.

X. ELECTRON-POSITRON ANNIHILATION INTO HADRONS

In this section we apply the same kind of speculations as in Sec. VIII to the vacuum expectation value of electromagnetic currents. We define, as in (3.1),

$$\sum_n \langle 0|j_\mu(0)|n\rangle\langle n|j_\nu(0)|0\rangle(2\pi)^4\delta^4(P_n-q) = (q_\mu q_\nu - g_{\mu\nu}q^2)\rho(q^2). \quad (10.1)$$

$\rho(q^2)$ is related to the total cross section for the process $e^+ + e^- \rightarrow$ hadrons at center-of-mass energy $\sqrt{q^2}$:

$$\sigma_{\text{tot}}(q^2) = \frac{16\pi^2\alpha^2\rho(q^2)}{q^2}.$$

By multiplying Eq. (10.1) by $E_n = q_0$ and integrating over q_0, we find, assuming that the double commutator exists,

$$\int d^3x \langle 0|[j_\mu(\mathbf{x}),[H,j_\nu(0)]]|0\rangle = (\eta_\mu\eta_\nu - g_{\mu\nu})\int \frac{dq^2}{2\pi}q^2\rho(q^2) = \frac{(\eta_\mu\eta_\nu - g_{\mu\nu})}{16\pi^2\alpha^2}\int \frac{dq^2}{2\pi}q^4\sigma_{\text{tot}}(q^2). \quad (10.2)$$

We go back to the quark Hamiltonian (8.10) and the evaluation of the double commutator (8.11). Already the kinetic energy term produces difficulties.

$$\int d^3x \langle 0|[j_i(x),[H_0,j_j(0)]]|0\rangle = -4i\langle 0|\psi^\dagger(\alpha_i\nabla_j - \delta_{ij}\boldsymbol{\alpha}\cdot\boldsymbol{\nabla})Q^2\psi|0\rangle = 4\int \frac{d^4k}{(2\pi)^4}\mathrm{Tr}(\gamma_i k_j - \delta_{ij}\boldsymbol{\gamma}\cdot\mathbf{k})Q^2 S(k), \quad (10.3)$$

where $S(k)$ is the propagator for the unrenormalized quark fields

$$S(k) = i\int \frac{dm^2[\not{k}\rho_1(m^2) + \rho_2(m^2)]}{k^2 - m^2}$$

and

$$\int dm^2\rho_1(m^2) = 1 \quad \rho_1 \geq 0. \quad (10.4)$$

It is clear that the double commutator diverges quartically.

$$\int d^3x \langle 0|[j_i(x),[H_0,j_j(0)]]|0\rangle \sim \int d^4k \int dm^2\rho_1(m^2). \quad (10.5)$$

The mass term is proportional to $\langle 0|\bar{\psi}Q^2M\psi|0\rangle$ which is related to

$$\langle 0|[Q_{ax}{}^+,\partial_\mu j_\mu{}^-(0)_{ax}]|0\rangle \sim \langle 0|\bar{\psi}\{\tau^+,\{\tau^-,M\}\}\psi|0\rangle \quad (10.6)$$

and is finite if we use the hypothesis of partially conserved axial-vector current and saturate the intermediate states with a pion.[1]

Unfortunately, the interaction term is less amenable to analysis; again, however, it is unlikely that it identically cancels off the quartic divergence from the first term, and we conclude

$$\int dq^2 q^4 \sigma_{\text{tot}}(q^2) = \infty. \quad (10.7)$$

If we demand that Eq. (10.7), like Eq. (10.5), diverge quartically, we find that (within logarithmic powers)

$$\sigma_{\text{tot}}(q^2) \sim 1/q^2 \quad \text{as} \quad q^2 \rightarrow \infty, \quad (10.8)$$

which is the perturbation theory result. The idea that the *total* hadron yield from colliding beams of given energy should be approximately the same as, say, the μ^+-μ^- yield is folklore[33]; we can consider this calculation as support (but certainly not a proof) of this point of view.[34]

Note added in proof. We have succeeded in verifying (6.21); cf. J. Bjorken, Phys. Rev. Letters **16**, 408 (1966).

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[33] B. Richter (private communication).

[34] A similar argument on the charged current vacuum expectation value leads to the conclusion that for large W-meson mass, the branching ratio for $W \rightarrow$ leptons is comparable to $W \rightarrow$ hadrons.