rejected with very high confidence. The additional assumptions of the calculation (beyond those mentioned in Sec. I) are twofold: (1) the assumption about threshold behavior, and (2) the smoothness assumption implicit in the numerical integration technique employed. A certain amount of independent evidence on these questions is available. The threshold behavior of the YLAM phase shifts can be examined in detail, and is found to deviate noticeably from the monomial form at 25 MeV, particularly for high J. (The same is true of the amplitudes arising from pole terms.) For this reason we have not tried to compute H (25 MeV). At higher energies, however, the contribution due to this region is rather small, and the approximation seems quite safe. The smoothness assumption can also be checked in part by comparison with similar computations using the energy-dependent phase shifts. The agreement was found to be satisfactory (see Ref. 22). The possibility of rapid oscillations, which would invalidate all these calculations, can only be ignored with horror.

I believe these assumptions are sound enough so that a dynamical model which does not meet the tests described in Sec. VI can be characterized as incompatible with the simultaneous requirements imposed by unitarity, analyticity, and the known experimental phase shifts. It should be noted that unitarity is incorporated here without any artificial imposition of elasticity beyond 310 MeV.

Although the remarks in the preceding section weigh somewhat against the value of pole models, the simplicity and clarity of the present situation more or less behoove us to make the quantitative and qualitative tests described in Sec. VI. The results of this work will be published elsewhere. It is worth mentioning that there is no reason in principle why the same ideas should not be applied to π-N scattering as well.

ACKNOWLEDGMENTS

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Consequences of Charge Independence in Strong Interactions: A Graphical Method

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The hypothesis of charge independence is studied using a simple graphical method. This method is applicable in the common situation when a particle reaction can proceed through several charge channels but only through two isospin channels. The branching ratios of the charge channels are plotted in a two-dimensional diagram which is possible because only two of these ratios are independent. The triangular inequalities are shown to be equivalent to a single inequality, and they determine in the diagram an elliptic boundary curve of a physical region. The plot is also useful for studying isospin properties of intermediate states in the reaction. The method is demonstrated with two examples, the reactions $K^+p \to KN\pi$ and $\pi N \to \pi N$.

I. INTRODUCTION

The hypothesis of charge independence has proven useful in various fields of nuclear science and particularly in high-energy physics. In this paper we shall study the consequences of this hypothesis using a graphical method which we find very useful for this purpose.

In the common situation when a particle reaction can proceed only through two isospin channels, we can, under the assumption of charge independence, analyze the cross sections for the charge channels in terms of only two amplitudes of definite isospin. From this it follows that not more than three of these can be independent, and these three furthermore must satisfy triangular inequalities.[1]

If instead of cross sections we consider branching ratios, we have the additional relation that the sum of these must be equal to unity, thus leaving only two independent. We can therefore plot these in a two-dimensional diagram. We shall see that the triangular inequalities are equivalent to one single inequality,

[1] D. Feldman, Phys. Rev. 89, 1159 (1953).

which determines an elliptic physical region in the diagram. In this way the diagram can be used for testing charge independence.

If the reaction proceeds through a resonance, one of the isospin channels dominates, and therefore definite branching ratios are expected, which correspond to a definite point in the diagram. Similarly, if the reaction is mediated by an exchange of a certain particle, the branching ratios can in many cases be computed, and they correspond to another point in the diagram. In this way the diagram can be used for studying intermediate states in the scattering process.

We shall demonstrate the method with two examples, the reactions $K^+p \to KN\pi$ and $\pi N \to \pi N$.

II. ISOSPIN CONSERVATION IN $K^+p \to KN\pi$

Let us consider the reaction $K^+p \to KN\pi$, where one pion is produced. We apply the isospin formalism to this reaction, which can proceed through three charge channels,

$$K^+p \to K^+p\pi^0, \tag{1}$$

$$K^+p \to K^0p\pi^+, \tag{2}$$

$$K^+p \to K^+n\pi^+. \tag{3}$$

The initial state is a pure isospin state with $T=1$ and $T_3=1$. In order to satisfy isospin conservation the final state must also have $T=1$. Because there are three outgoing particles, there are two linearly independent such states.[2] In order to find these states we can proceed in three different ways depending on the order in which the outgoing particles are coupled.

(a) We can first couple the kaon and the pion to an intermediate state of either $T=\frac{1}{2}$ or $T=\frac{3}{2}$ and then couple the nucleon in such a way that we obtain a three-particle state with $T=1$. Suppressing the space and spin part we find the following states:

$$|K_{1/2}\rangle = -(\sqrt{\tfrac{1}{3}})|K^+p\pi^0\rangle + (\sqrt{\tfrac{2}{3}})|K^0p\pi^+\rangle, \tag{4a}$$

$$|K_{3/2}\rangle = -(\sqrt{\tfrac{1}{6}})|K^+p\pi^0\rangle - (\sqrt{\tfrac{1}{12}})|K^0p\pi^+\rangle + (\sqrt{\tfrac{3}{4}})|K^+n\pi^+\rangle, \tag{4b}$$

where $|K_T\rangle$ denotes a final state which has an intermediate isospin T in the $K\pi$ system but an over-all isospin of unity. In general the final state is a linear combination of these states:

$$|KN\pi, T=1, T_3=1\rangle = a_{1/2}|K_{1/2}\rangle + a_{3/2}|K_{3/2}\rangle, \tag{5a}$$

where the coefficients a_T are the amplitudes of the two isospin channels. This representation of the final state is useful when studying resonances in the $K\pi$ system. Then one of the amplitudes $a_{1/2}$ or $a_{3/2}$ is expected to dominate.

(b) In a quite analogous way we obtain the states

$$|N_{1/2}\rangle = -(\sqrt{\tfrac{1}{3}})|K^+p\pi^0\rangle + (\sqrt{\tfrac{2}{3}})|K^+n\pi^+\rangle, \tag{6a}$$

$$|N_{3/2}\rangle = -(\sqrt{\tfrac{1}{6}})|K^+p\pi^0\rangle + (\sqrt{\tfrac{3}{4}})|K^0p\pi^+\rangle - (\sqrt{\tfrac{1}{12}})|K^+n\pi^+\rangle, \tag{6b}$$

if instead we first couple the nucleon to the pion and then the kaon to the πN system. $|N_T\rangle$ thus denotes the final state with an intermediate isospin T in the $N\pi$ system. In this case we find the following representation of the final state in terms of two amplitudes $b_{1/2}$ and $b_{3/2}$:

$$|KN\pi, T=1, T_3=1\rangle = b_{1/2}|N_{1/2}\rangle + b_{3/2}|N_{3/2}\rangle. \tag{5b}$$

This representation is useful when studying resonances in the $N\pi$ system.

(c) The third way of coupling the particles is first to couple the nucleon to the kaon and then the pion to the KN system. In this way we find the following states of definite isospin 0 and 1 in the KN system:

$$|Y_0\rangle = +(\sqrt{\tfrac{1}{2}})|K^0p\pi^+\rangle - (\sqrt{\tfrac{1}{2}})|K^+n\pi^+\rangle, \tag{7a}$$

$$|Y_1\rangle = -(\sqrt{\tfrac{1}{2}})|K^+p\pi^0\rangle + \tfrac{1}{2}|K^0p\pi^+\rangle + \tfrac{1}{2}|K^+n\pi^+\rangle. \tag{7b}$$

The representation of the final state is in this case

$$|KN\pi, T=1, T_3=1\rangle = c_0|Y_0\rangle + c_1|Y_1\rangle. \tag{5c}$$

The states in (4), (6), and (7) are related to each other by Racah coefficients. For example, the states in (6) and (7) are related to those in (4) by

$$\begin{aligned} |N_{1/2}\rangle &= \tfrac{1}{3}|K_{1/2}\rangle + \tfrac{2}{3}\sqrt{2}|K_{3/2}\rangle, \\ |N_{3/2}\rangle &= \tfrac{2}{3}\sqrt{2}|K_{1/2}\rangle - \tfrac{1}{3}|K_{3/2}\rangle, \\ |Y_0\rangle &= \tfrac{1}{3}|K_{1/2}\rangle - (\sqrt{\tfrac{2}{3}})|K_{3/2}\rangle, \\ |Y_1\rangle &= (\sqrt{\tfrac{2}{3}})|K_{1/2}\rangle + (\sqrt{\tfrac{1}{3}})|K_{3/2}\rangle. \end{aligned} \tag{8}$$

These states span the same two-dimensional subspace of the three-dimensional space spanned by $|K^+p\pi^0\rangle$, $|K^0p\pi^+\rangle$, and $|K^+n\pi^+\rangle$. The states in this subspace all have $T=1$. Orthogonal to this is the state with $T=2$, which is not allowed if isospin is conserved:

$$|KN\pi, T=2, T_3=1\rangle = (\sqrt{\tfrac{1}{2}})|K^+p\pi^0\rangle + \tfrac{1}{2}|K^0p\pi^+\rangle + \tfrac{1}{2}|K^+n\pi^+\rangle. \tag{9}$$

We can now write the expressions for the amplitudes of the charge channels in terms of the amplitudes a_T, b_T, or c_T. Substituting the expressions (4), (6), and (7) into (5a), (5b), and (5c), we find

$$A^{(1)} = -(\sqrt{\tfrac{1}{3}})a_{1/2} - (\sqrt{\tfrac{1}{6}})a_{3/2} = -(\sqrt{\tfrac{1}{3}})b_{1/2} - (\sqrt{\tfrac{1}{6}})b_{3/2} = -(\sqrt{\tfrac{1}{2}})c_1, \tag{10a}$$

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[2] Using group-theoretical methods we can see this by reducing the product of the representations of SU_2, which correspond to the K, N, and π:

$$D_{1/2}\times D_{1/2}\times D_1 = D_0 + D_1 + D_1 + D_2,$$

where the subscript refers to the isospin. We see that D_1 occurs twice. The representations D_2 and D_0 are excluded by isospin conversation (D_0 also by charge conservation).

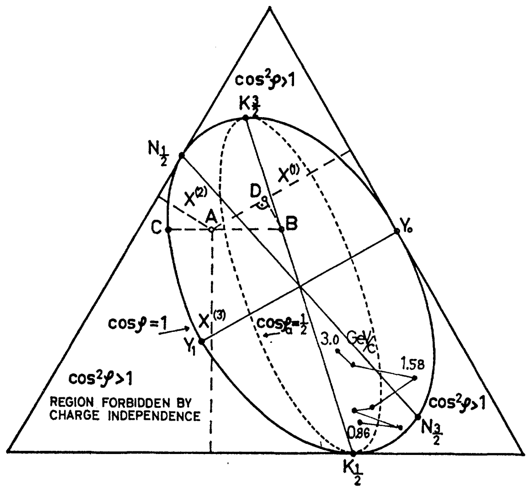

FIG. 1. The graphical method applied to the reaction $K^+p\to KN\pi$. The quantities $y^{(1)}$, $y^{(2)}$, and $y^{(3)}$ are the branching ratios for the three processes $K^+p\to K^+p\pi^0$, $K^+p\to K^0p\pi^+$, and $K^+p\to K^+n\pi^+$. In the right corner, points from data (Refs. 4 and 18) on total cross sections at different beam momenta (0.86, 0.96, 1.20, 1.36, 1.58, 1.96, and 3.0 GeV/c) are plotted.

$$A^{(2)}=(\sqrt{\tfrac{2}{3}})a_{1/2}-(\sqrt{\tfrac{1}{12}})a_{3/2}=(\sqrt{\tfrac{3}{4}})b_{3/2}=(\sqrt{\tfrac{1}{2}})c_0+\tfrac{1}{2}c_1, \quad (10b)$$

$$A^{(3)}=(\sqrt{\tfrac{3}{4}})a_{3/2}=(\sqrt{\tfrac{2}{3}})b_{1/2}-(\sqrt{\tfrac{1}{12}})b_{3/2}=-(\sqrt{\tfrac{1}{2}})c_0+\tfrac{1}{2}c_1, \quad (10c)$$

where the superscript refers to the three processes (1), (2), and (3), respectively. The corresponding cross sections are

$$\sigma^{(i)}=\sum|A^{(i)}|^2, \quad (11)$$

where the summation is performed over final states that have different space and spin parts and that are included in the cross section.

The amplitudes $A^{(i)}$ are related. We find

$$\sqrt{2}A^{(1)}+A^{(2)}+A^{(3)}=0. \quad (12)$$

From this the well-known triangular inequalities follow, which are

$$\sqrt{(2\sigma^{(1)})}\leqslant\sqrt{(\sigma^{(2)})}+\sqrt{\sigma^{(3)}},$$
$$\sqrt{(\sigma^{(2)})}\leqslant\sqrt{(2\sigma^{(1)})}+\sqrt{(\sigma^{(3)})}, \quad (13)$$
$$\sqrt{(\sigma^{(3)})}\leqslant(2\sigma^{(1)})+\sqrt{(\sigma^{(2)})}.$$

By squaring these three inequalities twice one can easily verify that they are equivalent to one single inequality [see (19) below]. We shall here derive this result directly from Eqs. (10) and (11). From these equations we can compute the following quantities:

$$\sum|a_{1/2}|^2=\sigma^{(1)}+\sigma^{(2)}-\tfrac{1}{3}\sigma^{(3)}, \quad (14)$$

$$\sum|a_{3/2}|^2=\tfrac{4}{3}\sigma^{(3)}, \quad (15)$$

$$\sum \mathrm{Re}\,a_{1/2}a_{3/2}^*=(\sqrt{\tfrac{1}{2}})(2\sigma^{(1)}-\sigma^{(2)}-\tfrac{1}{3}\sigma^{(3)}). \quad (16)$$

Similar relations are obtained for the amplitudes b_T and c_T. The quantities in (14) and (15) are the cross sections for definite isospin in the $K\pi$ system, and the third quantity (16) is an interference term. Taking the square of this term we find the inequality

$$(\sum \mathrm{Re}\,a_{1/2}a_{3/2}^*)^2=(\sum|a_{1/2}|\,|a_{3/2}|\cos\varphi_a)^2 \leqslant\sum|a_{1/2}|^2\sum(|a_{3/2}|^2\cos^2\varphi_a) \leqslant\sum|a_{1/2}|^2\sum|a_{3/2}|^2, \quad (17)$$

where φ_a is the phase angle between the amplitudes $a_{1/2}$ and $a_{3/2}$. In (17) we have used the Schwartz inequality and the condition

$$\cos^2\varphi_a\leqslant 1. \quad (18)$$

Substituting the expressions (14)–(16) into (17) we obtain an inequality which can be written in the form

$$\lambda(2\sigma^{(1)},\sigma^{(2)},\sigma^{(3)})\leqslant 0,$$

where

$$\lambda(x,y,z)=x^2+y^2+z^2-2xy-2xz-2yz. \quad (19)$$

In the case where we consider differential cross sections with definite values of the momenta and spins of the particles, there is no summation in (11). In this case the phase angle between the amplitudes $a_{1/2}$ and $a_{3/2}$ can be computed. We obtain the relation

$$\cos^2\varphi_a=\frac{(6d\sigma^{(1)}-3d\sigma^{(2)}-d\sigma^{(3)})^2}{8d\sigma^{(3)}(3d\sigma^{(1)}+3d\sigma^{(2)}-d\sigma^{(3)})}, \quad (20)$$

where $d\sigma^{(i)}$ denotes the differential cross sections. Similar relations are obtained for the phase angles between $b_{1/2}$ and $b_{3/2}$ or c_0 and c_1.

Let us consider the branching ratios for the three charge channels:

$$x^{(i)}=\sigma^{(i)}/(\sigma^{(1)}+\sigma^{(2)}+\sigma^{(3)}). \quad (21)$$

These are related by

$$x^{(1)}+x^{(2)}+x^{(3)}=1. \quad (22)$$

This relation allows us to plot the three branching ratios in a two-dimensional diagram. In Fig. 1 we have used an equilateral triangle and the fact that the sum of the perpendicular distances from an arbitrary point to the sides is a constant. The sides of this triangle correspond to the situation when one of the cross sections is zero. In the corners only one of the cross sections is different from zero.

We observe that the inequality (19) also holds if the $\sigma^{(i)}$'s are replaced with the $x^{(i)}$'s. This inequality which is of second degree in the $x^{(i)}$'s corresponds to the condition that the physically allowed points in the triangle must lie inside the outer ellipse shown in Fig. 1. The three triangular inequalities also correspond to the same condition. Loosely speaking each of these tests, in one corner of the triangle, whether we are inside or outside the ellipse.

Let us now consider the particular cases when one of the amplitudes $a_{1/2}$, $a_{3/2}$, $b_{1/2}$, $b_{3/2}$, c_0, or c_1 is zero. Then the final state is given by one of the states $|K_{1/2}\rangle$,

$|K_{3/2}\rangle$, $|N_{1/2}\rangle$, $|N_{3/2}\rangle$, $|Y_0\rangle$, or $|Y_1\rangle$. For these cases definite branching ratios for the three processes (1), (2) and (3) can be computed [see Eqs. (10) and (11)]. These correspond to definite points in the diagram. In Fig. 1 these points are labeled by the same symbol as the corresponding final state. Thus, e.g., the point $K_{1/2}$ corresponds to the situation when the final state is given by $|K_{1/2}\rangle$.

These points are useful when studying resonances produced in the reaction. If the reaction proceeds predominantly through a resonance in the $K\pi$, $N\pi$, or KN system, it is expected that the branching ratios should correspond to a point close to one of these points.

For example the resonances $K^*(878)$ $(T=1/2)$ and $N^*(1238)$ $(T=3/2)$ are known to be abundantly produced in this reaction in the 3-GeV/c region.[3,4] Thus if the branching ratios are computed in the resonance regions of the Dalitz plots, we expect to obtain points close to the points $K_{1/2}$ and $N_{3/2}$, respectively.

Let us finally consider some other features of the plot. For the case that the interference term $\sum a_{1/2}a_{1/2}^*$ is zero, the branching ratios lie on the line through the points $K_{1/2}$ and $K_{3/2}$. Similarily if $\sum b_{1/2}b_{3/2}^*=0$ they lie on the line through $N_{1/2}$ and $N_{3/2}$, and if $\sum c_0c_1^*=0$ on the line through Y_0 and Y_1.

The quantities in Eqs. (14) to (16) can be related to distances which are obtained by a simple geometrical construction. We find

$$\sum|a_{1/2}|^2/\sigma=K_{3/2}B, \tag{23}$$

$$\sum|a_{3/2}|^2/\sigma=K_{1/2}B, \tag{24}$$

$$\sum \mathrm{Re}\,a_{1/2}a_{3/2}^*/\sigma=AD, \tag{25}$$

where

$$\sigma=\sigma^{(1)}+\sigma^{(2)}+\sigma^{(3)},$$

and where $K_{1/2}B$, $K_{3/2}B$, and AD denotes the distances obtained in the construction shown in Fig. 1.

From the expression for $\cos^2\varphi_a$ [Eq. (20)], we find that a constant $|\cos\varphi_a|$ gives equations which are quadratic in the differential cross sections or in the corresponding branching ratios. Therefore we find that a constant $|\cos\varphi_a|$ corresponds in Fig. 1 to ellipses. These pass through the points $K_{1/2}$ and $K_{3/2}$. In Fig. 1 the ellipse for $|\cos\varphi_a|=\frac{1}{2}$ is drawn with a dashed line. The ellipse corresponding to $|\cos\varphi_a|=1$ coincides with the boundary curve. The quantity $\cos\varphi_a$ can be obtained from the distances AB and CB in the diagram. We find

$$\cos\varphi_a=\pm AB/CB,$$

where the sign depends on which side of the line joining $K_{1/2}$ and $K_{3/2}$ the point A is on. Analogous results are found for the phase angle between $b_{1/2}$ and $b_{3/2}$ or c_0 and c_1.

III. CHARGE INDEPENDENCE IN $\pi N\to\pi N$

As another example let us consider elastic and charge-exchange pion-nucleon scattering. These reactions can proceed through two isospin channels, $T=\frac{1}{2}$ and $T=\frac{3}{2}$. Under the hypothesis of charge independence the matrix elements are independent of the third component T_3. As a consequence all the ten different charge channels of this reaction can be described in terms of only two amplitudes $A_{1/2}$ and $A_{3/2}$. Experimentally, three of these channels are better known than the others. The amplitudes of these, expressed in terms of $A_{1/2}$ and $A_{3/2}$, are

$$A^+=A(\pi^+p\to\pi^+p)=A_{3/2}, \tag{26}$$

$$A^-=A(\pi^-p\to\pi^-p)=\tfrac{2}{3}A_{1/2}+\tfrac{1}{3}A_{3/2}, \tag{27}$$

$$A^{\mathrm{ex}}=A(\pi^-p\to\pi^0n)=-\tfrac{1}{3}\sqrt{2}(A_{1/2}-A_{3/2}). \tag{28}$$

The corresponding cross sections are

$$\sigma^+=\sum|A^+|^2,\quad \sigma^-=\sum|A^-|^2,\quad\text{and}\quad \sigma^{\mathrm{ex}}=\sum|A^{\mathrm{ex}}|^2, \tag{29}$$

where the summation is again performed over final states with different space and spin parts.

The seven other cross sections, which are more difficult or impossible to measure, are related to these:

$$\sigma(\pi^-n\to\pi^-n)=\sigma^+, \tag{30}$$

$$\sigma(\pi^+n\to\pi^+n)=\sigma^-, \tag{31}$$

$$\sigma(\pi^+n\to\pi^0p)=\sigma(\pi^0p\to\pi^+n)=\sigma(\pi^0n\to\pi^-p)=\sigma^{\mathrm{ex}}, \tag{32}$$

$$\sigma(\pi^0p\to\pi^0p)=\sigma(\pi^0n\to\pi^0n)=\tfrac{1}{2}(\sigma^++\sigma^--\sigma^{\mathrm{ex}})=\sigma^0. \tag{33}$$

The amplitudes A^+, A^-, and A^{ex} are related by

$$A^+-A^--\sqrt{2}A^{\mathrm{ex}}=0, \tag{34}$$

and as consequence the cross sections must satisfy the triangular inequalities

$$\begin{aligned}\sqrt{(\sigma^+)}&\leqslant\sqrt{(\sigma^-)}+\sqrt{(2\sigma^{\mathrm{ex}})},\\ \sqrt{(\sigma^-)}&\leqslant\sqrt{(\sigma^+)}+\sqrt{(2\sigma^{\mathrm{ex}})},\\ \sqrt{(2\sigma^{\mathrm{ex}})}&\leqslant(\sigma^+)+\sqrt{(\sigma^-)},\end{aligned} \tag{35a}$$

which we find, in a similar way to that in the previous case, are equivalent to

$$\lambda(\sigma^+,\sigma^-,2\sigma^{\mathrm{ex}})\leqslant 0. \tag{35b}$$

Let us consider the crossed channels $\pi\pi\to N\bar{N}$ and $\pi\bar{N}\to\pi\bar{N}$. The crossing relations imply that these channels can be described by the same amplitudes as the reaction $\pi N\to\pi N$. We can therefore introduce the hypothesis of charge independence in any of these three channels. In this way we find in addition to (26), (27),

[3] M. Ferro-Luzzi, R. George, Y. Goldschmith-Clermont, V. D. Henri, B. Jongejans, D. W. G. Leith, G. R. Lynch, F. Muller, and J.-M. Perreau. Nuovo Cimento **34**, 1101 (1955).

[4] P. Sällstrom, P. Otter, and G. Ekspong, Nuovo Cimento (to be published).

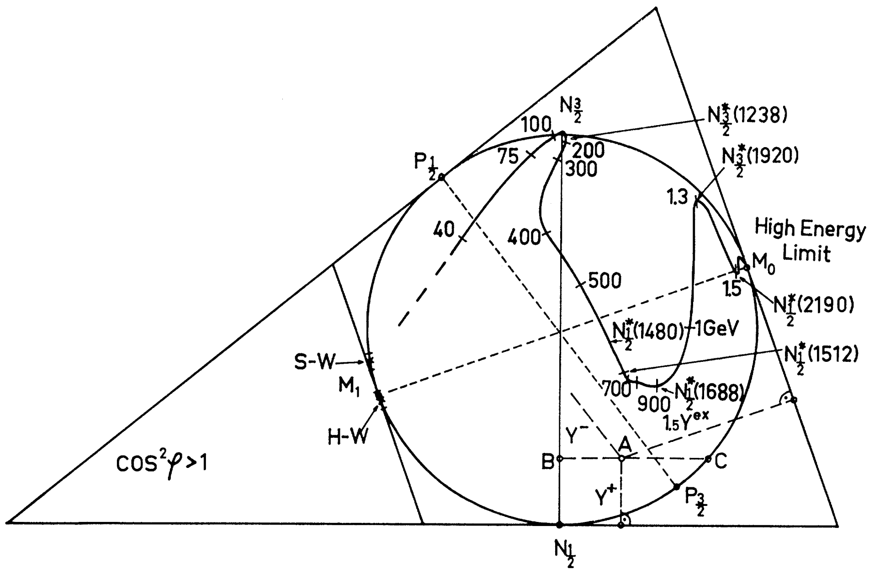

FIG. 2. Charge independence in elastic and charge-exchange pion-nucleon scattering. The curve inside the circle is obtained from data on total cross sections. The numbers along the curve are the beam energies in units of MeV or GeV.

and (28)

$$A^{+}=(\sqrt{\tfrac{1}{6}})A_0{}^t+\tfrac{1}{2}A_1{}^t=\tfrac{2}{3}A_{1/2}{}^u+\tfrac{1}{3}A_{3/2}{}^u, \quad (36)$$

$$A^{-}=(\sqrt{\tfrac{1}{6}})A_0{}^t-\tfrac{1}{2}A_1{}^t=A_{3/2}{}^u, \quad (37)$$

$$A^{\mathrm{ex}}=(\sqrt{\tfrac{1}{2}})A_1{}^t=\tfrac{1}{3}\sqrt{2}(A_{1/2}{}^u-A_{3/2}{}^u), \quad (38)$$

where $A_0{}^t$ and $A_1{}^t$ are the amplitudes of definite isospin $T=0$ and $T=1$ in the t channel ($\pi\pi\rightarrow N\bar{N}$) and $A_{1/2}{}^u$ and $A_{3/2}{}^u$ are the corresponding quantities in the u channel ($\pi\bar{N}\rightarrow\pi\bar{N}$).

The amplitudes A_T, $A_T{}^t$, and $A_T{}^u$ are related by crossing matrices. Expressing $A_T{}^t$ and $A_T{}^u$ in terms of the A_T we find

$$A_0{}^t=\tfrac{1}{3}\sqrt{6}(A_{1/2}+2A_{3/2}), \quad (39a)$$

$$A_1{}^t=-\tfrac{2}{3}(A_{1/2}-A_{3/2}), \quad (39b)$$

$$A_{1/2}{}^u=-\tfrac{1}{3}(A_{1/2}-4A_{3/2}), \quad (40a)$$

$$A_{3/2}{}^u=\tfrac{1}{3}(2A_{1/2}+A_{3/2}). \quad (40b)$$

Let us now turn to the graphical method. We define the following "branching ratios"

$$y^{+}=\sigma^{+}/\sigma, \quad y^{-}=\sigma^{-}/\sigma, \quad y^{\mathrm{ex}}=\sigma^{\mathrm{ex}}/\sigma, \quad (41)$$

where

$$\sigma=\sigma^{+}+\sigma^{-}+\sigma^{\mathrm{ex}}.$$

These satisfy the relation

$$y^{+}+y^{-}+y^{\mathrm{ex}}=1. \quad (42)$$

In a way similar to that used in the previously considered example we can plot these branching ratios in a two-dimensional diagram. It is of some advantage to choose the coordinate system in such a way that the boundary curve of the physical region becomes circular. This can be achieved if instead of an equilateral triangle we use a triangle with heights related as 2:2:3 (See Fig. 2). The sides of the triangle correspond to the situation when one of the cross sections is zero. The perpendicular distances from a point to the sides are proportional to y^+, y^-, and y^{ex}. The condition (35) is equivalent to the condition that only points inside the circle are allowed. Let us also consider the ratio

$$y^0=\sigma^0/\sigma=\tfrac{1}{2}(y^{+}+y^{-}-y^{\mathrm{ex}}), \quad (43)$$

which is related to the (unmeasurable) elastic cross section for neutral-pion–nucleon scattering [compare Eq. (33)]. This is zero along the line tangent to the circle in the point M_1, and the distance to this line is proportional to y^0. The cross sections of the remaining charge channels [Eqs. (30) to (32)] are equal to σ^+, σ^-, or σ^{ex}, and therefore their corresponding geometrical interpretation is also the same as that of y^+, y^-, or y^{ex}.

Let us now turn to some particular cases. If the reaction proceeds entirely through the $T=\frac{1}{2}$ or the $T=\frac{3}{2}$ channel (i.e., $A_{3/2}$ or $A_{1/2}$ is zero), we expect to obtain definite branching ratios. These correspond in the diagram to definite points, which in Fig. 2. are marked by $N_{1/2}$ and $N_{3/2}$ respectively. For example, in the resonance region of the $N^*(1238)$ the branching ratios are close to 9/12, 1/12, and 2/12, which are the coordinates of the point $N_{3/2}$ in Fig. 2.

On the other hand, if we consider the cross sections described in terms of amplitudes of definite isospin in the crossed channels, we find that the points M_0 and M_1 correspond to the situation when $A_0{}^t$ or $A_1{}^t$, respectively, is dominating, while the points $P_{1/2}$ and $P_{3/2}$ correspond to the situation when $A_{1/2}{}^u$ or $A_{3/2}{}^u$, respectively, is dominating. For example if the process is mediated predominantly by a $T=0$ meson exchange, a point close to M_0 would be expected. (The converse of course need not be true; there are more complicated processes which also give the same point).

In Fig. 2 we have plotted the branching ratios obtained from data[5–11] on total cross sections for the channels $\pi^+p \to \pi^+p$, $\pi^-p \to \pi^-p$, and $\pi^-p \to \pi^0 n$ at different beam energies.[12] In this way we obtained the curve inside the circle shown in Fig. 2. From this, many of the well-known isospin effects of pion-nucleon scattering can be seen. We observe that in the first resonance region ($N^*(1238)$) the curve passes very close to the point $N_{3/2}$. Here the experimental data give a few points which lie outside the circle, but the deviations are smaller than the errors (see also Ref. 13). At higher energies the oscillating behavior is due to the other resonances. The $T=\frac{1}{2}$ resonances tend to bring the curve downwards while the $T=\frac{3}{2}$ resonances tend to bring it upwards.

In the high-energy limit the curve approaches the point M_0, where the charge-exchange cross section is zero and $\sigma^+=\sigma^-$, as the Pomeranchuk theorem implies.

Again at very low energies, below the N_{33}^* resonance region there are experimental data from direct measurements down to 20 MeV. At the zero-energy limit the scattering can be described in terms of the S-wave scattering lengths a_1 and a_3, if charge independence is assumed,

$$\begin{aligned} \sigma^+ &= 4\pi a_3^2\,, \\ \sigma^- &= 4\pi\tfrac{1}{9}(2a_1+a_3)^2\,, \\ \sigma^{\text{ex}} &= 4\pi(2/9)(a_1-a_3)\,. \end{aligned} \tag{44}$$

Samarayanake and Woolcock[14] give the following numerical values, obtained from dispersion relations:

$$a_1-a_3=0.292\pm0.020\,,\quad a_1+2a_3=-0.035\pm0.012$$

(in units of $c=u=\hbar=1$). These data are consistent with those of Donald *et al.*[15] obtained from analysis of low-energy cross-section measurements. They correspond in the diagram to the point marked S-W. The earlier calculations by Hamilton and Woolcock[16] give the point marked by H-W. We observe that both these points lie close to the point M_1. This is a consequence of the smallness of a_1+2a_3. Hamilton *et al.*[17] explain this as due to an accidental cancellation of three different contributions: a short-range repulsive interaction, a $T=0$ $\pi\pi \to N\bar{N}$ effect, and rescattering. The quantity a_1-a_3, which is due to the $T=1$ $\pi\pi \to N\bar{N}$ channel, is mostly ρ-meson exchange.

IV. CONCLUDING REMARKS

The graphical method can be applied to a great number of different reactions. The form of the physical region varies from case to case and is determined by the Clebsch-Gordan coefficients which follow from coupling the isospins of the interacting particles. Unmodified, the technique is restricted to reactions with two isospin channels.

The same technique can also be used for studying invariance under other SU_2 subgroups of unitary symmetries, e.g., U spin. This work will be continued by applying the method to other data and other reactions.

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