Crossing Problem for the Backward Cone and the Threshold

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Recent high-energy pion-nucleon experiments with large momentum transfer have made the backward scattering problem attractive for theoretical investigation. In this paper we investigate the problem of crossing for particular regions of the physical s channel, namely, the backward cone and the threshold. Since the backward cone angle is a function of the energy, this is not a high-energy crossing problem alone, but the crossing region extends to a minimum energy given by $s=(M+1)^2$ in pion mass units, for which case the cone extends from backward direction to forward direction.

INTRODUCTION

THE difficulty in the application of the high- as well as the low-energy Regge crossing to the backward scattering of nonequal mass particles is connected with the kinematics of such processes. This problem has been discussed in detail in a separate paper.[1] The essential point is that the integral term in the Regge expressions for the amplitude contains a Legendre function and this term can be neglected in the case of forward scattering since the real part of the order of the Legendre function is $-\frac{1}{2}$ and for large values of the argument the Legendre function goes to zero. But large values of the argument correspond to the high-energy region in the physical channel so that the amplitude can be represented by a few pole terms of the form $s^{\alpha(t)}\beta(t)$. Here s is the energy squared in the physical channel, t the momentum transfer, and α, β the position and residue of the Regge pole. But we can show that there is a region, to be precise, a cone around the backward direction, which can not be reached with large values of the argument of the Legendre function in the u channel. The angle defining this backward cone is a function of the energy. Moreover, for crossing into the region outside of this cone, the mechanism is entirely different from the equal-mass case. Here there is a curve in the Mandelstam diagram given by $\cos\theta_u=\text{const}>+1$ which is asymptotic to the lines $u=0$ and $s=-\frac{1}{2}(1+\cos\theta_u)[u-2(M^2+m^2)]$. Here M is the nucleon mass and m the pion mass which we shall take to be unit from now on. As $\cos\theta_u$ increases, this curve recedes to high-energy regions of the s channel and approaches the $u=0$ line. For the present energies the value of $\cos\theta_u$ is too small for the integral term in the amplitude to be neglected. A numerical example for 8 BeV/c pion momentum at $\theta_{s\,\text{c.m.}}=170°$ (which corresponds to $u\cong-3m^2$) is $\cos\theta_u\cong3$, whereas in the forward direction $t=0$ at the same pion momentum $\cos\theta_t\cong57$. As for the inside of the backward cone, this region can be reached only with values $-1<\cos\theta_u<+1$.

[1] I. A. Sakmar, Nuovo Cimento **40**, 76 (1965).

CROSSING ONTO THE BACKWARD CONE

The relation between $\cos\theta_u$, u, and s is given by

$$\cos\theta_u=\frac{(M^2-1)^2-u[u-2(M^2+1)]-2us}{(M^2-1)^2+u[u-2(M^2+1)]}. \qquad (1)$$

Consider now π^+p scattering. The elastic differential cross section is given by

$$d\sigma/d\Omega=|f_1|^2+|f_2|^2+(\cos\theta_s)(f_1f_2^*+f_2f_1^*), \qquad (2)$$

where $f_1+(\cos\theta_s)f_2=$ spin nonflip amplitude and $f_2=$ spin-flip amplitude. For a well-defined isotopic spin (as in the π^+p(case), f_1 and f_2 are pure isotopic spin amplitudes ($f_1^{3/2}$ for the π^+p case).

θ_s is the c.m. scattering angle in the physical channel. f_1 and f_2 can be expressed in terms of the invariant amplitudes $A_s^{3/2}$ and $B_s^{3/2}$:

$$\begin{aligned} f_1^{s,3/2}&=\frac{E_s+M}{8\pi W_s}[A_s^{3/2}+(W_s-M)B_s^{3/2}],\\ f_2^{s,3/2}&=\frac{E_s-M}{8\pi W_s}[-A_s^{3/2}+(W_s+M)B_s^{3/2}], \end{aligned} \qquad (3)$$

where $E_s=(s+M^2-1)/2W_s$ is the total c.m. nucleon energy and $W_s=\sqrt{s}$.

The invariant amplitudes $A_s^{3/2}$ and $B_s^{3/2}$ can in turn be expressed as linear combinations of the invariant u-channel amplitudes $A_u^{1/2}$, $A_u^{3/2}$, $B_u^{1/2}$, and $B_u^{3/2}$ by making use of the crossing matrix

$$\begin{aligned} A_s^{3/2}&=\tfrac{2}{3}A_u^{1/2}+\tfrac{1}{3}A_u^{3/2},\\ -B_s^{3/2}&=\tfrac{2}{3}B_u^{1/2}+\tfrac{1}{3}B_u^{3/2}. \end{aligned} \qquad (4)$$

The u-channel invariant amplitudes can now be written in terms of the u-channel f_1 and f_2 amplitudes by the inverse relations of (3)

$$\begin{aligned} A_u^{I}&=4\pi\left[\frac{W_u+M}{E_u+M}f_1^{u,I}-\frac{W_u-M}{E_u-M}f_2^{u,I}\right],\\ B_u^{I}&=4\pi\left[\frac{1}{E_u+M}f_1^{u,I}+\frac{1}{E_u-M}f_2^{u,I}\right], \end{aligned} \qquad (5)$$

where $I=\frac{1}{2},\frac{3}{2}$.

After these substitutions, the amplitudes $f_1^{s,3/2}$ and $f_2^{s,3/2}$ are expressed in terms of $f_1^{u,1/2}$, $f_1^{u,3/2}$, $f_2^{u,1/2}$, and $f_2^{u,3/2}$. Inserting these expressions into the cross-section formula (2), one obtains

$$\begin{aligned}\frac{d\sigma}{d\Omega}=&\frac{1}{W_s^2(E_u+M)^2}[(E_s+M)^2(W_u-W_s+2M)^2\\&+(E_s-M)^2(W_u+W_s+2M)^2+2(\cos\theta_s)\\&\times(E_s^2-M^2)(W_s^2-(W_u+2M)^2)]x\\&+\frac{1}{W_s^2(E_u-M)^2}[(E_s+M)^2(W_s+W_u-2M)^2\\&+(E_s-M)^2(W_u-W_s-2M)^2+2(\cos\theta_s)\\&\times(E_s^2-M^2)(W_s^2-(W_u-2M)^2)]y\\&+\frac{1}{W_s^2(E_u^2-M^2)}[(E_s+M)^2((W_s-2M)^2-W_u^2)\\&+(E_s-M)^2((W_s+2M)^2-W_u^2)+2(\cos\theta)_s\\&\times(E_s^2-M^2)(W_u^2+W_s^2-4M^2)]z. \qquad (6)\end{aligned}$$

Here x, y, and z are combinations of the u-channel amplitudes

$$\begin{aligned}x=&(1/9)|f_1^{u,1/2}|^2+(1/36)|f_1^{u,3/2}|^2\\&+(1/18)(f_1^{u,1/2}f_1^{u,3/2*}+f_1^{u,1/2*}f_1^{u,3/2}),\\y=&(1/9)|f_2^{u,1/2}|^2+(1/36)|f_2^{u,3/2}|^2\\&+(1/18)(f_2^{u,1/2}f_2^{u,3/2*}+f_2^{u,1/2*}f_2^{u,3/2}),\\z=&(1/9)(f_1^{u,1/2}f_2^{u,1/2*}+f_1^{u,1/2*}f_2^{u,1/2})\\&+(1/18)(f_1^{u,1/2}f_2^{u,3/2*}+f_1^{u,1/2*}f_2^{u,3/2})\\&+(1/18)(f_2^{u,1/2}f_1^{u,3/2*}+f_2^{u,1/2*}f_1^{u,3/2})\\&+(1/36)(f_1^{u,3/2}f_2^{u,3/2*}+f_1^{u,3/2*}f_2^{u,3/2}). \qquad (7)\end{aligned}$$

All f_1 and f_2 amplitudes are functions of u and $\cos\theta_u$ and they are the amplitudes expressed in Regge form, that is, as the sum of an integral term and pole terms.

Let us now cross from the u channel into the s channel by taking $u=0$. According to Eq. (1), $u=0$ corresponds to $\cos\theta_u=+1$ and this represents a line in the physical s channel of the Mandelstam diagram. $u=0$ is the only value of u which makes $\cos\theta_u$ independent of s. For all other values of u, $\cos\theta_u$ is a function of s.

It is well known that f_1 and f_2 have singularities at $W_u=0$. But since the differential cross section is finite, the coefficients in the cross-section formula (6) which contain powers of W_u should cancel these singularities.

Now one may argue that the Legendre functions appearing in the Regge representation are functions of $\cos\theta_u$ and the partial waves functions of W_u, so that for $\cos\theta_u=1$ and $u=0$ there can be no s dependence coming from them. The amplitudes $f_{1,u}^{1/2,3/2}$ and $f_{2,u}^{1/2,3/2}$ are assigned constant values which depend only on the positions α_i and residues β_i of the Regge poles at $u=0$.

Thus one obtains the nice result that the differential cross sections on the backward cone are determined solely by the kinematical factors of the cross-section formula, the amplitude squares being the same at all energies. Given a certain energy s in the physical channel the condition $u=0$ determines an angle $\cos\theta_s$:

$$\cos\theta_s=\{M^2+1-2[(M^2+q_s^2)(1+q_s^2)]^{1/2}\}/2q_s^2. \qquad (8)$$

The cross section at a certain energy and angle is related to the cross section at a different energy and different angle, where the angles are given by Eq. (8). This angle is the one which defines the width of the backward cone. One may then conclude that the $f_{1,u}$ and $f_{2,u}$ amplitudes have the same values at all energies s, provided we are on the respective backward cones of these energies.

Expressing E_u,E_s and W_u,W_s in terms of u and s, the powers of W_u and W_s can be made explicit. Since $u=0$, $\cos\theta_s$ is given in terms of s alone;

$$\cos\theta_s=\frac{-s^2+2s(M^2+1)+(M^2-1)^2}{s^2-2s(M^2+1)+(M^2-1)^2}. \qquad (9)$$

With these substitutions Eq. (6) becomes

$$\begin{aligned}\frac{d\sigma}{d\Omega}=&\frac{4u^2}{(M^2-1)^2}\frac{s+2(M^2-1)}{s}(x+y-z)\\&+\frac{4u\sqrt{u}}{(M^2-1)^2}\frac{4M(M^2-1)}{s}(x-y)\\&+\frac{4u}{(M^2-1)^2}\frac{(M^2-1)^2-8M^3\sqrt{s}}{s}(x+y+z). \qquad (10)\end{aligned}$$

In this formula we have already set $u=0$ in the denominators where u appears as an additional term to M^2-1. Now the limit of this expression for $u=0$ should be formed. As we remarked earlier, here x, y, and z are functions of u and $\cos\theta_u$ only in the Regge representation. They do have singularities at $u=0$ which cancel with the powers of u in the numerators of Eq. (10) since the cross section is finite. Therefore the most general form one can obtain will be

$$d\sigma/d\Omega=a+b/\sqrt{s}+c/s,$$

where a, b, and c are constants. Since no approximations have been made in this derivation, such a form should be valid at all energies. The condition of being on the backward cone is built in, therefore the angle does not appear in Eq. (10). Some of the constants a, b, and c may be zero. The above form is the most general mathematical expression one can obtain without an explicit knowledge of the amplitudes, contained in x, y, and z. The $1/s$ behavior for high energies is predicted by Regge theory,[2] in which case a single pole is dominant. Experi-

[2] Virendra Singh, Phys. Rev. **129**, 1889 (1963).

Table I. π^+p differential cross section on the backward cone.[a]

s in m_π^2	92	110.94	115.56	126.82	143.68	154.94	185.85	209.35	216.23	246.96	287.34	431.68	816.82
p pion lab momentum (BeV/c)									1.76	2.08	2.5	4	8
T_π pion lab kinetic energy (MeV)	340	533	581	698	873	990	1311	1555					
$\cos\theta_s$	+1	−0.0278	−0.1578	−0.3829	−0.5816	−0.6650	−0.7981	−0.8524	−0.8642	−0.9028	−0.9327	−0.9737	−0.9934
$d\sigma/d\Omega$ (μb/sr)	11630	418	141	142	321	529	707	60	208	38	90	19	8

[a] All data have been interpolated to the backward cone angle $\cos\theta_s$. All data given in T_π are taken from Jerome A. Helland, Thomas J. Devlin, Donald E. Hagge, Michael J. Longo, Burton J. Moyer, and Calvin D. Wood, Phys. Rev. **134**, 1062 (1964), except $T_\pi=340$ MeV which was taken from the table given by G. Hoehler, G. Ebel, and J. Giesecke, Z. Physik **180**, 430 (1964). $p=1.76$ and 2.08 BeV/c data were taken from F. E. James, J. A. Johnson and H. L. Kraybill, Phys. Letters **19**, 72 (1965). $p=2.5$ BeV/c data are from V. Cook, B. Cork, W. R. Holley, and M. L. Perl, Phys. Rev. **130**, 762 (1963). Finally $p=4$ and 8 BeV/c data are from W. R. Frisken, A. L. Read, H. Ruderman, A. D. Krisch, J. Orear, R. Rubinstein, D. B. Scarl, and D. H. White, Phys. Rev. Letters **15**, 313 (1965).

mentally the same behavior has been suggested recently[3] as consequence of recent high-energy backward scattering experiments. At such high energies the backward cone is very narrow and close to the backward direction.

Even though the high-energy data are in agreement with this formula, the low-energy data seem to be in disagreement (see Table I). One explanation which comes into mind is that the amplitudes may have indeterminacy points at $u=0$ and $\cos\theta_u=1$. The indeterminacy apparently arises because of ratios which have independent variables in both the denominator and the numerator, so that the limit for zero is not determined. Thus it appears that u and $\cos\theta_u$ are not the proper variables of f_1 and f_2 for crossing onto the backward cone. These amplitudes are, rather, functions of $(1-\cos\theta_u)/u$ or, remembering the expression for s^1, the proper variables should be u and s. In view of the disagreement with experiment it seems one is forced to the conclusion that the Regge representations of f_1 and f_2 break down completely on the backward cone and one can not use this kind of crossing.

CROSSING TO THE THRESHOLD

An interesting point to cross from the u channel is the threshold of the physical s channel. For this point $s=(M+1)^2$ and $u=(M-1)^2$ in pion mass units. Since all curves $\cos\theta_u=$ const. pass through this point, when crossing from the u channel to this point the u-channel amplitudes should be $\cos\theta_u$-independent. If we write $f_1(u,\cos\theta_u)$ and $f_2(u,\cos\theta_u)$ in Regge representation, the integral terms have the form

$$\int f_{J\pm 1/2}{}^{e,o}\frac{P_{J\pm 1/2}'(\cos\theta_u)\pm P_{J\pm 1/2}'(-\cos\theta_u)}{\cos\pi J}dJ\,,$$

where $f_{J\pm 1/2}{}^{e,o}$ are even or odd continuations of the partial waves. P' are the derivatives of the Legendre functions. For the pole terms the amplitudes are replaced by the residues. Now since the cross section expressed in terms of the u-channel amplitudes should be $\cos\theta_u$-independent we should try to make both the pole terms and the integrals $\cos\theta_u$-independent. This can be achieved by choosing for the pole terms the derivatives of the Legendre functions as P_0', P_1', P_{-1}', and P_{-2}'. Thus the values of the nucleon, 1238-MeV N^*, and 1512-MeV N^* Regge trajectories at $W_u=M-1$ may be identified with the values $\alpha=(0,1,-1,-2)\pm\frac{1}{2}$. The integral term contains J as the integration variable and the vanishing of the integral implies certain symmetry properties of the partial waves with respect to the real axis in the complex J plane.

Let us evaluate the differential cross section for π^+p scattering.

$$f_2{}^{u,I}=\frac{E_u-M}{8\pi W_u}[-A_u{}^{3/2}+(W_u+M)B_u{}^{3/2}]\,. \quad (11)$$

Here

$$(E_u-M)/8\pi W_u=(u+M^2-1-2M\sqrt{u})/16\pi u\,. \quad (12)$$

At s-channel threshold $u=(M-1)^2$. This makes $f_2{}^{u,I}=0$. From Eq. (7) it follows that $y=0$ and $z=0$. Only x is different from 0. Evaluating the differential cross sections from Eq. (6) for $u=(M-1)^2$ and $s=(M+1)^2$ we find

$$\frac{d\sigma}{d\Omega}=4\left(\frac{M-1}{M+1}\right)^2[(1/9)|f_1{}^{u,1/2}|^2+(1/36)|f_1{}^{u,3/2}|^2+(1/18)(f_1{}^{u,1/2}f_1{}^{u,3/2*}+f_1{}^{u,1/2*}f_1{}^{u,3/2})]=|f_1{}^{s,3/2}|^2.$$

This can be written as

$$\left(\frac{M-1}{M+1}\right)^2\left|\tfrac{2}{3}f_1{}^{u,1/2}+\tfrac{1}{3}f_1{}^{u,3/2}\right|^2=|f_1{}^{s,3/2}|^2,$$

i.e.,

$$0.548(d\sigma/d\Omega)(\pi^-p\to\pi^-p)|_{u=(M-1)^2}=\frac{d\sigma}{d\Omega}(\pi^+p\to\pi^+p)\Big|_{s=(M+1)^2}. \quad (13)$$

[3] A. I. Alikhanov, G. L. Bayatyan, E. V. Brakhman, G. P. Eliseev, Yu. V. Galaktionov, L. G. Landsberg, V. A. Lyubimov, I. V. Sidoriv, F. A. Yetch, and O. Ya. Zeldovich, Phys. Letters **19**, 345 (1965).

The s-channel threshold is $s=(M+1)^2\cong 59m_\pi^2$. The pion total lab energy E_π is found from $s=M^2+1+2E_\pi M$ to be $E_\pi=1m_\pi$. Pion lab kinetic energy is $T_\pi=E_\pi-1=0$. At $s=(M+1)^2$

$$(d\sigma/d\Omega)(\pi^+p\rightarrow\pi^+p)=0.18\text{ mb/sr}.$$

From Eq. (13) one finds that at $u=(M-1)^2$

$$(d\sigma/d\Omega)(\pi^-p\rightarrow\pi^-p)=0.10\text{ mb/sr}.$$

Thus we have obtained the analytic continuation of the forward π^-p differential cross section at $u=(M-1)^2$. By a similar procedure one can also obtain the analytic continuation of the forward π^+p differential cross section for $s=(M-1)^2$.

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Application of Partial-Wave Analyticity to Nucleon-Nucleon Scattering*

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In order to give a quantitative discussion of models for nucleon-nucleon scattering which are founded on analyticity and unitarity of the S matrix, the fact that each partial-wave amplitude has a nonvanishing imaginary part in the physical region must be dealt with unambiguously. That goal is achieved by evaluating certain integrals over the physical region, using rigorous bounds when experimental data are unavailable. The resulting modified amplitudes have known experimental uncertainty, and can validly be compared directly with dynamical models which do not themselves satisfy the requirements of unitarity, such as the popular single-boson-exchange models. The modified amplitudes are tabulated and various tests for models are discussed.

I. INTRODUCTION

THE hope has been expressed many times[1] that a knowledge of the analytic properties of strong-interaction scattering amplitudes may provide a partial or perhaps even a complete framework for dynamical calculations involving the strong interactions. In practice a common approach is to make use of crossing symmetry and unitarity to approximate the scattering amplitudes of interest in some unphysical region, and to "derive" from this approximate information some estimates of the amplitudes in the physical region. These "derivations" always involve assumptions additional to those mentioned above, which make the interpretation of such calculations less clean than one would like. The present work is a brief examination of some of these auxiliary assumptions[2] and a detailed discussion of an alternative procedure whose auxiliary assumptions have particularly uncomplicated physical interpretation. Finally, this procedure is applied to the nucleon-nucleon problem to provide functions which can be directly compared with the approximate amplitudes derived by crossing symmetry, and the accuracy obtainable by this method is discussed. The precise analyticity assumption used is that each partial wave amplitude has only those singularities given by the Mandelstam representation.[3] It should be clearly understood that the method to be discussed does *not* provide for the "derivation" of amplitudes in the physical region. However, in my opinion it has not yet been established clearly that the popular single-boson-exchange models give an accurate enough description of the true state of affairs to justify attempting such derivation. The present method is particularly well suited to determining whether this is the case, and hence, in spite of its limitations, will be very useful in the present situation.

The presentation is divided into seven parts, dealing with: kinematics (Sec. II), dynamics (Sec. III), the problem of unitarization (Sec. IV), removal of the unitarity cut (Sec. V), detailed numerical results (Sec. VI), remarks concerning pole models (Sec. VII), and conclusions (Sec. VIII).

* A portion of this work was reported earlier in Phys. Rev. Letters **12**, 52 (1964).

[1] See, for example, G. F. Chew, *S-Matrix Theory of Strong Interactions* (W. A. Benjamin, Inc., New York, 1961).

[2] A more detailed discussion with some numerical examples has been given elsewhere [P. B. Kantor, Ann. Phys. (N.Y.) **33**, 196 (1965)]. In particular it is shown there that various methods do not agree in practice.

[3] The Mandelstam analyticity for N-N is discussed by M. L. Goldberger, M. T. Grisaru, S. W. MacDowell, and D. Y. Wong. Phys. Rev. **120**, 2250 (1960).