Space Inversion, Time Reversal, and Other Discrete Symmetries in Local Field Theories*

T. D. LEE AND G. C. WICK
Department of Physics, Columbia University, New York, New York
(Received 28 February 1966)

The general algebraic relations between space inversion, time reversal, and the internal symmetry group are analyzed within the framework of a Lorentz-invariant local field theory. The problem of unitary representations of the full Poincaré group including the space and time reflection operators has been studied by Wigner, and the representations are classified into 4 cases. It is shown that, with the added assumption of the local field theory, Wigner's cases 2, 3, and 4 either do not not occur or can be reduced to his case 1. The concept of minimal group extension is introduced and the related mathematical analysis is given. The symmetry properties under space inversion, time reversal, and other discrete operators such as charge conjugation are analyzed separately for each of the three known interactions: strong, electromagnetic, and weak.

I. INTRODUCTION

OUR views of discrete symmetries such as space inversion P, time reversal T, and charge conjugation C have undergone great changes in recent years. Since 1957, it has been well established[1] that both P and C symmetries are only approximately valid. Rather naturally, this discovery led immediately to speculations as to the possible need of a profound revision of some of our concepts. Until the recent discovery[2] of the $\pi^+\pi^-$ decay mode of the long-lived K_2^0 meson, however, it was possible to retain intact the structure of the full Poincaré group and in particular to believe in the essential symmetry between left and right by using CP, instead of P. From a fundamental point of view, therefore, the strong evidence now existing of a CP violation in K_2^0 decay, and the concurrent indirect conclusion that T is also not an exact symmetry, are a more decisive blow to our notions on geometric symmetry principles than earlier results.

Strictly speaking, since the very existence of symmetry operators such as T and P, or CP, is deduced from the alleged equivalence of certain reference systems, the presence of phenomena violating that equivalence implies that the operators themselves cannot be exactly defined. The language employed most often in describing the situation is in fact somewhat inconsistent. One says: there is a P or CP, etc., operator and in specific theories one often proceeds to give exact definitions of these operators; subsequently one assumes a Hamiltonian containing terms which do not commute with P, CP or T, etc. This is the same as saying, however, that these operators do not satisfy exactly the multiplication laws of the coordinate transformations they are alleged to represent within the full Poincaré group. More explicitly, the transformation

$$\mathbf{r}\rightarrow-\mathbf{r},\quad t\rightarrow+t,$$

corresponding to P or CP, should commute with a time translation, represented infinitesimally by H, and the transformation

$$\mathbf{r}\rightarrow+\mathbf{r},\quad t\rightarrow-t,$$

corresponding to T, transforms the time translation $t\rightarrow t+\tau$ into $t\rightarrow t-\tau$ which, because of the anti-unitary nature[3] of T, again implies commutativity of the corresponding operators. Thus the "exact" definition of the symmetry operators is a purely formal convention, that in fact does not satisfy the basic geometrical requirements. One can similarly think of other possibilities of "exact" definitions. For example, one might start with the *physical* single-particle state $|\mathbf{k},\lambda\rangle$, and define P and T to be, respectively, the unitary and anti-unitary operators that satisfy

$$P|\mathbf{k},\lambda\rangle=\eta_P|-\mathbf{k},-\lambda\rangle,\tag{1.1}$$

and

$$T|\mathbf{k},\lambda\rangle=\eta_T|-\mathbf{k},+\lambda\rangle,\tag{1.2}$$

where $\mathbf{k}$ and λ are, respectively, the momentum and the helicity of the particle, and η_T and η_P are phase factors which may depend on λ. (Helicity is defined to be the spin component along the direction of $\mathbf{k}$.)

In equations such as (1.1) and (1.2), the states $|\mathbf{k},\lambda\rangle$ and $|-\mathbf{k},\pm\lambda\rangle$ refer to the same physical particle. The equations, however, have an unambiguous meaning only for an exactly stable particle. The state vector of an unstable particle must sooner or later develop components corresponding to the decay products. No satisfactory generalization of (1.1) or (1.2) exists for such

* This research was supported in part by the U. S. Atomic Energy Commission.

[1] The first experiment that conclusively established the approximate nature of P and C symmetries was made on β decay by C. S. Wu, E. Ambler, R. W. Hayward, D. D. Hoppes, and R. P. Hudson, Phys. Rev. **105**, 1413 (1957). This was immediately followed by the observation of the same noninvariance properties in π, μ decays by R. L. Garwin, L. M. Lederman, and M. Weinrich, Phys. Rev. **105**, 1415 (1957) and by J. J. Friedman and V. L. Telegdi, Phys. Rev. **105**, 1681 (1957). The possibility that P, C, and T are only approximate symmetries was suggested theoretically by T. D. Lee and C. N. Yang, Phys. Rev. **104**, 254 (1956) and further discussed by T. D. Lee, R. Oehme, and C. N. Yang, Phys. Rev. **106**, 340 (1957).

[2] J. H. Christenson, J. W. Cronin, V. L. Fitch, and R. Turlay, Phys. Rev. Letters **13**, 138 (1964). See also A. Abashian *et al.*, Phys. Rev. Letters **13**, 243 (1964).

[3] E. P. Wigner, Gött. Nach. Math. Naturw. Kl., p. 546 (1932). For a general discussion of anti-unitary operators see, e.g., E. P. Wigner, *Group Theory* (Academic Press Inc., New York, 1959), Chap. 26.

a state, if the decay violates either space-inversion or time-reversal symmetry. Similar difficulties arise for multiparticle states, if the interaction involves symmetry violations.[4]

The notion that a certain "fuzziness" is attached to the definition of the symmetry operators P, PC, T is not new and is not likely to be questioned; but the situation is clearly an unsatisfactory one from a fundamental point of view. The following is an attempt to characterize the ambiguity in a mathematically more precise way; if this can be done, at least some order can be brought into the subject.

There is already a completely different source of ambiguity in the definition of symmetries which has been analyzed in some detail. This is the possible existence of superselection rules.[5] Such rules imply that the Hilbert space of a system (say a system of interacting fields) should split into "noncoherent subspaces." Thus, every symmetry operator can be multiplied (without detectable consequences) in each subspace by a phase factor, which is independent of the phase factors in the other subspaces. It is of course quite possible that there are no superselection rules, the rules now generally accepted being the result of an inadequate sensitivity of our experimental techniques. This latter point of view may in fact present some conceptual advantages, but it clearly leads to the same physical results as the superselection rules, so long as the violation is either unobserved or can be neglected.

It seems practical at the present stage to accept the view that there are superselection rules associated with N (baryon number), Q (electric charge) and perhaps also the leptonic numbers. The independent phase factors in each noncoherent subspace mentioned previously are adequately replaced by the gauge group generated by these quantities. It is then easy to summarize the situation by stating that each geometrical symmetry, if it exists, is represented not just by one operator, but by a coset of the gauge group (as will be discussed in more detail later). The *geometrical* group then appears as the quotient of the complete symmetry group by the above mentioned gauge-group. This formulation has been used, for example, by Michel for the full Poincaré group.[6] This discussion is still based, however, on the assumption that the discrete space-time symmetries are exact.

The approximate nature of the symmetries suggests, in our opinion, that the process of redefinition must be carried a step further. Let us consider a model Hamiltonian H which may be constructed from the real Hamiltonian H_{total} by setting some of its parameters equal to zero:

$$H=\lim_{g\to 0} H_{\text{total}}, \tag{1.3}$$

where g represents a set of parameters which can be certain coupling constants and maybe mass differences. By a suitable choice of the set of parameters g, both space inversion and time reversal can be regarded as exact symmetries for the solutions of the model Hamiltonian H.

As an example, we may take $H=H_{\text{free}}+H_{\text{st}}$ where

$$H_{\text{free}}+H_{\text{st}}=\lim_{e=G_{\text{wk}}=0} H_{\text{total}}. \tag{1.4}$$

The parameters e and G_{wk} are, respectively, the electron charge and the weak-interaction coupling constant. In this limit, the mass differences between particles within the same isospin multiplet are all assumed to be zero. The H_{free} is the free-particle Hamiltonian, and H_{st} the strong-interaction Hamiltonian. The existing experimental results strongly support the assumption that, so far as this model Hamiltonian H is concerned, both space inversion and time reversal are exact symmetries. Equations (1.1) and (1.2) may still be used to define the operators P and T provided $|\mathbf{k},\lambda\rangle$ is regarded as the single-particle eigenstate of the model Hamiltonian. Similar requirements for P and T can be easily extended to the multiparticle eigenstates of H. However, in the limit $e=G_{\text{wk}}=0$, it is not possible to distinguish between different members of the same isospin multiplet. Thus, if P is the unitary space-inversion operator, an equally good choice can be the product SP where S is any isospin transformation operator (or, any other internal symmetry operator). The same holds for the time-reversal operator T.

In general, for any given model Hamiltonian H, there exists an *internal symmetry group* $\mathcal{G}$:

$$\mathcal{G}=\{S\}, \tag{1.5}$$

where the group element S is called an internal symmetry operator. S will be defined later (in Sec. II) to be any unitary operator that satisfies

$$SHS^{-1}=H, \tag{1.6}$$

[4] For example, for the multiparticle states one might try to define P to be the unitary operator which transforms, say, any physical incoming-wave state $|\mathbf{k}_1,\lambda_1;\mathbf{k}_2,\lambda_2\cdots\rangle^{\text{in}}$ to $|-\mathbf{k}_1,-\lambda_1;-\mathbf{k}_2,-\lambda_2\cdots\rangle^{\text{in}}$, multiplied by an appropriate phase factor. Such an operator, which we may call P^{in}, has only matrix elements between states of the same energy; therefore, it commutes with the total Hamiltonian. The observed nonconservation of parity implies that a different operator, called P^{out}, would result if the incoming-wave states are replaced by the outgoing-wave states. At any finite time t, neither P^{in} nor P^{out} transforms the field operators at $\mathbf{r}$ to those at $-\mathbf{r}$. These two operators P^{in} and P^{out} are, respectively, connected only with the transformation properties of the initial and the final configurations of the system; such operators are, in general, rather useless (unless space inversion were an exact symmetry).

[5] G. C. Wick, A. S. Wightman, and E. P. Wigner, Phys. Rev. **88**, 101 (1952). See also G. Feinberg and S. Weinberg, Nuovo Cimento **14**, 571 (1959).

[6] L. Michel, in *Group Theoretical Concepts and Methods in Elementary Particles*, edited by F. Gürsey (Gordon and Breach Publishers, New York, 1964), pp. 135–200. A discussion of group extensions is also given in this paper. See also F. Kamber and N. Straumann, Helv. Phys. Acta **37**, 563 (1964), and L. C. Biedenharn, J. Nuyts, and H. Ruegg, Commun. Math. Phys. (to be published).

and is not connected with any coordinate transformations. We note that if H is invariant under the charge conjugation C, then, but only then, by definition, $\mathcal{G}$ contains C. The structure of $\mathcal{G}$ depends on the particular H under consideration. A full discussion of the properties of the space-inversion operator and the time-reversal operator cannot be given without studying their algebraic relations with the internal symmetry group $\mathcal{G}$.

In Sec. II, the general defintions of the space-inversion operator and the time-reversal operator are given for any Lorentz-invariant local field theory. It follows from these general definitions that if a unitary operator $\mathcal{P}$ is a space-inversion operator, then any member in the set $\{S\mathcal{P}\}$ can also be used as a space-inversion operator. Similarly, if an anti-unitary operator $\mathcal{T}$ satisfies this general definition of the time reversal operator, then any member in the set $\{S\mathcal{T}\}$ also satisfies the same definition. For clarity, we use the script letters $\mathcal{P}$ and $\mathcal{T}$ to denote an arbitrary choice of these operators. The general algebraic relations between these discrete elements $\mathcal{P}$, $\mathcal{T}$ and the internal-symmetry group $\mathcal{G}$ are then investigated.

As we shall see later, an immediate consequence of the usual "CPT" theorem[7] is that for any model Hamiltonian H, if $\mathcal{P}$ exists, then $\mathcal{T}$ exists and vice versa. Furthermore, for any $\mathcal{P}$ in the set $\{S\mathcal{P}\}$ there exists a $\mathcal{T}$ in the set $\{S\mathcal{T}\}$ such that

$$\mathcal{P}^2=\mathcal{T}^2, \tag{1.7}$$

$$\mathcal{P}\mathcal{T}=(-1)^{2J}\mathcal{T}\mathcal{P}, \tag{1.8}$$

and

$$\mathcal{P}^{-1}\mathcal{T}=\mathcal{I}, \tag{1.9}$$

where J is the spin of the local field operator and $\mathcal{I}$ is the usual CPT operator which satisfies

$$\mathcal{I}^2=(-1)^{2J}, \tag{1.10}$$

and

$$\mathcal{I}S=S\mathcal{I}, \tag{1.11}$$

for all internal-symmetry operators S. Thus, the algebraic relations between $\{S\mathcal{T}\}$ and the internal symmetry group $\mathcal{G}$ are completely determined if those between $\{S\mathcal{P}\}$ and $\mathcal{G}$ are known (and vice versa).

The set of operators $\{S\mathcal{P}\}$ and $\{S\}$ forms a new group $\mathcal{E}$ which we shall call a *minimal extension* of $\mathcal{G}$; $\mathcal{G}$ is an invariant subgroup of $\mathcal{E}$ and the factor group is the two-element group Z_2. The mathematical analysis of the minimal extension for any given group $\mathcal{G}$ is an elementary case of the general theory, and is given in Sec. III. All such minimal extensions can be classified into two general types A and B, depending on whether the correspondence $S\leftrightarrow\mathcal{P}S\mathcal{P}^{-1}$ is an inner or outer automorphism of $\mathcal{G}$. Some simple illustrative examples are given in Sec. IV.

The problem of unitary representations of the full Poincaré group including the space and time-reflection operators has been discussed by Wigner,[8] and the representations are classified into 4 cases, depending on the algebraic properties of the space-time reflection operators; these classifications are, however, made without explicitly considering the internal symmetry group. Equations (1.7)–(1.11) show that, with the added assumption of the local-field theory, Wigner's cases 2, 3 and 4 either do not occur or can be reduced to his case 1 by using different operators in the sets $\{S\mathcal{P}\}$ and $\{S\mathcal{T}\}$.

The notion of minimal extension is also useful in studying the question of possible discrete elements in the internal symmetry group. It has been suggested[9] recently that each of the three known interactions (the strong interaction $H_{\rm st}$, the electromagnetic interaction H_γ, and the weak interaction $H_{\rm wk}$) is invariant under its own C_i, P_i and T_i where $i=$strong, γ, or weak. In all these cases, the internal symmetry group $\mathcal{G}_i$ of the particular H_i is itself a minimal extension of another group $\mathcal{G}_i^0$ which does not contain the discrete element C_i.

Applications to these interactions are given in Secs. V and VI. The general solutions of $C_{\rm wk}$, $P_{\rm wk}$, and $T_{\rm wk}$

[7] W. Pauli, *Niels Bohr and the Development of Physics* (Pergamon Press, London, 1955); J. Schwinger, Phys. Rev. **91**, 720 (1953); **94**, 1366 (1953). G. Lüders, Kgl. Danske Videnskab. Selskab Mat. Fys. Medd. **28**, No. 5 (1954). See also R. Jost, Helv. Phys. Acta **30**, 409 (1957); R. F. Streater and A. S. Wightman, *PCT, Spin and Statistics, and All That* (W. A. Benjamin, Inc., New York, 1964).

It should be pointed out that in the framework of the present paper, the expression "CPT" may be misleading because either C does not exist as a symmetry, or if C exists then there is no clear distinction between our script letter $\mathcal{P}$ and $C\mathcal{P}$ (likewise for $\mathcal{T}$). Therefore, the CPT theorem could be called the $\mathcal{P}\mathcal{T}$ theorem.

[8] E. P. Wigner, in *Group Theoretical Concepts and Methods in Elementary Particles*, Ref. 6, pp. 37–80. See also R. M. F. Houtappel, H. Van Dam, and E. P. Wigner, Rev. Mod. Phys. **37**, 595 (1965). The identification of Wigner's four cases in terms of the language of field theory is not without certain possibilities of ambiguity. Here, we assume that each symmetry operator $\mathcal{P}$, or $\mathcal{T}$, or $\mathcal{I}$, is normalized so that the vacuum state is an eigenstate with eigenvalue $+1$. Let the signs ϵ_θ, ϵ_I be defined by $\mathcal{I}^2=\epsilon_I(-1)^{2J}$ and $\mathcal{P}\mathcal{T}=\epsilon_\theta\epsilon_I(-1)^{2J}\mathcal{T}\mathcal{P}$. [By using (1.9), this implies $\mathcal{T}^2=\epsilon_\theta\mathcal{P}^2$.] The four cases 1, 2, 3, 4 correspond to $(\epsilon_\theta,\epsilon_I)=(1,1)$, $(1,-1)$, $(-1,1)$ and $(-1,-1)$, respectively. Equations. (1.7)–(1.9) show that the choice $\epsilon_\theta=\epsilon_I=1$ is always possible.

The above normalization condition merely gives a convenient handle, to connect the transformation properties, under $\mathcal{P}$, $\mathcal{T}$, and $\mathcal{I}$, of a composite system to the product of the transformations of its constituents. The existence of a "composition law" expressing the connection between the space-time reflection properties of a multiparticle state and those of the single particle states is, of course, a natural consequence of local-field theory, but we must emphasize that *some* statement about this connection is an integral part of any usable formulation of the transformation laws; for example it is needed if one considers only the asymptotic states of any physical reaction.

The analysis of the possible irreducible representations or co-representations of the group of reflection operators, and the ensuing reduction, of the subspace of all single-particle states, into types of irreducible spaces, may be a logical step in a certain way of analyzing the problem. But the distinction between inequivalent representations is devoid of observable consequences, unless it is coupled with a specific prescription for the "composition law." We have adopted the rules of field theory, applied if one wishes only to the "in" and "out" states of S-matrix theory, as by far the most natural and simple way to formulate the necessary connections.

[9] T. D. Lee, Phys. Rev. **140**, B959 (1965).

are analyzed. It can be shown that $C_{\rm wk}$, $P_{\rm wk}$, and $T_{\rm wk}$ also commute with the SU_3-invariant part of the strong interaction. The various physical consequences of these discrete symmetries are discussed.

The operator C_i is an element of $\mathcal{G}_i$. An interesting, though perhaps academic, question is whether C_i (in particular $C_{\rm st}$) can be continuously connected to the unit element of $\mathcal{G}_i$. In Sec. VII, it is found that, at least in some explicit models, this is possible. For example, if we consider all Feynman graphs which contain only the usual strong interaction vertices between the nucleons and any $G=+1$ mesons such as ρ, η^0, η'^0 etc., then the internal symmetry group automatically contains a subgroup SO_4 which, in turn, contains the usual isospin group SU_2. The $C_{\rm st}$ is one of the elements of this SO_4 group; consequently, it is connected to the unit element.

For practical applications, this SO_4 symmetry is violated by the strong interaction between the nucleons and the pions. Some possible uses of this broken SO_4 symmetry are discussed.

II. GENERAL DISCUSSION OF $\mathcal{P}$ AND $\mathcal{T}$

1. Spin-0 and Spin-$\frac{1}{2}$ Fields

We consider first a model Hamiltonian H which depends on n Hermitian[10] spin 0 fields $\phi_1(x), \cdots, \phi_n(x)$ and m Hermitian spin-$\frac{1}{2}$ fields $\psi_1(x), \cdots, \psi_m(x)$ where

$$\phi_a(x)=\phi_a{}^\dagger(x)\,,$$
$$[\psi_a(x)]_\mu=[\psi_a{}^\dagger(x)]_\mu\,, \qquad (2.1)$$

$x=(\mathbf{r},t)$, and $a=1, 2, \cdots, n$ (or m). Each ψ_a is a 4-component spinor, and its spinor components are labeled by μ which varies from 1 to 4. The $\phi_a(x)$ and $\psi_a(x)$ satisfy the usual equal-time commutation and anticommutation relations:

$$[\phi_a(\mathbf{r},t),\phi_b(\mathbf{r}',t)]=[\phi_a(\mathbf{r},t),\psi_b(\mathbf{r}',t)]=0\,, \qquad (2.2)$$

and, if there is no derivative coupling,

$$[\phi_a(\mathbf{r},t),(d/dt)\phi_b(\mathbf{r}',t)]=i\delta_{ab}\delta^3(\mathbf{r}-\mathbf{r}')\,,$$

and

$$[\psi_a(\mathbf{r},t)]_\mu[\psi_b(\mathbf{r}',t)]_\nu+[\psi_b(\mathbf{r}',t)]_\nu[\psi_a(\mathbf{r},t)]_\mu =\delta_{ab}\delta_{\mu\nu}\delta^3(\mathbf{r}-\mathbf{r}')\,; \qquad (2.3)$$

otherwise, these relations become more complicated. Our general discussions, of course, do not depend on whether derivative couplings exist or not. For clarity, we assume here that the masses of the spin-$\frac{1}{2}$ fields are all different from zero. The zero-mass case will be discussed in Appendix A.

Definition 1. A unitary operator S in the Hilbert space is called an *internal-symmetry operator* if the Hamiltonian H is invariant under the transformation

$$S\phi(x)S^{-1}=\tilde{s}_0\phi(x)\,,$$

and

$$S\psi(x)S^{-1}=\tilde{s}_{1/2}\psi(x)\,, \qquad (2.4)$$

where

$$\phi=\begin{bmatrix}\phi_1\\ \vdots\\ \phi_n\end{bmatrix}, \quad \psi=\begin{bmatrix}\psi_1\\ \vdots\\ \psi_m\end{bmatrix},$$

and s_0, $s_{1/2}$ are, respectively, $(n\times n)$ and $(m\times m)$ matrices. From the Hermiticity condition and the commutation relations, both s_0 and $s_{1/2}$ must be real and orthogonal. Throughout the paper, the dagger and tilde denote, respectively, Hermitian conjugation and transposition. In (2.4), the transpose is introduced so that if the matrices s_J and s_J' are associated with the internal-symmetry operators S and S', then the matrix associated with (SS') is (s_Js_J'), where $J=0$ or $\frac{1}{2}$. For any given matrices s_0 and $s_{1/2}$, if S satisfies (2.4), so does $e^{i\theta}S$. Throughout our discussion we assume that there exists a *single* lowest eigenstate of H, called the vacuum state $|\text{vac}\rangle$. The arbitrary phase factor in $(e^{i\theta}S)$ will always be chosen so that

$$S|\text{vac}\rangle=|\text{vac}\rangle\,. \qquad (2.5)$$

Definition 2. The group of *all* the internal symmetry operators is defined to be the *internal-symmetry group* $\mathcal{G}$:

$$\mathcal{G}=\{S\}\,. \qquad (2.6)$$

Definition 3. The theory is said to be invariant under a *space inversion* if there exists a unitary operator $\mathcal{P}$ such that

$$\mathcal{P}\phi(\mathbf{r},t)\mathcal{P}^{-1}=\tilde{u}_0{}^P\phi(-\mathbf{r},\,t)\,,$$
$$\mathcal{P}\psi_\mu(\mathbf{r},t)\mathcal{P}^{-1}=\tilde{u}_{1/2}{}^P[i\gamma_4\psi(-\mathbf{r},\,t)]_\mu\,, \qquad (2.7)$$

and

$$\mathcal{P}H\mathcal{P}^{-1}=H\,,$$

where $u_0{}^P$ and $u_{1/2}{}^P$ are, respectively, $(n\times n)$ and $(m\times m)$ matrices, $\psi_\mu(x)$ is a $(m\times 1)$ matrix whose components are $[\psi_1(x)]_\mu, \cdots, [\psi_m(x)]_\mu$. Throughout the paper, the Majorana representation of the Dirac matrices will be used; i.e.,

$$\gamma_1,\quad \gamma_2,\quad \gamma_3\quad \text{are real}\,,$$

and

$$\gamma_4,\quad \gamma_5\quad \text{are imaginary}\,. \qquad (2.8)$$

All the γ's are (4×4) Hermitian matrices. It follows from Eqs. (2.1)–(2.3) that $u_J{}^P$ must be real and orthogonal, where $J=0$ or $\frac{1}{2}$. For any given matrices $u_0{}^P$ and $u_{1/2}{}^P$, (2.7) only determines the unitary operator $\mathcal{P}$ up to a phase factor. We will choose this phase factor so that

$$\mathcal{P}|\text{vac}\rangle=|\text{vac}\rangle\,. \qquad (2.9)$$

Definition 4. The theory is said to be invariant under a *time reversal* if there exists an anti-unitary operator

[10] Any non-Hermitian field can always be decomposed into a sum of two Hermitian fields. The use of Hermitian fields simplifies some of the general discussions, and there is no loss of generality.

$\mathcal{T}$ such that

$$\mathcal{T}\phi(\mathbf{r},t)\mathcal{T}^{-1}=\tilde{u}_0{}^T\phi(\mathbf{r},-t),$$
$$\mathcal{T}\psi_\mu(\mathbf{r},t)\mathcal{T}^{-1}=\tilde{u}_{1/2}{}^T[\gamma_1\gamma_2\gamma_3\psi(-\mathbf{r},t)]_\mu, \quad (2.10)$$

and

$$\mathcal{T}H\mathcal{T}^{-1}=H,$$

where in order to satisfy Eqs. (2.1)–(2.3), the matrices $u_0{}^T$ and $u_{1/2}{}^T$ must both be real and orthogonal.

Under $\mathcal{T}$, the state $|\text{vac}\rangle$ remains the same vacuum state, hence

$$\mathcal{T}|\text{vac}\rangle=e^{ia}|\text{vac}\rangle \quad (2.11)$$

where e^{ia} is a phase factor. Since $\mathcal{T}$ is an anti-unitary operator, independently of the phase factor e^{ia} we must have

$$\mathcal{T}^2|\text{vac}\rangle=|\text{vac}\rangle. \quad (2.12)$$

For any given matrices $u_0{}^T$ and $u_{1/2}{}^T$, if $\mathcal{T}$ satisfies (2.10), so does $e^{-ia}\mathcal{T}$. Thus we can always choose the phase factor so that

$$\mathcal{T}|\text{vac}\rangle=|\text{vac}\rangle. \quad (2.13)$$

[An alternative way to transform away the phase factor e^{ia} in Eq. (2.11) is to adopt $e^{\frac{1}{2}ia}|\text{vac}\rangle$ as the vacuum state.]

We note that if a unitary operator $\mathcal{P}$ satisfies the definition of the space-inversion operator, then any member in the set of unitary operators $\{S\mathcal{P}\}$ also satisfies the definition of space inversion, where S is any internal symmetry operator. Conversely, if $\mathcal{P}$ and $\mathcal{P}'$ both satisfy the definition of space inversion, then $\mathcal{P}'\mathcal{P}^{-1}$ must be a member of the internal symmetry group $\mathcal{G}$. Thus, the set $\{S\mathcal{P}\}$ contains all possible solutions of the space-inversion operators. Similarly, if an anti-unitary operator $\mathcal{T}$ satisfies the definition of the time-reversal operator, so does any member in the set $\{S\mathcal{T}\}$. The set $\{S\mathcal{T}\}$ also contains all possible time-reversal operators.

In the following, we assume that the local-field theory under consideration is invariant under the inhomogeneous proper Lorentz transformation. The CPT theorem[7] states that the theory must also be invariant under an anti-unitary operator $\mathcal{J}$ where

$$\mathcal{J}\phi(\mathbf{r},t)\mathcal{J}^{-1}=+\phi(-\mathbf{r},-t),$$
$$\mathcal{J}\psi_\mu(\mathbf{r},t)\mathcal{J}^{-1}=[i\gamma_5\psi(-\mathbf{r},-t)]_\mu, \quad (2.14)$$

and

$$\mathcal{J}H\mathcal{J}^{-1}=H. \quad (2.15)$$

The anti-unitary operator $\mathcal{J}$ is usually called the CPT operator. From (2.15), it follows that

$$\mathcal{J}|\text{vac}\rangle=e^{i\theta}|\text{vac}\rangle.$$

Consequently,

$$\mathcal{J}^2|\text{vac}\rangle=|\text{vac}\rangle. \quad (2.16)$$

The phase factor $e^{i\theta}$ can be transformed to unity by either transforming $\mathcal{J}\to e^{-i\theta}\mathcal{J}$ or by using $e^{\frac{1}{2}i\theta}|\text{vac}\rangle$ as the vacuum state. For convenience, we adopt the choice

$$\mathcal{J}|\text{vac}\rangle=|\text{vac}\rangle. \quad (2.17)$$

The operator $\mathcal{J}^2$ is unitary, and it satisfies

$$\mathcal{J}^2\phi(\mathbf{r},t)\mathcal{J}^{-2}=+\phi(\mathbf{r},t),$$
$$\mathcal{J}^2\psi(\mathbf{r},t)\mathcal{J}^{-2}=-\psi(\mathbf{r},t);$$

therefore, we may write

$$\mathcal{J}^2=(-1)^{2J}. \quad (2.18)$$

Any state of total angular momentum J is an eigenstate of $(-1)^{2J}$; the eigenvalue is $+1$ or -1 depending on whether J is an integer or a half-integer. The operator $\mathcal{J}^2$ is, by definition, an element of the internal-symmetry group $\mathcal{G}$.

By using the above definitions of $\mathcal{P}$, $\mathcal{T}$, $\mathcal{J}$ and S, one can easily verify that

$$\mathcal{J}S=S\mathcal{J}, \quad (2.19)$$
$$\mathcal{J}\mathcal{P}=(-1)^{2J}(\mathcal{P}\mathcal{J}), \quad (2.20)$$

and

$$\mathcal{J}\mathcal{T}=(-1)^{2J}(\mathcal{T}\mathcal{J}). \quad (2.21)$$

The operators $\mathcal{P}^2$, $\mathcal{T}^2$, $\mathcal{P}S\mathcal{P}^{-1}$, and $\mathcal{T}S\mathcal{T}^{-1}$ are all members of the internal-symmetry group $\mathcal{G}$, the operator $(\mathcal{P}\mathcal{J})$ belongs to the set $\{S\mathcal{T}\}$ and the operator $(\mathcal{T}\mathcal{J})$ belongs to the set $\{S\mathcal{P}\}$.

Theorem 1. If $\mathcal{P}$ exists, then $\mathcal{T}$ exists, and vice versa. Furthermore, for every given $\mathcal{P}$ there exists a $\mathcal{T}$ in the set $\{S\mathcal{T}\}$ such that

$$\mathcal{T}^2=\mathcal{P}^2, \quad (2.22)$$

and

$$\mathcal{T}\mathcal{P}=(-1)^{2J}\mathcal{P}\mathcal{T}. \quad (2.23)$$

Proof. For any given $\mathcal{P}$ in the set $\{S\mathcal{P}\}$, we can choose $\mathcal{T}=\mathcal{J}\mathcal{P}$. By using the definitions of $\mathcal{T}$, $\mathcal{P}$ and $\mathcal{J}$, we find

$$u_J{}^P=u_J{}^T, \quad (2.24)$$

where $J=0$ or $\frac{1}{2}$. Theorem 1, then, follows. So far, only the spin-0 and spin-$\frac{1}{2}$ fields are considered. As we shall see, Theorem 1 is valid for any Lorentz-invariant local-field theory.

2. Generalizations to Higher Spin Fields

All the above considerations can be readily applied to higher spin fields. Let χ_1, χ_2, $\cdots$, χ_n be n Hermitian field operators. Each χ_a describes a spin-J field. The general definitions of S, $\mathcal{P}$, $\mathcal{T}$ can be most easily obtained by noting that the formal transformation properties of the spin-J fields are the same as that of the appropriate derivatives (or the appropriate sum of derivatives) of the spin-0, or spin-$\frac{1}{2}$, fields. For example, the transformation properties of a set of n $J=1$ fields $[\chi_a(x)]_\mu$ are formally the same as those of $\partial\phi_a(x)/\partial x_\mu$ where $a=1, \cdots, n$ and ϕ_a is a spin-0 field. The definitions of S, $\mathcal{P}$, $\mathcal{T}$, and $\mathcal{J}$ are the same as before except that Eqs.

(2.4), (2.7), (2.10), and (2.14) are replaced by

$$\mathcal{S}\chi_\mu(\mathbf{r},t)\mathcal{S}^{-1}=\tilde{s}_1\chi_\mu(\mathbf{r},t)\,, \tag{2.25}$$

$$\mathcal{P}\chi_\mu(\mathbf{r},t)\mathcal{P}^{-1}=\tilde{u}_1{}^P[-\eta_{\mu\nu}\chi_\nu(-\mathbf{r},\,t)]\,, \tag{2.26}$$

$$\mathcal{T}\chi_\mu(\mathbf{r},t)\mathcal{T}^{-1}=\tilde{u}_1{}^T[\eta_{\mu\nu}\chi_\nu(\mathbf{r},\,-t)]\,, \tag{2.27}$$

$$\mathcal{g}\chi_\mu(\mathbf{r},t)\mathcal{g}^{-1}=-\chi_\mu(-\mathbf{r},\,-t)\,, \tag{2.28}$$

where s_1, $u_1{}^P$ and $u_1{}^T$ are all $(n\times n)$ real orthogonal matrices,

$$\chi_\mu=\begin{bmatrix}(\chi_1)_\mu\\ \vdots\\ (\chi_n)_\mu\end{bmatrix},$$

μ, ν vary from 0 to 3, $(\chi_a)_\mu=(\chi_a)_\mu{}^\dagger$, and $\eta_{\mu\nu}=1$ if $\mu=\nu\neq 0$, -1 if $\mu=\nu=0$, and zero otherwise. [Throughout, we adopt $x_0=t$. Sometimes μ, ν may vary from 1 to 4. In such cases, $x_4=ix_0=it$.]

All previous considerations can be directly applied to the $J=1$ case, and in a similar way to any other spin-J case. The phases of these operators $\mathcal{S}$, $\mathcal{P}$, $\mathcal{T}$, and $\mathcal{g}$ will always be chosen so that

$$\mathcal{S}|\text{vac}\rangle=\mathcal{P}|\text{vac}\rangle=\mathcal{T}|\text{vac}\rangle=\mathcal{g}|\text{vac}\rangle=|\text{vac}\rangle. \tag{2.29}$$

One can readily verify that, for any spin-J case, Eqs. (2.18)–(2.21) remain valid; and for any given $\mathcal{P}$, the general validity of Theorem 1 can be established by choosing the element $\mathcal{T}=\mathcal{g}\mathcal{P}$ in the set $\{\mathcal{S}\mathcal{T}\}$. The theory is, then, invariant under the group

$$\{\mathcal{G},\mathcal{G}\mathcal{P},\mathcal{G}\mathcal{T},\mathcal{G}\mathcal{g}\}\,. \tag{2.30}$$

The algebraic relations between $\{\mathcal{S}\mathcal{T}\}$ and $\mathcal{G}$ are completely determined by those between $\{\mathcal{S}\mathcal{P}\}$ and $\mathcal{G}$, and vice versa. Thus, the symmetry property of any Lorentz-invariant local-field theory can be studied in two steps by examining (i) the structure of the internal-symmetry group $\mathcal{G}$ and (ii) if $\mathcal{P}$ exists, the algebraic relations between $\{\mathcal{S}\mathcal{P}\}$ and $\mathcal{G}$.

We note that while $\mathcal{P}$ is not a member of $\mathcal{G}$, both $\mathcal{P}^2$ and $\mathcal{P}\mathcal{S}\mathcal{P}^{-1}$ are. The set of operators $\{\mathcal{S},\mathcal{S}\mathcal{P}\}$, therefore, form a group $\mathcal{E}$ which is a minimal group extension of $\mathcal{G}$. The mathematical problem of classifying all possible minimal extensions of a given group $\mathcal{G}$ will be discussed in the next section.

III. MINIMAL EXTENSIONS

Consider a given group $\mathcal{G}$ of finite order n. The order of any group $\mathcal{E}$ which contains $\mathcal{G}$ as a subgroup must be a multiple of n. If the order is $2n$ we call $\mathcal{E}$ a "minimal extension."

Putting it differently, $\mathcal{G}$ possesses in $\mathcal{E}$ only one coset, which is therefore a right- and left-coset at the same time; $\mathcal{G}$ is therefore an invariant or normal subgroup. This definition does not require $\mathcal{G}$ to be of finite order and is the one preferred in the following.

An obvious minimal extension is the direct product of $\mathcal{G}$ by the cyclical group Z_2, consisting of the elements 1 and ρ, with $\rho^2=1$. There are, however, in general also other "minimal extensions" and our purpose is to analyze them.

Let us designate by Latin characters

$$a,\quad b,\quad c,\quad \cdots \tag{3.1}$$

the elements of the subgroup $\mathcal{G}$. Let ρ be an arbitrarily picked element of the coset of $\mathcal{G}$. The elements of the coset are then

$$a\rho,\quad b\rho,\quad c\rho,\quad \cdots. \tag{3.2}$$

We could also write them as ρa, ρb, $\cdots$ etc. but $\rho a=a'\rho$ where, in general, $a'\neq a$. In fact, $a'=\rho a\rho^{-1}$ defines an automorphism F of the group $\mathcal{G}$; we write

$$a'\equiv Fa=\rho a\rho^{-1}. \tag{3.3}$$

In that notation, F appears as an "operator" on the elements of $\mathcal{G}$. An automorphism, of course, preserves the group laws

$$Fa\cdot Fb=F(ab);\quad Fa^{-1}=(Fa)^{-1}. \tag{3.4}$$

In addition, the square of ρ must be an element of $\mathcal{G}$, which we designate by f:

$$\rho^2=f. \tag{3.5}$$

Thus, to a given minimal extension $\mathcal{E}$, we associate an automorphism F and an element f of the group $\mathcal{G}$. However, F and f are not uniquely determined by $\mathcal{E}$, since they depend on the choice of the element ρ. If we replace ρ by another element ρ' of the set (3.2), say $\rho'=g\rho$, the automorphism F is replaced by F'

$$F'a=\rho'a\rho'^{-1}=g(\rho a\rho^{-1})g^{-1}=g(Fa)g^{-1}.$$

The transformation

$$a\rightarrow gag^{-1} \tag{3.6}$$

is an "inner automorphism" of $\mathcal{G}$, which we designate by I_g and treat as an operator: $a\rightarrow I_ga$. We then see that changing ρ to $\rho'=g\rho$ induces the transformation[11]

$$F'=I_gF. \tag{3.7}$$

Likewise, $f=\rho^2$ becomes $f'=\rho'^2=g\rho g\rho=g(\rho g\rho^{-1})\rho^2$. One has, therefore,

$$f'=g(Fg)f. \tag{3.8}$$

Obviously, the pair F', f' is associated to the extension $\mathcal{E}$ in the same sense as the pair F, f.

We now ask: given an automorphism F and an element f of the group $\mathcal{G}$, is there a minimal extension $\mathcal{E}$, to which F and f are associated in the way described? We first notice that F and f are subject to certain conditions. The transformation F^2 must be an inner automorphism:

$$F^2a=\rho(\rho a\rho^{-1})\rho^{-1}=\rho^2a\rho^{-2}=faf^{-1}.$$

[11] One says that F and F' differ only by an inner automorphism or "belong to the same class." An inner automorphism thus belongs to the same class as the "identity automorphism:" $a\rightarrow a$.

In fact, F^2 is more precisely the automorphism generated by f

$$F^2 = I_f. \tag{3.9}$$

Moreover, the element f is obviously a fixed-point of F:

$$Ff = \rho f \rho^{-1} = f \tag{3.10}$$

since $f=\rho^2$. Thus F and f cannot be chosen arbitrarily. We can show, however, that an $\mathcal{E}$ exists, if (3.9) and (3.10) are satisfied.

To begin with, the structure of $\mathcal{E}$ is completely defined once the subgroup [i.e., the set (3.1) and the corresponding multiplication table] and the associated pair F, f are given. By definition, $\mathcal{E}$ consists of the subgroup $\mathcal{G}$ and its coset (3.2), where ρ satisfies Eq. (3.5). We may regard $a\rho$, $b\rho$, $\cdots$ etc. as "symbols" for the elements of the coset. The multiplication table for $\mathcal{E}$ follows from Eqs. (3.3) and (3.5), since F is given. In fact, we have

$$a\cdot b\rho = (ab)\rho, \tag{3.11a}$$

$$a\rho\cdot b = a(\rho b\rho^{-1})\rho = (a\cdot Fb)\rho. \tag{3.11b}$$

The right-hand sides are elements of the set (3.2) unambiguously defined by the multiplication table for $\mathcal{G}$. Finally, the product of two elements of the coset is

$$a\rho\cdot b\rho = a(\rho b\rho^{-1})\rho^2 = a\cdot Fb\cdot f, \tag{3.12}$$

which is the product of three elements of set (3.1) and thus belongs to $\mathcal{G}$.

To complete the proof that $\mathcal{E}$ exists, we must show that the multiplication law thus defined is associative and that each element of $\mathcal{E}$ has an inverse. Each element a of (3.1) has, of course, an inverse a^{-1} since $\mathcal{G}$ is a group; an element $a\rho$ of (3.2) has the inverse $\bar{a}\rho$ where $\bar{a}=f^{-1}\cdot Fa^{-1}$. One easily verifies, in fact, by means of (3.12), (3.4), and (3.10), that

$$a\rho\cdot\bar{a}\rho = a\cdot F\bar{a}\cdot f = a\cdot Ff^{-1}\cdot F^2a^{-1}\cdot f = a\cdot f^{-1}\cdot I_f a^{-1}\cdot f = 1, \tag{3.13a}$$

$$\bar{a}\rho\cdot a\rho = \bar{a}\cdot Fa\cdot f = f^{-1}\cdot Fa^{-1}\cdot Fa\cdot f = 1, \tag{3.13b}$$

where "1" is the unit element in the set (3.1). We verify the associative law for the product of three elements of (3.2) and leave the other cases to the reader. We have, because of (3.12), (3.4), and (3.9)

$$(a\rho\cdot b\rho)\cdot c\rho = (a\cdot Fb\cdot f)\cdot c\rho = (a\cdot Fb\cdot f\cdot c)\rho,$$

$$a\rho\cdot(b\rho\cdot c\rho) = a\rho\cdot(b\cdot Fc\cdot f) = aF(b\cdot Fc\cdot f)\rho = (a\cdot Fb\cdot F^2c\cdot Ff)\rho,$$

$$= (a\cdot Fb\cdot f\cdot c)\rho.$$

The result is the same. That the minimal extension thus defined is associated to the pair F, f is, of course, trivally implied by the construction.

We have thus seen that a pair F, f satisfying conditions (3.9) and (3.10) defines a minimal extension. Two pairs F, f and F', f' related by the transformation (3.7), (3.8) where g is some element of (3.1) may be said to define equivalent extensions. In fact, the groups $\mathcal{E}$ and $\mathcal{E}'$ constructed according to the above procedure only differ in the choice of the element ρ in the coset (3.2).

Our problem is to classify all possible *inequivalent* extensions. To this end we may use a transformation (3.7) and (3.8) to reduce F, f to the simplest possible form. Firse we make a major distinction between case A: F is an inner automorphism of $\mathcal{G}$, or case B: F is an outer automorphism.

Case A: One can choose g in Eq. (3.7) so that F is transformed into the identity automorphism ϵ. Let us therefore assume that $F=\epsilon$ to begin with. Condition (3.10) is now satisfied by any element of the group, but (3.9) tells us that $I_f=\epsilon$, i.e., that f commutes with all elements of $\mathcal{G}$ (belongs to the center). A possible value of f is, of course, $f=1$; also if $f=f_1^2$ where f_1 is another element of the center, we can apply a transformation (3.7), (3.8) with $g=f_1^{-1}$, which leaves $F=\epsilon$ unchanged, and transforms f to $f=1$. Thus, after all transformations we may have case A_1: $F=\epsilon$, $f=1$. This means $\rho^2=1$ and ρ commutes with all elements of the group. This is the obvious solution: the direct product of $\mathcal{G}$ by the cyclic group of order 2, consisting of the elements 1 and ρ, with $\rho^2=1$.

There is also case A_2: $F=\epsilon, f=f_1^2$. In this case, there is an f_1 satisfying this equation outside the center, but not one in the center. Then by using a transformation (3.7), (3.8), with $g=f_1^{-1}$, we transform to the values

$$F=I_g, \quad f=1.$$

Here I_g is an inner automorphism generated by an element g such that g^2 is an element of the center which is different from the square of any element of the center (in particular $g^2\neq 1$). The product laws (3.11) and (3.12) can now be summarized by writing

$$a\alpha\cdot b\beta = (aF_\alpha b)\alpha\beta, \tag{3.14}$$

where α and β are two-valued symbols with the possible values 1 and ρ, $F_\alpha=\epsilon$, if $\alpha=1$, and $=F$ if $\alpha=\rho$. This solution is known as a semidirect product. Another case of semidirect product will be encountered later.

Finally, we have case A_3: $F=\epsilon$, $f\neq f_1^2$. That is, f belongs to the center; there is no "square root of f" in the group $\mathcal{G}$. Clearly the existence of Cases A_2 and A_3 depends on whether the center of $\mathcal{G}$ has elements that have a square root only outside the center, or no square root at all.

Case B: Clearly F cannot be transformed to $F=\epsilon$, but only to other members of the same class[11] of outer automorphisms. According to (3.9), F^2 must be an inner automorphism. If this is true for an element of a certain class, it is also true for every other element of the same class. In fact, if $F=I\psi$ where I is an inner automorphism, $F^2=I\psi I\psi$. But: $\psi I\psi^{-1}=I'$ which is another inner automorphism, since the inner automorphisms are an invariant subgroup of the group of all the automorphisms; therefore, $F^2=II'\psi^2=I''\psi^2$. Hence the assertion. (We have in fact proved that the squares of all elements of the same class belong to one class.) Therefore, there is not in general a solution for every class of outer automorphism.[12] We shall assume from now on that F belongs to an admissible class. In particular, there may be in the class an F such that $F^2=\epsilon$, the identity automorphism. Then, clearly, $f=1$ is a possible solution of Eqs. (3.9) and (3.10). We have then case B_1: F is an outer automorphism, such that $F^2=\epsilon$; $f=1$. This case leads to a semidirect product, with a multiplication law exactly similar to Eq. (3.14). Only the nature of F is different.

There is next case B_2: F is an outer automorphism and $F^2=\epsilon$; f satisfies (3.10), but it cannot be transformed to unity. The element f is an element of the center. Last, we have case B_3: F and f satisfy Eqs. (3.9) and (3.10). By using transformation (3.8) f cannot be transformed to any element of the center. Consequently, F^2 cannot be transformed to ϵ.

In the following sections, we will encounter several examples of cases A_1, A_2 and B_1, B_2, but not A_3 and B_3.

IV. SOME EXAMPLES

Before discussing the realistic cases, we first illustrate the results of the previous sections by considering some simple examples.

1. Example: $\mathcal{G}=U_1$

The elements of the U_1 group are $e^{i\theta}$ where θ varies from $-\pi$ to $+\pi$. The U_1 group is Abelian; therefore, it has no other inner automorphism except the identity automorphism. For problems of physical interest, we consider only the automorphism

$$e^{i\theta}\to e^{ig(\theta)}, \quad (4.1)$$

where $g(\theta)$ is a continuous function of θ. Since $e^{i\theta_1}\cdot e^{i\theta_2}=e^{i(\theta_1+\theta_2)}$, we must have

$$g(\theta_1)+g(\theta_2)=g(\theta_1+\theta_2). \quad (4.2)$$

Thus, $g(\theta)$ is a linear and homogeneous function of θ; i.e.,

$$g=K\theta. \quad (4.3)$$

In order to satisfy Eq. (3.9),

$$K^2=1. \quad (4.4)$$

The group U_1 has only one continuous outer automorphism $e^{i\theta}\to e^{-i\theta}$, and in order to satisfy Eq. (3.10) f must be $+1$ or -1.

The minimal extension $\mathcal{E}$ of U_1 falls into the following three cases:

$$A_1:\quad F=\epsilon,\quad f=1;$$
$$B_1:\quad F^2=\epsilon,\quad f=1;$$
$$B_2:\quad F^2=\epsilon,\quad f=-1.$$

To study the physical content of each of these different cases, let us consider a local-field theory whose internal-symmetry group consists *only* of U_1. To each element $e^{i\theta}$ there is an internal-symmetry operator $S(\theta)$. It is convenient to define the Hermitian operator Q by

$$Q=-i(dS/d\theta)\quad \text{at}\quad \theta=0. \quad (4.5)$$

It follows from the group property of U_1 that

$$S(\theta)=e^{iQ\theta}. \quad (4.6)$$

The operator Q commutes with the Hamiltonian H, and since $S(0)=S(2\pi)$ the eigenvalues of Q must be integers. The automorphism $Fe^{i\theta}=e^{ig}$ becomes $\mathcal{P}S(\theta)\mathcal{P}^{-1}=S(g)$. The three cases of minimal extensions can be written in the following alternative forms:

$$A_1:\quad \mathcal{P}Q-Q\mathcal{P}=0, \quad (4.7)$$
$$\mathcal{P}^2=1; \quad (4.8)$$
$$B_1:\quad \mathcal{P}Q+Q\mathcal{P}=0, \quad (4.9)$$
$$\mathcal{P}^2=1; \quad (4.10)$$
$$B_2:\quad \mathcal{P}Q+Q\mathcal{P}=0, \quad (4.11)$$
$$\mathcal{P}^2=S(\pi)=(-1)^Q. \quad (4.12)$$

Case A_1 corresponds to a theory which, in the usual language, is invariant under $\mathcal{P}=P$ and $\mathcal{T}=CT$. The theory is, by assumption, not invariant under the usual C where C, should it exist, must anticommute with Q.

The operator $\mathcal{g}^2$ must be a member of the center of the internal symmetry group $\mathcal{G}$ which is assumed to be U_1. Assuming that the theory contains some half-integer spin fields, then by using Eq. (2.18), all half-integer spin-particle states in this theory must belong to states of odd Q, and all integer spin-particle states belong to that of even Q. Otherwise, $\mathcal{G}$ cannot be U_1. Thus,

$$\mathcal{g}^2=S(\pi)=(-1)^Q=(-1)^{2J}. \quad (4.13)$$

Case B_2 corresponds to a usual theory which is invariant under $\mathcal{P}=CP$ and $\mathcal{T}=T$. For both cases A_1 and B_2, the *Hermitian* field operators associated with $Q\neq 0$ must exist in pairs, say ψ_1 and ψ_2. Under $S(\theta)$, the ψ_1 and ψ_2 transform as

$$S(\theta)\psi(x)S^{-1}(\theta)=\tilde{s}\psi(x), \quad (4.14)$$

$$\psi=\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}, \quad (4.15)$$

where

$$s=\exp(iq\tau_2\theta), \quad (4.16)$$

$$\tau_2=\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}, \quad (4.17)$$

[12] It is known that the classes may be considered as elements of a factor group A/G where A is the group of automorphisms, G the subgroup of inner automorphisms. The problem reduces, therefore, to finding, in this factor group, the classes that are "square roots of unity" or the involuntary elements of the factor group.

and $q\neq 0$ is an eigenvalue of Q. The states of $Q=q$ are degenerate with those of $Q=-q$, and this degeneracy[13] is independent of $\mathcal{P}$, or $\mathcal{T}$, invariance.

Case B_1 is an unusual case.[14] An explicit model is given in Appendix B, in which the Hermitian field operators of half-integer spins must occur in double pairs. The theory now has an additional degeneracy because of $\mathcal{P}$, or $\mathcal{T}$, invariance.

2. Example: $\mathcal{G}=\{U_2, U_2\cdot C\}$

Next, we consider another example where the internal symmetry group contains an SU_2 group whose generators are called the "isospin" $\mathbf{I}$, a U_1 group whose generator is called the "hypercharge" Y, and a discrete element called the "charge-conjugation" operator C. The operators $\mathbf{I}$, Y, and C satisfy the same commutation relations as those of the corresponding operators in the realistic case:

$$[I_i,I_j]=i\epsilon_{ijk}I_k\,, \tag{4.18}$$

$$[I_i,Y]=0\,, \tag{4.19}$$

$$[G,I]=0\,, \tag{4.20}$$

$$GY+YG=0\,, \tag{4.21}$$

and

$$G=C\exp(i\pi I_2)\,,\quad C^2=1\,, \tag{4.22}$$

where i, j, k vary from 1 to 3, $\epsilon_{ijk}=+1$, or -1, or 0 depending on whether (ijk) is an even permutation of (123), or an odd permutation, or otherwise.

The element of the U_2 group is represented by $\exp(i\mathbf{I}\cdot\boldsymbol{\theta}+iY\alpha)$ where $\boldsymbol{\theta}$ and α are real parameters. In order that it is a U_2 group and not the direct product $SU_2\times U_1$, we assume that[15] all field operators with half-integer I have odd Y; i.e.,

$$(-1)^Y=(-1)^{2I}. \tag{4.23}$$

The elements of the U_2 group and the coset $U_2C=U_2G$ form the internal symmetry group $\mathcal{G}=\{S\}=\{U_2,U_2G\}$. The group $\mathcal{G}$ is a minimal extension (Case B_1) of the U_2 group.

In order to study the minimal extension of $\mathcal{G}$ due to $\mathcal{P}$ invariance, we first establish the following Theorem:

Theorem 2. If the theory is invariant under the space inversion, then among the set $\{S\mathcal{P}\}$ there is a space inversion operator $\mathcal{P}$ which satisfies

$$[\mathcal{P},\mathbf{I}]=[\mathcal{P},Y]=[\mathcal{P},C]=0.$$

A consequence of $[\mathcal{P},\mathbf{I}]=0$ is that different members of the same isospin multiplet must have the same transformation under $\mathcal{P}$.

Proof. Let us assume that at least in the neighborhood of $\boldsymbol{\theta}=0$ and $\alpha=0$, the transformed operator $\mathcal{P}\exp(i\mathbf{I}\cdot\boldsymbol{\theta}+iY\alpha)\mathcal{P}^{-1}$ is continuous and differentiable in the parameters θ_i and α. In this neighborhood, we have

$$\mathcal{P}\exp(i\mathbf{I}\cdot\boldsymbol{\theta}+iY\alpha)\mathcal{P}^{-1}=\exp(i\mathbf{I}\cdot\boldsymbol{\theta}'+iY\alpha')\,. \tag{4.24}$$

By differentiating with respect to θ_i and α, and by setting $\boldsymbol{\theta}=0$ and $\alpha=0$, (4.24) becomes

$$\mathcal{P}I_i\mathcal{P}^{-1}=a_{ij}I_j+b_iY\,, \tag{4.25}$$

and

$$\mathcal{P}Y\mathcal{P}^{-1}=c_iI_i+dY\,, \tag{4.26}$$

where

$$a_{ij}=(\partial\theta_j'/\partial\theta_i)_0\,,\quad b=(\partial\alpha'/\partial\theta_i)_0\,,$$
$$c_i=(\partial\theta_i'/\partial\alpha)_0\,,\quad d=(\partial\alpha'/\partial\alpha)_0$$

are all real numbers, and all repeated indices are to be summed over. By using (4.25) and the commutation relation (4.18), we find

$$a_{li}a_{mj}\epsilon_{ijk}I_k=\epsilon_{lmn}(a_{nk}I_k+b_nY)\,.$$

Thus,

$$b_n=0\,,$$

and the (3×3) matrix (a_{ij}) must be real orthogonal and with determinant=1. Consequently, there exists an element $\exp(i\mathbf{I}\cdot\boldsymbol{\theta}_a)$ in $\mathcal{G}$ such that

$$\exp(i\mathbf{I}\cdot\boldsymbol{\theta}_a)I_i\exp(-i\mathbf{I}\cdot\boldsymbol{\theta}_a)=a_{ij}I_j\,.$$

By choosing $\exp(-i\mathbf{I}\cdot\boldsymbol{\theta}_a)\mathcal{P}$ as $\mathcal{P}$, we establish

$$[\mathcal{P},I_i]=0\,. \tag{4.27}$$

The commutation relation (4.19) requires that

$$c_i=0$$

in Eq. (4.26); i.e.,

$$\mathcal{P}Y\mathcal{P}^{-1}=dY\,. \tag{4.28}$$

Now, $\mathcal{P}^2$ is a member of $\mathcal{G}$. Since the square of any element in $\mathcal{G}$ commutes with Y, so must $(\mathcal{P}^2)^2$; therefore, $d^4=1$. The real constant d must be $+1$ or -1. If $d=+1$, $\mathcal{P}$ commutes with Y. If $d=-1$, we can select $G\mathcal{P}$ as the new $\mathcal{P}$. Thus,

$$[\mathcal{P},Y]=0\,. \tag{4.29}$$

The operator $\mathcal{P}G\mathcal{P}^{-1}$ anticommutes with Y, so it must belong to the set $\exp(i\mathbf{I}\cdot\boldsymbol{\theta}+iY\alpha)G$. It also commutes

---

[13] In the literature, the mass degeneracy between states of charge $Q=\pm q$ (e.g., e^+ and e^-) is often established by using invariance under $\mathcal{J}$ (i.e., the usual "CPT" operator). It is important to notice that in such a proof Q-conservation is implicitly assumed. In a Lorentz-invariant local-field theory, $\mathcal{J}$ invariance is automatically satisfied; the degeneracy between ψ_1 and ψ_2, or between states $Q=\pm q$, is insured by the invariance under the internal symmetry group U_1. Such degeneracy can be removed if one adds to the Hamiltonian an additional $\mathcal{J}$-invariant term H_1 which violates the U_1 invariance (i.e., Q conservation). As an example, we may mention the well-known case of K_1^0 and K_2^0 which are degenerate if $H_{\rm wk}=0$, and, therefore, the strangeness S is conserved. $H_{\rm wk}$ does not alter the $\mathcal{J}$ invariance, but it violates S conservation. As a result, K_1^0 and K_2^0 do have different masses.

[14] If the theory contains only integer spin fields, then instead of (4.13) and (4.31) we have $\mathcal{J}^2=1$. In example 1, case B_1 corresponds to a usual theory which is invariant under $\mathcal{P}$="CP" and $\mathcal{T}$="T," while case B_2 corresponds to an unusual theory. Similarly, if there are only integer spin fields in the theory, then in example 2, case A_1 corresponds to a usual "C," "P," "T" separately invariant theory, and case A_2 corresponds to an unusual theory.

[15] Relations such as these have been emphasized by L. Michel, Ref. 6.

with **I**. By using Eq. (4.18), we find

$$\mathcal{P}G\mathcal{P}^{-1}=e^{iY\alpha}G. \tag{4.30}$$

We may choose $e^{-\frac{1}{2}iY\alpha}\mathcal{P}$ as the new $\mathcal{P}$, and this new $\mathcal{P}$ commutes with G. The theorem is, then, proved.

Since $\mathcal{P}$ commutes with all elements in $\mathcal{G}$, the automorphism $\mathcal{P}S\mathcal{P}^{-1}$ corresponds to $F=\epsilon$. The center of the group $\mathcal{G}$ consists of two elements: 1 and $(-1)^Y=(-1)^{2I}$. Thus, according to the discussion given in Sec. III, the minimal extension can only be one of the two cases:

case A_1

$$\mathcal{P}^2=1,$$

and

case A_2

$$\mathcal{P}^2=(-1)^Y=(-1)^{2I}.$$

The element $\mathcal{J}^2$ belongs to the center of the internal-symmetry group, where $\mathcal{J}$ is given by (2.14). Hence, it is either 1 or $(-1)^Y$. If the theory contains some half-integer spin fields, then we must have

$$\mathcal{J}^2=(-1)^{2J}=(-1)^Y; \tag{4.31}$$

i.e., all half-integer spin particles are of odd Y and all integer spin particles are of even Y. We may define

$$P=\mathcal{P}\exp(i\tfrac{1}{2}\pi Y); \tag{4.32}$$

P satisfies

$$[P,\mathbf{I}]=[P,Y]=0, \tag{4.33}$$

and

$$PC=(-1)^{2J}CP. \tag{4.34}$$

For case A_1,

$$P^2=(-1)^Y \tag{4.35}$$

and for case A_2

$$P^2=1. \tag{4.36}$$

Thus, case A_2 corresponds to a usual theory with separate C, P and T invariances. Case A_1 corresponds to an unusual theory[14] which has an additional degeneracy due to $\mathcal{P}$ or $\mathcal{T}$ invariance, somewhat like the case B_1 of the preceding example. An explicit model of such a case A_1 is given in Appendix B.

3. Remarks

1. Both case B_1 of example 1 and case A_1 of example 2 are theories in which $\mathcal{P}$ (or $\mathcal{T}$) invariance implies additional degeneracies. At present, there does not exist in nature any known example of such degeneracies. However, since $\mathcal{P}$ and $\mathcal{T}$ invariances are approximate symmetries, such degeneracy, if it exists, must also be only approximate. It is difficult to predict whether such approximate degeneracies might find some applications in the future.

2. In physical problems, one often deals with the U_n and SU_n groups. The above examples are concerned only with $n=1$ and 2. To study the general case of arbitrary n, it is useful to know the following mathematical theorem[16]:

The group of all $(n\times n)$ unitary matrices u with determinant 1 is the SU_n group. If $u\rightarrow F(u)$ is an automorphism of SU_n and $n\geqq 3$, then there exists a v in SU_n such that for every u in SU_n

$$F(u)=vuv^{-1},$$

or, for every u

$$F(u)=vu^*v^{-1}$$

where u^* is the complex conjugate of u.

We will not discuss the detailed analysis of the minimal extensions of these groups. Such analysis is greatly simplified by using the above theorem. For example, consider a theory which satisfies $\mathcal{P}$ invariance and whose internal-symmetry group $\mathcal{G}$ is the SU_n group ($n\geqq 3$). Let the (n^2-1) generators of the SU_n group be $F_1, F_2, \cdots, F_{n^2-1}$. The above theorem shows that there must exist an element $\mathcal{P}$ in the coset $\mathcal{G}\mathcal{P}$ such that either

$$[\mathcal{P},F_i]=0, \tag{4.37}$$

for all $i=1, 2, \cdots, (n^2-1)$, or

$$\mathcal{P}F_i\mathcal{P}^{-1}=-F_i^*, \tag{4.38}$$

for all i.

Similar conclusions can be obtained if the internal symmetry group $\mathcal{G}$ is the U_n group. Let $F_1, F_2, \cdots, F_{n^2}$ be the n^2 generators of the U_n group where $F_1, F_2, \cdots, F_{n^2-1}$ satisfy the same algebraic relations as those of the SU_n group, and F_{n^2} commutes with all other F_i. If the theory satisfies $\mathcal{P}$ invariance, then there exists an element $\mathcal{P}$ in the coset $\mathcal{G}\mathcal{P}$ such that either (4.37) holds for all $i=1, 2, \cdots, (n^2-1)$, or (4.38) holds; in addition, $\mathcal{P}$ either commutes, or anticommutes, with F_{n^2}. Thus, $\mathcal{P}^2$ must belong to the center of the internal symmetry group. The detailed classification of the theory into the different cases of minimal extensions depends on the element $\mathcal{P}^2$ and the commutation relations between $\mathcal{P}$ and the various generators. Furthermore, depending on the relation between $\mathcal{P}^2$ and $\mathcal{J}^2=(-1)^{2J}$, the theory may, or may not, contain some unusual degeneracies similar to those in the above case B_1 of example 1 and case A_1 of example 2.

In a similar way and by following the arguments used in example 2, we can readily extend such analysis to the case in which the internal symmetry group $\mathcal{G}$ is itself a minimal extension of either the SU_n or the U_n group (due to the presence of other discrete internal-symmetry operators, such as the charge-conjugation operator).

[16] In proving this theorem, one does not have to assume that the automorphism preserves continuity. We wish to thank Professor E. Kolchin (private communication) for showing us the proof.

V. WEAK INTERACTIONS AND SU_3-INVARIANT STRONG INTERACTIONS

In this section, we consider the following Hamiltonian H:

$$H=H_{\text{free}}+H_{\text{st}}(SU_3)+H_{\text{wk}}, \tag{5.1}$$

where H_{wk} is the usual weak interaction, $H_{\text{st}}(SU_3)$ is the SU_3 invariant part[17] of the usual strong interaction and H_{free} is the free-particle Hamiltonian in which the masses of the different hadrons within the same SU_3 multiplet are considered to be the same, and the masses of all leptons are set to be zero; i.e.,

$$m_e=m_\mu=m_{\nu_e}=m_{\nu_\mu}=0. \tag{5.2}$$

The H_{wk} can be separated into three parts:

$$H_{\text{wk}}=H_{ll}+H_{lh}+H_{hh}, \tag{5.3}$$

where H_{ll}, H_{hh} and H_{lh} describe respectively the pure leptonic weak interaction, the pure hadronic weak interaction, and the weak interaction between leptons and hadrons. Among these, only the leptonic part is reasonably well understood:

$$H_{ll}=(G_\mu/\sqrt{2})\int j_\lambda^* j_\lambda d^3r, \tag{5.4}$$

and

$$H_{lh}=\int [J_\lambda^* j_\lambda+J_\lambda j_\lambda^*]d^3r, \tag{5.5}$$

where G_μ is real and denotes the usual μ-decay coupling constant,

$$j_\lambda=i\psi_e^\dagger\gamma_4\gamma_\lambda(1+\gamma_5)\psi_{\nu_e}+i\psi_\mu^\dagger\gamma_4\gamma_\lambda(1+\gamma_5)\psi_{\nu_\mu}, \tag{5.6}$$

$$j_\lambda^*=i\psi_{\nu_e}^\dagger\gamma_4\gamma_\lambda(1+\gamma_5)\psi_e+i\psi_{\nu_\mu}^\dagger\gamma_4\gamma_\lambda(1+\gamma_5)\psi_\mu, \tag{5.7}$$

ψ_e, ψ_μ, ψ_{ν_e}, and ψ_{ν_μ} are the field operators for e^-, μ^-, ν_e, and ν_μ, J_λ is the hadron current, $J_\lambda^*=J_\lambda^\dagger$ if $\lambda\neq 4$ and $J_4^*=-J_4^\dagger$. Apart from some general selection rules and certain matrix elements, the detailed forms of the hadron-dependent parts, J_λ and H_{hh}, are not known.

For clarity, a simplified model of the hadrons will be considered first. The symmetry properties derived for the simple model can be applied to the realistic case, and it leads to some consequences which may be tested experimentally.

1. A Simple Model

In this model, the hadrons are assumed to consist of only three *hypothetical* spin-$\frac{1}{2}$ fields ψ_1, ψ_2 and ψ_3 which correspond to p, n and Λ^0 in the Sakata model.[18] The ψ_1, ψ_2 and ψ_3 are of equal, but nonzero, masses; their strong interactions are invariant under the U_3 group of transformations between these three fields. For the weak interaction, we will consider only H_{ll} and H_{lh}; i.e.,

$$H_{\text{wk}}=H_{ll}+H_{lh}. \tag{5.8}$$

Furthermore, the hadron current in H_{lh} is assumed to be

$$J_\lambda=i2^{-1/2}G_\mu\psi_b^\dagger\gamma_4\gamma_\lambda(1+\gamma_5)\psi_a \tag{5.9}$$

where

$$\psi_a=\psi_1, \tag{5.10}$$

$$\psi_b=(\cos\theta)\psi_2+(\sin\theta)\psi_3, \tag{5.11}$$

and θ is the equivalent of the Cabibbo angle in this model.[19] It is useful to introduce

$$\psi_c=-(\sin\theta)\psi_2+(\cos\theta)\psi_3. \tag{5.12}$$

The fields ψ_a, ψ_b and ψ_c are all of unit baryon number, $N=1$; their charges are, respectively, $+e$, 0 and 0.

For the leptons, since only the zero-mass limit will be considered, we may require

$$\gamma_5\psi_\alpha=\psi_\alpha \tag{5.13}$$

where $\alpha=e$, μ, ν_e and ν_μ. Each ψ_α describes a two-component field. The Hamiltonian H is given by Eq. (5.1). In this model, H depends only on 4 zero-mass two-component fields ψ_e, ψ_μ, ψ_{ν_e}, ψ_{ν_μ} and 3 four-component hadron fields ψ_a, ψ_b and ψ_c. It is clear that H commutes with the charge Q, the baryon number N and the two usual lepton numbers L_e and L_μ where the eigenvalues of L_e and L_μ are given by

$$\begin{aligned} L_l&=-1 \quad \text{for} \quad l^-, \nu_l, \\ &=+1 \quad \text{for} \quad l^+, \bar\nu_l, \\ &=0 \quad\;\; \text{for all other particles}, \end{aligned} \tag{5.14}$$

and $l=e$ or μ; H also commutes with another Hermitian operator S' whose eigenvalues are given by

$$\begin{aligned} S'&=-1 \quad \text{for} \quad c, \\ &=1 \quad\;\; \text{for} \quad \bar c, \\ &=0 \quad\;\; \text{for all other particles}, \end{aligned} \tag{5.15}$$

and c, $\bar c$ refer to the particles described, respectively, by ψ_c and $\psi_c^\dagger$. For a general discussion of the symmetry properties, it is convenient to use, in place of Q, L_e, and L_μ, three alternative Hermitian operators L, M, and T_3:

$$L=L_e+L_\mu, \tag{5.16}$$

$$T_3=\tfrac{1}{2}(L_\mu-L_e), \tag{5.17}$$

and

$$Q=\tfrac{1}{2}[L+M+N+S']. \tag{5.18}$$

The operators L, M, N, S', T_3 mutually commute.

Theorem 3. The internal symmetry group $\mathcal{G}_{\text{wk}}$ consists of a connected subgroup $\mathcal{G}_{\text{wk}}^0=U_2\times U_1\times U_1\times U_1$ and a coset $\mathcal{G}_{\text{wk}}^0 C_{\text{wk}}$:

$$\mathcal{G}_{\text{wk}}=\{\mathcal{G}_{\text{wk}}^0, \mathcal{G}_{\text{wk}}^0 C_{\text{wk}}\} \tag{5.19}$$

---

[17] M. Gell-Mann, Phys. Rev. **125**, 1067 (1962); Y. Ne'eman, Nucl. Phys. **26**, 222 (1961).

[18] Z. Maki M. Nakagawa, Y. Ohnuki, and S. Sakata, Progr. Theoret. Phys. (Kyoto) **23**, 1174 (1960).

[19] N. Cabibbo, Phys. Rev. Letters **10**, 531 (1963).

where the element of $\mathcal{G}_{wk}{}^0$ is given by

$$\exp[i(\mathbf{T}\cdot\boldsymbol{\theta}+L\xi+M\eta+N\zeta+S'\phi)],$$

$\boldsymbol{\theta}$, ξ, η, ζ, ϕ are all real parameters. The element C_{wk} and the generator $\mathbf{T}$ satisfy the following algebraic relations:

$$C_{wk}M+MC_{wk}=0, \tag{5.20}$$

$$[C_{wk},\mathbf{T}]=[C_{wk},L]=[C_{wk},N]=[C_{wk},S']=0, \tag{5.21}$$

$$[T_i,T_j]=i\epsilon_{ijk}T_k, \tag{5.22}$$

$$C_{wk}{}^2=1, \tag{5.23}$$

and $\mathbf{T}$ commutes with L, M, N, and S'.

The group $\mathcal{G}_{wk}$ is a minimal extension of $\mathcal{G}_{wk}{}^0$, and it belongs to case B_1. Table I lists the eigenvalues $\mathbf{T}^2=T(T+1)$, T_3, L, M, N, S' for the various particles in the model.

TABLE I. Quantum numbers for the various particles in the simple model. The field operators for the particles a, b, c are given by Eqs. (5.10)–(5.12).

Particles	T	T_3	L	M	N	S'
e^-	$\frac{1}{2}$	$\frac{1}{2}$	-1	-1	0	0
μ^-	$\frac{1}{2}$	$-\frac{1}{2}$	-1	-1	0	0
ν_e	$\frac{1}{2}$	$\frac{1}{2}$	-1	1	0	0
ν_μ	$\frac{1}{2}$	$-\frac{1}{2}$	-1	1	0	0
a	0	0	0	1	1	0
b	0	0	0	-1	1	0
c	0	0	0	0	1	-1

Proof. Eq. (5.13) and the anticommutation relations between the lepton fields require that any internal symmetry operator S must satisfy

$$S\psi S^{-1}=u\psi, \tag{5.24}$$

where

$$\psi=\begin{bmatrix}\psi_e\\ \psi_\mu\\ \psi_{\nu_e}\\ \psi_{\nu_\mu}\end{bmatrix},$$

and u is a (4×4) *unitary* matrix. It is convenient to introduce the matrices $\boldsymbol{\sigma}$ and $\boldsymbol{\rho}$:

$$\rho_1=\begin{bmatrix}0&0&1&0\\0&0&0&1\\1&0&0&0\\0&1&0&0\end{bmatrix},\quad \rho_2=\begin{bmatrix}0&0&-i&0\\0&0&0&-i\\i&0&0&0\\0&i&0&0\end{bmatrix},$$

$$\sigma_1=\begin{bmatrix}0&1&0&0\\1&0&0&0\\0&0&0&1\\0&0&1&0\end{bmatrix},\quad \sigma_2=\begin{bmatrix}0&-i&0&0\\i&0&0&0\\0&0&0&-i\\0&0&i&0\end{bmatrix},$$

and

$$\rho_3=-i\rho_1\rho_2,\quad \sigma_3=-i\sigma_1\sigma_2. \tag{5.25}$$

In terms of ψ and these matrices, the lepton currents can be written as

$$j_\lambda=i\psi^\dagger\gamma_4\gamma_\lambda(\rho_1+i\rho_2)\psi, \tag{5.26}$$

and

$$j_\lambda{}^*=i\psi^\dagger\gamma_4\gamma_\lambda(\rho_1-i\rho_2)\psi. \tag{5.27}$$

Since H_{lh} is a linear function of j_λ and $j_\lambda{}^*$, the invariance property $SH_{lh}S^{-1}=H_{lh}$ implies that j_λ must become a linear combination of j_λ and $j_\lambda{}^*$ under S; i.e., under the transformation (5.24),

$$u^\dagger\rho_1u=A_{11}\rho_1+A_{12}\rho_2, \tag{5.28}$$

and

$$u^\dagger\rho_2u=A_{21}\rho_1+A_{22}\rho_2. \tag{5.29}$$

The coefficients A_{ij} form a (2×2) matrix A. The Hermiticity conditions of ρ_1 and ρ_2 require A to be real, and their commutation relations require A to be orthogonal. Thus, A must be of the form

$$\begin{pmatrix}\cos\eta & \sin\eta\\ \mp\sin\eta & \pm\cos\eta\end{pmatrix}, \tag{5.30}$$

where η is real. Let us define u':

$$u'=u\exp(-\tfrac{1}{2}i\eta\rho_3)$$

if $\det A=+1$, and

$$u'=u[\exp(-\tfrac{1}{2}i\eta\rho_3)]\rho_1$$

if $\det A=-1$; then,

$$u'^\dagger\rho_1u'=\rho_1,$$

and

$$u'^\dagger\rho_2u'=\rho_2. \tag{5.31}$$

All (4×4) unitary matrices can be written as a linear function of the sixteen (4×4) matrices: 1, ρ_i, σ_j, and $\rho_i\sigma_j$. In order to satisfy (5.31), u' must be a linear function of only 1 and σ_j; consequently, u is of the form

$$u=\exp[i(\tfrac{1}{2}\boldsymbol{\theta}\cdot\boldsymbol{\sigma}+\xi+\eta\rho_3)], \tag{5.32}$$

or

$$u=\rho_1\exp[i(\tfrac{1}{2}\boldsymbol{\theta}\cdot\boldsymbol{\sigma}+\xi+\eta\rho_3)]. \tag{5.33}$$

Under the transformation (5.32), $j_\lambda\to e^{-2i\eta}j_\lambda$ and $j_\lambda{}^*\to e^{2i\eta}j_\lambda{}^*$; under (5.33), $j_\lambda\to e^{2i\eta}j_\lambda{}^*$ and $j_\lambda{}^*\to e^{-2i\eta}j_\lambda$.

For the hadron fields, it can be shown that, for the present case, we need only consider the transformations

$$S\begin{pmatrix}\psi_a\\ \psi_b\end{pmatrix}S^{-1}=v\begin{pmatrix}\psi_a\\ \psi_b\end{pmatrix}$$

and

$$S\psi_cS^{-1}=e^{i\phi}\psi_c. \tag{5.34}$$

In order that the current J_λ is transformed into a linear combination of J_λ and $J_\lambda{}^*$, the (2×2) matrix v must be of the form

$$v=\exp[-i(\eta\tau_3+\zeta)] \tag{5.35}$$

or

$$v=\tau_1\exp[-i(\eta\tau_3+\zeta)], \tag{5.36}$$

where

$$\tau_1=\begin{pmatrix}0&1\\1&0\end{pmatrix}\quad\text{and}\quad \tau_3=\begin{pmatrix}1&0\\0&-1\end{pmatrix}. \tag{5.37}$$

Under the transformation (5.35), $J_\lambda \to e^{-2i\eta}J_\lambda$ and $J_\lambda^* \to e^{2i\eta}J_\lambda^*$; under (5.36), $J_\lambda \to e^{2i\eta}J_\lambda^*$ and $J_\lambda^* \to e^{-2i\eta}J_\lambda$. Theorem 3 is, then, proved.

The discrete element $C_{\rm wk}$ is identified with the transformation $u=\rho_1$, $v=\tau_1$ and $\phi=0$ in (5.24) and (5.34). Under $C_{\rm wk}$

$$C_{\rm wk}j_\lambda C_{\rm wk}^{-1}=j_\lambda^*,\qquad C_{\rm wk}J_\lambda C_{\rm wk}^{-1}=J_\lambda^*. \tag{5.38}$$

While the symmetry group $\mathcal{G}_{\rm wk}$ has been discussed in the literature,[20,21] Theorem 3 establishes that, at least for this simple model, Eq. (5.19) gives all possible internal-symmetry operators.

Next, we investigate the question of $\mathcal{P}$ and $\mathcal{T}$ invariance for the model. Let us define

$$\mathcal{P}[\psi_n(\mathbf{r},t)]_\alpha\mathcal{P}^{-1}=(i\gamma_4)_{\alpha\beta}[\psi_n^\dagger(-\mathbf{r},t)]_\beta, \tag{5.39}$$

$$\mathcal{T}[\psi_n(\mathbf{r},t)]_\alpha\mathcal{T}^{-1}=(\gamma_1\gamma_2\gamma_3)_{\alpha\beta}[\psi_n(\mathbf{r},-t)]_\beta, \tag{5.40}$$

where n denotes the various particles e, μ, ν_e, ν_μ, a, b and c; the subscripts α and β denote the spinor indices which vary from 1 to 4. The γ_μ matrices are in the Majorana representation (2.8). Applying $\mathcal{P}$ and $\mathcal{T}$ to j_λ and J_λ, one finds

$$\mathcal{P}j_\lambda(\mathbf{r},t)\mathcal{P}^{-1}=j_\lambda^\dagger(-\mathbf{r},t),$$
$$\mathcal{P}J_\lambda(\mathbf{r},t)\mathcal{P}^{-1}=J_\lambda^\dagger(-\mathbf{r},t),$$
$$\mathcal{T}j_\lambda(\mathbf{r},t)\mathcal{T}^{-1}=-j_\lambda(\mathbf{r},-t),$$

and

$$\mathcal{T}J_\lambda(\mathbf{r},t)\mathcal{T}^{-1}=-J_\lambda(\mathbf{r},-t). \tag{5.41}$$

The Hamiltonian H is, therefore, invariant under $\mathcal{P}$ and $\mathcal{T}$. (In this model, $H_{\rm st}$ is assumed to be invariant under $\mathcal{P}$ and $\mathcal{T}$.)

In the above discussion, because of familiarity we use the field operators ψ_e, ψ_μ, $\cdots$ which are all non-Hermitian. Identical results can be derived if, instead, the appropriate Hermitian operators are used. Such a treatment would conform more to the general notations used in Sec. II; the details are given in Appendix A.

It is convenient to introduce

$$P_{\rm wk}\equiv C_{\rm wk}^{-1}e^{-i\pi T_2}\mathcal{P}, \tag{5.42}$$

and

$$T_{\rm wk}\equiv\mathcal{T}e^{i\pi T_2}. \tag{5.43}$$

Thus,

$$[P_{\rm wk},\mathbf{T}]=[P_{\rm wk},M]=0, \tag{5.44}$$

$$P_{\rm wk}L+LP_{\rm wk}=P_{\rm wk}N+NP_{\rm wk}=P_{\rm wk}S'+S'P_{\rm wk}=0, \tag{5.45}$$

$$P_{\rm wk}^2=(-1)^{N+S'}, \tag{5.46}$$

and

$$C_{\rm wk}P_{\rm wk}T_{\rm wk}=\mathcal{J} \tag{5.47}$$

where $\mathcal{J}$ is given by Eq. (2.14).

[20] G. Feinberg and F. Gürsey, Phys. Rev. **128**, 378 (1962).

[21] T. D. Lee, Nuovo Cimento **35**, 954 (1965). See also Ref. 9.

The minimal extension of the internal symmetry group $\mathcal{G}_{\rm wk}$ is

$$\mathcal{E}=\{\mathcal{G}_{\rm wk},\mathcal{G}_{\rm wk}\mathcal{P}\}=\{\mathcal{G}_{\rm wk},\mathcal{G}_{\rm wk}P_{\rm wk}\} \tag{5.48}$$

which belongs to case B_2.

2. Applications of $C_{\rm wk}$, $P_{\rm wk}$ and $T_{\rm wk}$ Symmetries

The above $C_{\rm wk}$, $P_{\rm wk}$ and $T_{\rm wk}$ symmetries can be readily applied to the Hamiltonian (5.1) for the general case, without the triplet model of the hadrons and without the special assumptions (5.8) and (5.9). In place of Eq. (5.9), the hadron current J_μ is assumed to transform like an octet member[19] under the SU_3 transformations:

$$J_\lambda=\cos\theta(O_\lambda)_1{}^2+\sin\theta(O_\lambda)_1{}^3, \tag{5.49}$$

where $(O_\lambda)_1{}^2$ and $(O_\lambda)_1{}^3$ transform, respectively, like π^- and K^- under the usual SU_3 group of transformations. From Eq. (5.49), it follows that J_λ^* must also transform like an octet member under the SU_3 transformations. Without any further assumption, J_λ^* might belong to a *different* octet from J_λ, and their matrix elements would not be connected by any SU_3 transformation. The requirement of $C_{\rm wk}$ symmetry, however, links J_λ with J_λ^*.

To illustrate this, let us consider the matrix elements of J_λ between two single-baryon states. Such matrix elements can be reduced to the usual D_μ and F_μ by using the following relation:

$$\langle B_l{}^i(p')|(O_\lambda)_m{}^j|B_n{}^k(p)\rangle=d_{lmn}{}^{ijk}U'^\dagger D_\lambda U+f_{lmn}{}^{ijk}U'^\dagger F_\lambda U, \tag{5.50}$$

where all indices vary from 1 to 3, $|B_n{}^k(p)\rangle$ is the (k,n)th member of a baryon octet with 4-momentum p_μ, $\langle B_l{}^i(p')|$ is the Hermitian conjugate of $|B_l{}^l(p')\rangle$,

$$d_{lmn}{}^{ijk}=(4/9)\delta_l{}^i\delta_m{}^j\delta_n{}^k+\delta_n{}^i\delta_l{}^j\delta_m{}^k+\delta_m{}^i\delta_n{}^j\delta_l{}^k-\tfrac{2}{3}(\delta_l{}^i\delta_n{}^j\delta_m{}^k+\delta_n{}^i\delta_m{}^j\delta_l{}^k+\delta_m{}^i\delta_l{}^j\delta_n{}^k), \tag{5.51}$$

$$f_{lmn}{}^{ijk}=\delta_n{}^i\delta_l{}^j\delta_m{}^k-\delta_m{}^i\delta_n{}^j\delta_l{}^k, \tag{5.52}$$

$\delta_j{}^i$ is the Kronecker δ symbol, U and U' are, respectively, the appropriate free Dirac spinors with 4-momentum p and p'. From Lorentz invariance, we find

$$D_\lambda=i\gamma_4[\gamma_\lambda D_1{}^V+i(p'+p)_\lambda D_2{}^V+i(p'-p)_\lambda D_3{}^V]+i\gamma_4\gamma_5[\gamma_\lambda D_1{}^A+i(p'-p)_\lambda D_2{}^A+i(p'+p)_\lambda D_3{}^A], \tag{5.53}$$

and

$$F_\lambda=i\gamma_4[\gamma_\lambda F_1{}^V+i(p'+p)_\lambda F_2{}^V+i(p'-p)_\lambda F_3{}^V]+i\gamma_4\gamma_5[\gamma_\lambda F_1{}^A+i(p'-p)_\lambda F_2{}^A+i(p'+p)_\lambda F_3{}^A], \tag{5.54}$$

where the functions $D_i{}^V$, $D_i{}^A$, $F_i{}^V$, $F_i{}^A$ $(i=1,\cdots 3)$ depend only on $q^2=(p'-p)^2$. The octet current hypothesis (5.49) is satisfied for arbitrary twelve *complex* functions $D_i{}^V$, $\cdots$, $F_i{}^A$.

By using Eq. (5.38), the C_{wk} symmetry requires that

$$D_1{}^\alpha,\quad D_2{}^\alpha,\quad F_1{}^\alpha,\quad F_2{}^\alpha \quad \text{are real},$$

and

$$D_3{}^\alpha,\quad F_3{}^\alpha \quad \text{are imaginary}, \tag{5.55}$$

where $\alpha = V$, or A. Expression (5.55), at least in principle, can be tested experimentally.

In order that C_{wk} commute with $H_{\text{st}}(SU_3)$, we may identify the transformation of hadrons under C_{wk} to be the same as that under the SU_3 group element

$$\begin{bmatrix} 0 & \cos\theta & \sin\theta \\ \cos\theta & -\sin^2\theta & \sin\theta\cos\theta \\ \sin\theta & \sin\theta\cos\theta & -\cos^2\theta \end{bmatrix}, \tag{5.56}$$

where θ is the Cabibbo angle. Thus, if C_{wk} symmetry holds, by applying (5.56) to (5.49), one finds

$$J_\lambda{}^* = \cos\theta (O_\lambda)_2{}^1 + \sin\theta (O_\lambda)_3{}^1. \tag{5.57}$$

Conversely, if Eq. (5.57) holds, then C_{wk} symmetry follows and the condition (5.55) is valid.

We note that in Eq. (5.57) the strangeness-conserving part $(O_\mu)_2{}^1$ of the hadron current satisfies the charge symmetry condition; i.e., $(O_\mu)_2{}^1$ is related to $(O_\mu)_1{}^2$ by a 180° isospin rotation. Consequences of the charge-symmetry condition have been discussed in the literature.[22]

The C_{wk} symmetry can also be applied to the pure hadronic weak interaction H_{hh}. For example, if H_{hh} transforms like an octet $\mathcal{H}_j{}^i$ under SU_3, then C_{wk} symmetry requires H_{hh} to be of the form

$$(\cos^2\theta)\mathcal{H}_3{}^3 + (\sin^2\theta)\mathcal{H}_2{}^2 - \sin\theta\cos\theta(\mathcal{H}_3{}^2 + \mathcal{H}_2{}^3), \tag{5.58}$$

which connects the strangeness-nonconserving part of H_{hh} to the strangeness-conserving part. [In (5.58), the trace $\mathcal{H}_i{}^i$ is taken to be zero.] Similar results can also be obtained if different SU_3 transformation properties of H_{hh} are assumed.

The C_{wk} symmetry is broken by the mass difference between l^- and ν_l, by the SU_3-violating strong interaction and by the electromagnetic interaction.

At present, it remains an open question whether the weak interaction does, or does not, satisfy the $\mathcal{T}$ invariance.[23] If we assume that it does satisfy $\mathcal{T}$ invariance, then Theorem 1 states that it also satisfies $\mathcal{P}$ invariance, and we may define P_{wk} and T_{wk} by using Eqs. (5.39)–(5.43).

A consequence of $\mathcal{T}$ invariance is that the 12 form factors $D_i{}^V, \cdots, F_i{}^A$ in Eqs. (5.53) and (5.54) must all be real. Thus, if both $\mathcal{T}$ invariance and C_{wk} symmetry hold, we must have

$$D_3{}^V = D_3{}^A = F_3{}^V = F_3{}^A = 0. \tag{5.59}$$

[22] See T. D. Lee and C. S. Wu, Ann. Rev. Nucl. Sci. **15**, 408 (1965).

[23] See, e.g., the discussions given by T. D. Lee, in *Proceedings of the Oxford International Conference on Elementary Particles, 1965* (Rutherford High Energy Laboratory, Harwell, England, 1966).

VI. APPLICATIONS TO OTHER INTERACTIONS

1. Electromagnetic Interaction

Let us consider the model Hamiltonian

$$H = H_{\text{free}} + H_\gamma, \tag{6.1}$$

where H_γ is the electromagnetic interaction and H_{free} is the free Hamiltonian of all charged particles and the electromagnetic field. For simplicity, we assume that H_γ is given by the minimal electromagnetic interaction[24] of spin-0 and spin-$\frac{1}{2}$ charged particles only; the masses of these charged particles are all assumed to be *different* and nonzero. Each charged particle is described by two Hermitian fields, say, $\psi_{\alpha,1}$ and $\psi_{\alpha,2}$ where $\alpha = 1, \cdots, N$ labels the different particles. The explicit form of (6.1) is well known; its symmetry properties will be summarized in the following.

It can be readily verified that the internal symmetry group $\mathcal{G}_\gamma$ of (6.1) must contain a subgroup

$$\mathcal{G}_\gamma{}^0 = U_1 \times U_1 \times \cdots \times U_1. \tag{6.2}$$

For each charged particle α there is a group $U_1 = \{S_\alpha(\theta)\}$:

$$S_\alpha(\theta)\begin{pmatrix}\psi_{\beta,1}\\ \psi_{\beta,2}\end{pmatrix} S_\alpha{}^{-1}(\theta) = \exp(i\delta_{\alpha\beta}\tau_2\theta)\begin{pmatrix}\psi_{\beta,1}\\ \psi_{\beta,2}\end{pmatrix}, \tag{6.3}$$

where τ_2 is given by Eq. (4.17) and $\delta_{\alpha\beta}$ is the Kronecker δ symbol. The group $\mathcal{G}_\gamma{}^0$ is the direct product of these U_1 groups. Let Q_α be the generator of the group $\{S_\alpha(\theta)\}$. The total charge of the system is (in units of e)

$$Q = \sum Q_\alpha.$$

The group U_1 generated by Q insures that states with $Q = +q$ and $-q$ are degenerate.[13]

The Hamiltonian (6.1) conserves not only the total charge, but also each Q_α separately; it is also invariant under the usual *charge-conjugation* operator C_γ where

$$C_\gamma Q_\alpha + Q_\alpha C_\gamma = 0 \tag{6.4}$$

for all α, and

$$C_\gamma{}^2 = 1. \tag{6.5}$$

The internal symmetry group $\mathcal{G}_\gamma$ is a minimal extension of $\mathcal{G}_\gamma{}^0$:

$$\mathcal{G}_\gamma = \{\mathcal{G}_\gamma{}^0, \mathcal{G}_\gamma{}^0 C_\gamma\} \tag{6.6}$$

and the extension belongs to case B_1.

The minimal electromagnetic interactions of spin-0 and spin-$\frac{1}{2}$ particles are well known to satisfy both $\mathcal{P}$ and $\mathcal{T}$ invariances. Furthermore, there exists an element $\mathcal{P}$ in the coset $\mathcal{G}_\gamma\mathcal{P}$ which satisfies (2.7) with

[24] The minimal electromagnetic interaction of a charged particle can be generated by replacing $(\partial/\partial x_\mu) \to (\partial/\partial x_\mu) - ieA_\mu$ in its free Lagrangian, where e is its charge. For the spin-0 or spin-$\frac{1}{2}$ particles, such a minimal electromagnetic interaction is always invariant under C_γ, P_γ, and T_γ. But, this is not true for the spin-1 particles. See T. D. Lee, Phys. Rev. **140**, B967 (1965).

$u_0{}^P = u_{1/2}{}^P = 1$. Consequently, $\mathcal{P}$ also satisfies

$$[\mathcal{P},Q_i]=[\mathcal{P},C_\gamma]=0\,, \tag{6.7}$$

and

$$\mathcal{P}^2=(-1)^{2J}. \tag{6.8}$$

Thus, the minimal extension

$$\mathcal{E}=\{\mathcal{G}_\gamma,\mathcal{G}_\gamma\mathcal{P}\} \tag{6.9}$$

belongs to case A_2.

We note that $\mathcal{P}$ commutes with C_γ; therefore, it does not correspond to the customary choice of the parity operator. Let us define the parity operator P_γ and the time-reversal operator T_γ by

$$P_\gamma=\mathcal{P}\prod_\beta S_\beta(\pi/2)\,, \tag{6.10}$$

and

$$C_\gamma P_\gamma T_\gamma=\mathcal{J}\,, \tag{6.11}$$

where $\mathcal{J}$ is given by (1.9), and the product $\prod_\beta$ extends over all half-integer spin fields. The operator P_γ satisfies

$$[P_\gamma,Q_\alpha]=0\,, \tag{6.12}$$

$$P_\gamma C_\gamma=(-1)^{2J}C_\gamma P_\gamma\,, \tag{6.13}$$

and

$$P_\gamma{}^2=1\,. \tag{6.14}$$

The electromagnetic interaction of the two charged leptons e and μ is well known to be invariant under C_γ, P_γ, and T_γ; it is also invariant under the group $\{S_e(\theta)\}\times\{S_\mu(\theta)\}$ where $S_e(\theta)$ and $S_\mu(\theta)$ are given by Eq. (6.3) with $\alpha=e$ and μ, respectively. The electromagnetic interaction of the hadrons is found to be invariant[25] under P_γ, $\mathcal{J}$ and the various U_1 groups generated by charge, hypercharge, and baryon number. It seems esthetically appealing to assume that it is also invariant under C_γ and T_γ. However, this is only a theoretical supposition and is, as yet, without experimental basis.[26]

[25] The best experiments establishing that H_{st} and H_γ are invariant under the same parity operator, $P_{st}=P_\gamma$, are from nuclear transitions: F. Boehm and E. Kankeleit, California Institute of Technology Report No. Calt-63-13 (unpublished); Yu. G. Abov, P. A. Krupchitsky, and Yu. A. Oratovsky, *Comptes Rendus du Congrès Internationale de Physique Nucleaire, Paris, 1964*; (Editions du Centre National de la Recherche Scientifique, Paris, 1965); L. Grodzins and F. Genovese, Phys. Rev. **121**, 228 (1961); R. E. Segel *et al.*, *ibid.* **123**, 1382 (1961); D. E. Alburger *et al.*, Phil. Mag. **6**, 171 (1961); R. Haas, L. B. Leipuner, and R. K. Adair, Phys. Rev. **116**, 1221 (1959); F. Boehm and U. Hauser, Nucl. Phys. **14**, 615 (1959); D. A. Bromley *et al.*, Phys. Rev. **114**, 758 (1959).

Evidence for C_{st} and T_{st} invariances of H_{st} comes from the $\bar{p}p$ annihilation measurements [C. Baltay *et al.*, Phys. Rev. Letters **15**, 591 (1965)], the pp scattering experiment [A. Abashian and E. M. Hafner, Phys. Rev. Letters **1**, 225 (1958); C. F. Hwang, T. R. Ophel, E. H. Thorndike, and R. Wilson, Phys. Rev. **119**, 352 (1960)], and the reciprocity relations [L. Rosen and J. E. Brolley, Jr., Phys. Rev. Letters **2**, 98 (1959); D. Bodansky *et al.*, *ibid.* **2**, 101 (1959)].

[26] If the electromagnetic interaction of the hadrons is not invariant under C_γ and T_γ, then through virtual emissions and absorptions of photons all strong-interaction processes would have small C_{st}- and T_{st}-violating amplitudes (proportional to α). However, the converse statement is not true. Such a small C_{st} and T_{st} noninvariant amplitude could appear if there is a mismatch between C_{st} and C_γ, even though the electromagnetic interaction is invariant under C_γ and T_γ. For a detailed discussion, see Ref. 9.

2. Strong Interactions

We now turn to the symmetry problem of the strong interaction. Let

$$H=H_{\text{free}}+H_{st}\,, \tag{6.15}$$

where H_{st} contains both the SU_3-invariant and the SU_3-breaking strong interactions, and H_{free} contains the free-particle Hamiltonian of all hadrons. The masses of different particles in the same isospin multiplet are assumed to be the same. It is well known that H commutes with the isospin $\mathbf{I}$, the charge Q and the baryon number N. The internal symmetry group, denoted by $\mathcal{G}_{st}$, must contain these generators $\mathbf{I}$, Q and N. [For simplicity, we do not include the leptons, nor the electromagnetic field, in H_{free}.] It is useful to introduce the hypercharge

$$Y=2(Q-I_3)\,. \tag{6.16}$$

All known hadrons satisfy[15]

$$(-1)^{2I}=(-1)^Y\,, \tag{6.17}$$

and

$$(-1)^{2J}=(-1)^{2N}\,. \tag{6.18}$$

Thus, Y and $\mathbf{I}$ are the generators of a U_2 group, N is the generator of a U_1 group; the element $\mathcal{J}^2=(-1)^{2J}$ belongs to this U_1 group. The internal symmetry group $\mathcal{G}_{st}$ must contain a subgroup $\mathcal{G}_{st}{}^0$ which is the direct product of these two groups:

$$\mathcal{G}_{st}{}^0=U_2\times U_1\,. \tag{6.19}$$

In addition, H is invariant under a discrete internal symmetry operator[27] G which satisfies the following algebraic relations:

$$G^2=(-1)^Y\,, \tag{6.20}$$

$$[G,\mathbf{I}]=0\,, \tag{6.21}$$

and

$$GY+YG=GN+NG=0\,. \tag{6.22}$$

Sometimes, it is convenient to use C_{st}, which is related to G by

$$C_{st}=G\exp(-i\pi I_2)\,. \tag{6.23}$$

Although it was always accepted that C_{st} anticommutes with Q, this relation has been questioned recently.[9]

We note that if

$$C_{st}Q+QC_{st}=0\,, \tag{6.24}$$

and if H does not have any further internal symmetry properties, then $\mathcal{G}_{st}$ is a minimal extension of $\mathcal{G}_{st}{}^0$:

$$\mathcal{G}_{st}=\{\mathcal{G}_{st}{}^0,\mathcal{G}_{st}{}^0C_{st}\}\,, \tag{6.25}$$

where the coset $\mathcal{G}_{st}{}^0C_{st}=\mathcal{G}_{st}{}^0G$. Since $C_{st}{}^2=1$, this minimal extension belongs to case B_1.

On the other hand, if

$$C_{st}Q+QC_{st}\neq 0\,, \tag{6.26}$$

[27] T. D. Lee and C. N. Yang, Nuovo Cimento **3**, 749 (1956); L. Michel, *ibid.* **10**, 319 (1953).

then the group $\mathcal{G}_{st}$ must be a different one. Let us define

$$Q_J=\tfrac{1}{2}(Q-C_{st}QC_{st}^{-1}), \quad (6.27)$$

and

$$Q_K=\tfrac{1}{2}(Q+C_{st}QC_{st}^{-1}). \quad (6.28)$$

Thus,

$$C_{st}Q_J+Q_JC_{st}=0, \quad (6.29)$$

and

$$C_{st}Q_K-Q_KC_{st}=0. \quad (6.30)$$

Since H commutes with Q and C_{st}, it must also commute with Q_J and Q_K. The internal symmetry group $\mathcal{G}_{st}$ now contains, in addition to the generators $\mathbf{I}$, N, and $Q=Q_J+Q_K$, also (Q_J-Q_K). The exact structure of $\mathcal{G}_{st}$ depends on the specific form of the model Hamiltonian. In Appendix C, a *nonderivative-coupling* model of such a strong interaction is given. For the model,

$$\mathcal{G}_{st}=\{\mathcal{G}_{st}^{0\prime},\mathcal{G}_{st}^{0\prime}C_{st}\}, \quad (6.31)$$

and

$$\mathcal{G}_{st}^{0\prime}=U_2\times U_1\times U_1, \quad (6.32)$$

where the generators of the U_2 group are $\mathbf{I}$ and $2(Q_J-I_3)$, the generators of the two U_1 groups are, respectively, N and Q_K.

There exists strong evidence[25] that $(H_{\text{free}}+H_{st})$ satisfies $\mathcal{P}$ invariance, and consequently also $\mathcal{T}$ invariance; furthermore, in the set $\{S\mathcal{P}\}$ there exists an element, called P_{st}, which is the same operator as P_γ defined by $(H_{\text{free}}+H_\gamma)$. We have

$$P_{st}=P_\gamma, \quad (6.33)$$

and it satisfies

$$[P_{st},\mathbf{I}]=[P_{st},Q]=[P_{st},N]=0, \quad (6.34)$$

$$P_{st}C_{st}=(-1)^{2J}C_{st}P_{st}, \quad (6.35)$$

and

$$P_{st}^2=1. \quad (6.36)$$

Let us define

$$\mathcal{P}=P_{st}\exp(\tfrac{1}{2}i\pi N). \quad (6.37)$$

By using (6.18) and (6.34)–(6.36), we find $\mathcal{P}$ commutes with all internal-symmetry operators, and

$$\mathcal{P}^2=(-1)^N. \quad (6.38)$$

Thus, the minimal extension

$$\mathcal{E}=\{\mathcal{G}_{st},\mathcal{G}_{st}P_{st}\} \quad (6.39)$$

belongs to case A_2.

It is customary to define T_{st} by

$$C_{st}P_{st}T_{st}=\mathcal{J}. \quad (6.40)$$

The strong interaction is invariant under C_{st}, P_{st}, and T_{st}. The weak interaction H_{wk} violates both C_{st} and P_{st} symmetries, and the electromagnetic interaction H_γ is invariant under P_{st}. The important questions, whether H_γ is invariant under C_{st} and T_{st}, and whether H_{wk} is invariant under T_{st} and $(C_{st}P_{st})$, are, at present, still unanswered.

VII. REMARKS ON C_i AND THE EXAMPLE OF SO_4 SYMMETRY

As discussed in the two preceding sections, our present knowledge is consistent with the assumption that the model Hamiltonians (5.1), (6.1), and (6.15) are separately invariant under the group

$$\{\mathcal{G}_i,\mathcal{G}_iP_i,\mathcal{G}_iT_i,\mathcal{G}_i\mathcal{J}\}, \quad (7.1)$$

where $i=$ weak, γ, and strong, respectively. The group $\mathcal{G}_i$ is the internal symmetry group, and it consists of a connected invariant subgroup $\mathcal{G}_i^0$ and a coset $\mathcal{G}_i^0C_i$:

$$\mathcal{G}_i=\{\mathcal{G}_i^0,\mathcal{G}_i^0C_i\}. \quad (7.2)$$

From the definitions of $\mathcal{P}$, $\mathcal{T}$, and $\mathcal{J}$, it is clear that in (7.1) no element in the cosets $\mathcal{G}_iP_i$, $\mathcal{G}_iT_i$, and $\mathcal{G}_i\mathcal{J}$ can be continuously connected to the unit element. The same is also true for the elements in the coset $\mathcal{G}_i^0C_i$. However, it is less transparent why it should not be possible to connect the element C_i, or the elements in the coset $\mathcal{G}_i^0C_i$, to the unit element in a continuous way. As we shall see, this is perhaps due to the particular forms of (5.1), (6.1) and (6.15), rather than due to any general underlying principle.

To illustrate this, we shall discuss some explicit strong interaction models. These models are all invariant under the usual C_{st}, P_{st} and T_{st}, but unlike in (7.2), the element C_{st} can be continuously connected to the unit element.

Let us consider the virtual transitions

$$N\rightleftarrows N+\rho, \quad (7.3)$$

and

$$N\rightleftarrows N+\eta^0 \quad (\text{or } \eta'^0), \quad (7.4)$$

where N stands for n or p and ρ for $\rho^\pm$ or ρ^0. Such transitions can be represented, at least phenomenologically, by vertices of the form

$$ig_\rho[N^\dagger(x)\gamma_4\gamma_\mu\boldsymbol{\tau}N(x)]\boldsymbol{\varrho}_\mu(x), \quad (7.5)$$

and

$$ig_\eta[N^\dagger(x)\gamma_4\gamma_5N(x)]\eta(x), \quad (7.6)$$

where g_ρ and g_η are the appropriate strong-interaction coupling constants which, because of Hermiticity, must be real, $\boldsymbol{\tau}$ are the usual (2×2) Pauli matrices for isospin [which are the same as those given by (4.17) and (5.37)],

$$N(x)=\begin{pmatrix}\psi_p(x)\\ \psi_n(x)\end{pmatrix} \quad (7.7)$$

is the field operator for n and p, $\boldsymbol{\varrho}_\mu(x)$ and $\eta(x)$ are, respectively, the field operators for ρ and η (or η'). The fields $\boldsymbol{\varrho}_\mu(x)$ and $\eta(x)$ are Hermitian, but $\psi_n(x)$ and $\psi_p(x)$ are not. It is clear that both interactions (7.5) and (7.6) commute with the isospin $\mathbf{I}$ and the baryon number N; they also satisfy the usual C_{st}, P_{st} and T_{st} invariances. As we shall see, these two interactions are also invariant under a much bigger symmetry group.

To study the full extent of the symmetry group, we follow the general procedure outlined in Sec. II and express ψ_p and ψ_n in terms of Hermitian fields:

$$\psi_p = 2^{-1/2}(\psi_{p1} + i\psi_{p2}), \tag{7.8}$$

$$\psi_n = 2^{-1/2}(\psi_{n1} + i\psi_{n2}), \tag{7.9}$$

where

$$\psi_{aj} = \psi_{aj}^\dagger, \tag{7.10}$$

$a=n$ or p and $j=1$ or 2. If S is an internal symmetry operator, then according to (2.4),

$$S\psi S^{-1} = \tilde{s}_{1/2}\psi, \tag{7.11}$$

where

$$\psi = \begin{pmatrix} \psi_{p1} \\ \psi_{p2} \\ \psi_{n1} \\ \psi_{n2} \end{pmatrix}, \tag{7.12}$$

and $s_{1/2}$ is a (4×4) real orthogonal matrix. Any such matrix with determinant $=+1$ can always be expressed in the form

$$\exp[i(\mathbf{I}\cdot\boldsymbol{\theta} + \mathbf{I}'\cdot\boldsymbol{\theta}')], \tag{7.13}$$

where $\boldsymbol{\theta}$ and $\boldsymbol{\theta}'$ are real parameters, the components of $\mathbf{I}$ and $\mathbf{I}'$ are related to those of $\boldsymbol{\rho}$ and $\boldsymbol{\sigma}$ matrices introduced in (5.25) by

$$2\mathbf{I} = (-\sigma_2\rho_1,\ \rho_2,\ -\sigma_2\rho_3), \tag{7.14}$$

and

$$2\mathbf{I}' = (\rho_2\sigma_1,\ \rho_2\sigma_3,\ -\sigma_2). \tag{7.15}$$

These operators are so chosen that $\mathbf{I}$ is the same as the isospin operator,

$$N = 2I_3', \tag{7.16}$$

$$G = \exp(-i\pi I_2'), \tag{7.17}$$

and

$$C_{\rm st} = \exp[-i\pi(I_2 + I_2')]. \tag{7.18}$$

where N is the baryon number, and $C_{\rm st}$ and G are the same operators as those used in (6.20)–(6.23).

It can be easily verified that $\mathbf{I}$ commutes with $\mathbf{I}'$, and they are the generators of two independent SU_2 groups. Their direct product $SU_2\times SU_2$ has an invariant subgroup Z_2 whose elements are 1 and $(-1)^{2I+2I'}$; its factor group is the SO_4 group which comprises all such (4×4) matrices given by (7.13). The irreducible representations of the SO_4 group can be labeled as (I,I') where $I(I+1)$ and $I'(I'+1)$ are, respectively, the eigenvalues of the operators $\mathbf{I}^2$ and $\mathbf{I}'^2$, and

$$(-1)^{2I+2I'} = 1. \tag{7.19}$$

The nucleon belongs to the representation $(\frac{1}{2},\frac{1}{2})$.

Theorem 4. This SO_4 symmetry is invariant under any virtual transition

$$N \rightleftarrows N + (G=+1 \text{ meson}), \tag{7.20}$$

but is violated by

$$N \rightleftarrows N + (G=-1 \text{ meson}). \tag{7.21}$$

Both (7.3) and (7.4) are examples of (7.20), and the usual pion-nucleon vertex is an example of (7.21).

Proof. Under the SO_4 group of transformations, any meson transforms like an appropriate nucleon-antinucleon system; therefore, it belongs to $I_3'=0$ and $I'=0$, or 1. From (7.17), it follows that the additional condition $G=+1$ implies $I'=0$. Hence, reaction (7.20) conserves $\mathbf{I}'$. Since $\mathbf{I}$ conservation is implicitly assumed, reaction (7.20) is invariant under the entire SO_4 group of transformations. Any $G=-1$ meson transforms like the $I_3'=0$ member of an $I'=1$ multiplet. Thus, reaction (7.21) violates $\mathbf{I}'$ conservation. Theorem 4 is, then, proved.

It can be shown that this SO_4 symmetry is also violated by the known interactions between the strange particles and the nucleon. The usefulness of this SO_4 symmetry depends on whether we can separate out in any physical process the contributions due to the $G=+1$ mesons (ρ, η, etc.) from the symmetry-breaking terms such as those due to the $G=-1$ mesons (π, ω, etc.). [For example, in the theoretical study of the vector and axial-vector form factors of the nucleons, the contributions due to $G=+1$ and $G=-1$ mesons can be conveniently separated.] In the following, such separations are assumed to be useful, and we will consider those Feynman graphs which consist of *only* SO_4-invariant vertices.

For practical applications, it is convenient to use, instead of (7.12),

$$\begin{pmatrix} p \\ n \\ \bar{n} \\ -\bar{p} \end{pmatrix}, \tag{7.22}$$

as the base vectors of a (4×4) representation of the SO_4 group. In (7.22), the field operators for $\bar{n}$ and $\bar{p}$ are $\psi_n^\dagger$ and $\psi_p^\dagger$, respectively. In this representation, the generators $\mathbf{I}$ and $\mathbf{I}'$ become simply $\frac{1}{2}\boldsymbol{\sigma}$ and $\frac{1}{2}\boldsymbol{\rho}$, respectively [instead of (7.14) and (7.15)].

First, let us consider reactions

$$N + \bar{N} \to \text{mesons}, \tag{7.23}$$

and

$$N + \text{mesons} \to N + \text{mesons}, \tag{7.24}$$

where "mesons" represents an arbitrary distribution of $G=+1$ mesons. In (7.23), only the $I'=0$ amplitudes contribute; in (7.24), on both sides only the nucleon carries a nonzero $I'=\frac{1}{2}$. Thus, all useful consequences of SO_4 invariance can be obtained by using only the usual $\mathbf{I}$ conservation and G symmetry.

However, for reactions such as

$$N + N \to N + N + \text{mesons} \tag{7.25}$$

and

$$N + \bar{N} \to N + \bar{N} + \text{mesons}, \tag{7.26}$$

the full SO_4 symmetry can yield additional relations between different amplitudes. The states NN and $N\bar{N}$

TABLE II. Eigenstates of I, I_3, I', and I_3' where I and I' are the generators of the SO_4 group discussed in Sec. VII. The order of the nucleons and antinucleons in the state, say, $\alpha\beta$ (where α and β can be p, n, $\bar{p}$ or $\bar{n}$) indicates that α has momentum $\mathbf{k}$ and helicity λ, and β has momentum $\mathbf{k}'$ and helicity λ'.

I	I_3	I'	I_3'	State
1	1	1	1	pp
1	1	1	0	$2^{-1/2}(p\bar{n}+\bar{n}p)$
1	1	1	−1	$\bar{n}\bar{n}$
1	0	1	1	$2^{-1/2}(pn+np)$
1	0	1	0	$\frac{1}{2}(n\bar{n}+\bar{n}n-p\bar{p}-\bar{p}p)$
1	0	1	−1	$-2^{-1/2}(\bar{p}\bar{n}+\bar{n}\bar{p})$
1	−1	1	1	nn
1	−1	1	0	$-2^{-1/2}(n\bar{p}+\bar{p}n)$
1	−1	1	−1	$\bar{p}\bar{p}$
0	0	1	1	$2^{-1/2}(pn-np)$
0	0	1	0	$\frac{1}{2}(-n\bar{n}+\bar{n}n-p\bar{p}+\bar{p}p)$
0	0	1	−1	$2^{-1/2}(\bar{p}\bar{n}-\bar{n}\bar{p})$
1	1	0	0	$2^{-1/2}(p\bar{n}-\bar{n}p)$
1	0	0	0	$\frac{1}{2}(n\bar{n}-\bar{n}n-p\bar{p}+\bar{p}p)$
1	−1	0	0	$2^{-1/2}(-n\bar{p}+\bar{p}n)$
0	0	0	0	$\frac{1}{2}(n\bar{n}+\bar{n}n+p\bar{p}+\bar{p}p)$

can be resolved into eigenstates of I, I_3, I' and I_3'; these eigenstates are listed in Table II. By using the explicit forms of these eigenstates, it is straightforward to obtain the various consequences of the SO_4 symmetry for reactions (7.25) and (7.26). For example,

$$A(p+p\to p+p+\eta^0)=\tfrac{1}{2}[A(p+\bar{n}\to p+\bar{n}+\eta^0)+A(p+\bar{n}\to \bar{n}+p+\eta^0)], \quad (7.27)$$

where A denotes the appropriate amplitude, and the state $(p+\bar{n}+\eta^0)$ is related to $(\bar{n}+p+\eta^0)$ by an exchange of the momenta and helicities of p and $\bar{n}$. Relations such as (7.27) cannot be derived by using only I and G.

Apart from such possible applications which is bound to be limited because of known violations of this SO_4 symmetry, the above discussions at least give a general class of strong interaction models in which the element C_{st} is connected, in a continuous way, to the unit element of the internal symmetry group.

ACKNOWLEDGMENTS

We wish to thank Professor G. Feinberg and Professor R. Serber for discussions.

APPENDIX A

In this Appendix, we will illustrate the use of Hermitian fields for a set of zero-mass spin-$\frac{1}{2}$ particles. For definiteness, let us take the simple model discussed in Sec. V. In this model, only the zero-mass limit of all known leptons is considered. Each lepton α is described by a non-Hermitian field ψ_α which satisfies Eq. (5.13); it can also be described by a Hermitian field ϕ_α defined by

$$(\phi_\alpha)_\lambda=(\psi_\alpha)_\lambda+(\psi_\alpha{}^\dagger)_\lambda, \quad (A1)$$

where $\alpha=e$, μ, ν_e or ν_μ, λ denotes the spinor index and it varies from 1 to 4. From (2.8), (5.13), and (A1), one finds

$$\psi_\alpha=\tfrac{1}{2}(1+\gamma_5)\phi_\alpha \quad (A2)$$

and that the components of $\psi_\alpha{}^\dagger$ are identical with those of $\frac{1}{2}(1-\gamma_5)\phi_\alpha$. The fields ψ_α and ϕ_α, therefore, represent two equivalent descriptions of the same two-component theory[28] for a spin-$\frac{1}{2}$ particle. By using (A1) and (A2), any transformation of ψ_α can be readily translated into a corresponding transformation of ϕ_α, and vice versa.

Consider, e.g., the gauge transformation

$$\psi_\alpha\to e^{i\xi}\psi_\alpha. \quad (A3)$$

The same transformation in terms of ϕ_α becomes

$$\phi_\alpha\to e^{i\gamma_5\xi}\phi_\alpha. \quad (A4)$$

As another example, we may consider the SU_2 transformation

$$\psi\to(\exp i\boldsymbol{\sigma}\cdot\boldsymbol{\theta})\psi \quad (A5)$$

where

$$\psi=\begin{bmatrix}\psi_e\\ \psi_\mu\\ \psi_{\nu_e}\\ \psi_{\nu_\mu}\end{bmatrix},$$

and $\boldsymbol{\sigma}$ is given by (5.25). In terms of ϕ_α, (A5) becomes

$$\phi\to e^{i(\gamma_5\sigma_1\theta_1+\sigma_2\theta_2+\gamma_5\sigma_3\theta_3)}\phi, \quad (A6)$$

where

$$\phi=\begin{bmatrix}\phi_e\\ \phi_\mu\\ \phi_{\nu_e}\\ \phi_{\nu_\mu}\end{bmatrix}.$$

Similarly, the transformation

$$\psi\to e^{i\rho_3\eta}\psi$$

becomes

$$\phi\to e^{i\gamma_5\rho_3\eta}\phi,$$

where ρ_3 is given by (5.25). Thus, Theorem 3 can be proved by following the same proof as that given in Sec. V, but, using the Hermitian fields ϕ_α.

APPENDIX B

Some explicit models will be given in this Appendix to illustrate the two unusual cases discussed in Sec. IV (case B_1 of example 1 and case A_1 of example 2).

[28] The use of a non-Hermitian field for the two-component spin-$\frac{1}{2}$ theory was first introduced by H. Weyl, Z. Physik **56**, 330 (1929); it was later applied to the leptons by T. D. Lee and C. N. Yang, Phys. Rev. **105**, 1671 (1957), L. Landau, Nucl. Phys. **3**, 127 (1957) and A. Salam, Nuovo Cimento **5**, 299 (1957). The use of a Hermitian field to describe a two-component spin-$\frac{1}{2}$ particle was made by E. Majorana, *ibid.* **14**, 171 (1937). See the discussion given by K. M. Case, Phys. Rev. **107**, 307 (1957). Both descriptions can be applied to the nonzero mass case as well as the zero mass case. That the Majorana theory is applicable to the nonzero mass case is obvious, since the Dirac equation is real in the Majorana representation (2.8). That the Weyl description is also applicable to the nonzero mass case follows from Eq. (A2). For a detailed discussion, see the review article by T. D. Lee and C. S. Wu (Ref. 22). If the mass is zero, then transformation (A3), or (A4), is an internal symmetry transformation; otherwise, it is not.

1. First, we consider the example $\mathcal{G}=U_1$. Let us introduce one spin-1 Hermitian field ϕ_μ and four spin-$\frac{1}{2}$ Hermitian fields ψ_1, ψ_2, ψ_3, and ψ_4. Under the internal symmetry operation (4.6), these fields transform as follows:

$$S(\theta)\phi_\mu S^{-1}(\theta)=\phi_\mu$$
$$S(\theta)\psi S^{-1}(\theta)=e^{i\sigma_2\theta}\psi \tag{B1}$$

where σ_2 is given by (5.25) and

$$\psi=\begin{bmatrix}\psi_1\\ \psi_2\\ \psi_3\\ \psi_4\end{bmatrix}. \tag{B2}$$

As an example of case B_1, we may assume that the total Hamiltonian is $H=H_{\text{free}}+H_{\text{int}}$ in which the interaction Hamiltonian is

$$H_{\text{int}}=\int j_\lambda\phi_\lambda d^3r. \tag{B3}$$

The current j_λ is given by

$$j_\lambda=il^\dagger\gamma_4(g_1+g_2\gamma_5)\gamma_\lambda l'+il'^\dagger\gamma_4(g_1{}^*+g_2{}^*\gamma_5)\gamma_\lambda l +ig_3(l^\dagger\gamma_4\gamma_\lambda l-l'^\dagger\gamma_4\gamma_\lambda l'), \tag{B4}$$

where $l(x)$, $l'(x)$ are non-Hermitian fields related to $\psi_1, \cdots, \psi_4$ by

$$l(x)=2^{-1/2}[\psi_1(x)+i\psi_2(x)], \tag{B5}$$

and

$$l'(x)=2^{-1/2}[\psi_3(x)+i\psi_4(x)]. \tag{B6}$$

The coupling constants g_1, g_2 are *not* relatively real, *nor* relatively imaginary, but the coupling constant g_3 is real. The interaction Hamiltonian H_{int} is invariant under the transformation (B1). In order that the total-Hamiltonian H is invariant under $S(\theta)$ we must have the bare mass (consequently, also the physical mass) of ψ_1 to be the same as that of ψ_2 and the mass of ψ_3 to be the same as that of ψ_4. Then, the internal symmetry group $\mathcal{G}$ is $U_1=\{S(\theta)\}$.

To satisfy $\mathcal{P}$ invariance, we assume all four fields $\psi_1, \cdots, \psi_4$ to be degenerate. It can be verified that H is invariant under any space-inversion operator in the coset $\mathcal{G}\mathcal{P}$ where

$$\mathcal{P}\phi_\mu(\mathbf{r},t)\mathcal{P}^{-1}=-\eta_{\mu\nu}\phi_\nu(-\mathbf{r},t), \tag{B7}$$

$$\mathcal{P}\psi(\mathbf{r},t)\mathcal{P}^{-1}=\tilde{u}_{1/2}{}^P(i\gamma_4)\psi(-\mathbf{r},t). \tag{B8}$$

The matrix $u_{1/2}{}^P$ is given by

$$u_{1/2}{}^P=\begin{bmatrix}0&0&0&1\\0&0&1&0\\0&-1&0&0\\-1&0&0&0\end{bmatrix},$$

and $\eta_{\mu\nu}=1$ if $\mu=\nu\neq4$, $\eta_{\mu\nu}=-1$ if $\mu=\nu=4$, and zero otherwise. In terms of $l(x)$ and $l'(x)$, (B8) becomes

$$\mathcal{P}l(\mathbf{r},t)\mathcal{P}^{-1}=\gamma_4\bar{l}'(-\mathbf{r},t),$$

and

$$\mathcal{P}l'(\mathbf{r},t)\mathcal{P}^{-1}=-\gamma_4\bar{l}(-\mathbf{r},t),$$

where

$$\bar{l}(x)=2^{-1/2}[\psi_1(x)-i\psi_2(x)],$$

and

$$\bar{l}'(x)=2^{-1/2}[\psi_3(x)-i\psi_4(x)].$$

The operator $\mathcal{P}$ satisfies Eqs. (4.9) and (4.10), which are the conditions for case B_1. From Theorem 1, it follows that the theory is invariant under any time-reversal operator in the coset $\mathcal{G}\mathcal{T}$. In this case, the unusual degeneracy between $l(x)$ and $l'(x)$ is due to $\mathcal{P}$ or $\mathcal{T}$ invariance.

2. For the second example, $\mathcal{G}=\{U_2,U_2C\}$, it is case A_1 that has an unusual degeneracy.[14] As an illustration, let us consider a model which consists of one spin 0, $Y=0$, $I=1$ triplet ϕ and two spin $\frac{1}{2}$, $Y=1$, $I=\frac{1}{2}$ doublets N and N' where

$$\boldsymbol{\phi}=(\phi_1,\phi_2,\phi_3), \tag{B9}$$

$$N=2^{-1/2}\begin{pmatrix}\psi_1+i\psi_2\\ \psi_3+i\psi_4\end{pmatrix}, \tag{B10}$$

and

$$N'=2^{-1/2}\begin{pmatrix}\psi_1'+i\psi_2'\\ \psi_3'+i\psi_4'\end{pmatrix}. \tag{B11}$$

The fields $\phi_1, \cdots, \psi_1, \cdots, \psi_1', \cdots, \psi_4'$ are all Hermitian fields. The different fields in the same isomultiplet are assumed to be degenerate. Let the interaction Hamiltonian be given by

$$H_{\text{int}}=\int \mathbf{J}\cdot\boldsymbol{\phi}d^3r, \tag{B12}$$

where

$$\mathbf{J}=N^\dagger\gamma_4(g_1+ig_2\gamma_5)\boldsymbol{\tau}N+N'^\dagger\gamma_4(g_1-ig_2\gamma_5)\boldsymbol{\tau}N' +ig_3(N^\dagger\gamma_4\gamma_5\boldsymbol{\tau}N'+N'^\dagger\gamma_4\gamma_5\boldsymbol{\tau}N),$$

the coupling constants g_1, g_2, g_3 are all real, and $\boldsymbol{\tau}=(\tau_1,\tau_2,\tau_3)$ are the usual three (2×2) Pauli matrices. It can be verified that the internal symmetry group is $\mathcal{G}=\{U_2,U_2C\}$ where $U_2=\{\exp(i\mathbf{I}\cdot\boldsymbol{\theta}+iY\alpha)\}$, and C satisfies

$$CN(x)C^{-1}=\bar{N}(x)=2^{-1/2}\begin{pmatrix}\psi_1-i\psi_2\\ \psi_3-i\psi_4\end{pmatrix},$$

$$CN'(x)C^{-1}=\bar{N}'(x)=2^{-1/2}\begin{pmatrix}\psi_1'-i\psi_2'\\ \psi_3'-i\psi_4'\end{pmatrix},$$

and

$$C\boldsymbol{\phi}(x)C^{-1}=\boldsymbol{\phi}(x).$$

To satisfy $\mathcal{P}$ invariance, we assume that the two isospin doublets $N(x)$ and $N'(x)$ are degenerate. The theory is invariant under any space-inversion operator in the coset $\mathcal{G}\mathcal{P}$ where

$$\mathcal{P}\phi(\mathbf{r},t)\mathcal{P}^{-1}=\phi(-\mathbf{r},t),$$
$$\mathcal{P}N(\mathbf{r},t)\mathcal{P}^{-1}=i\gamma_4 N'(-\mathbf{r},t),$$

and

$$\mathcal{P}N'(\mathbf{r},t)\mathcal{P}^{-1}=-i\gamma_4 N(-\mathbf{r},t).$$

The operator $\mathcal{P}$ commutes with $\mathbf{I}$, Y, and C, and it satisfies $\mathcal{P}^2=1$, which is the condition for case A_1. By using Theorem 1, one finds that the theory is also invariant under any time-reversal operator in the coset $\mathcal{G}\mathcal{T}$.

Both of these examples are quite artificial; they are constructed to illustrate the physical contents of these unusual cases. At present, no such degeneracy which is the result of $\mathcal{P}$, or $\mathcal{T}$, invariance has been observed.

APPENDIX C

In this Appendix, we will give an explicit example[29] of a strong interaction with no derivative coupling, and the internal symmetry group is (6.31). Let us assume, in addition to the known hadrons, the existence of three hypothetical charged particles, called $a_1^\pm$, $a_2^\pm$, and $a_3^\pm$. The masses of these three particles are all *different.* There exist strong interactions between the $a_i^\pm$ particles and an $I=0$ meson, say the known η_0' (which is also an SU_3 singlet); these interactions are described by the Hamiltonian density $H_{\rm st}{}^a$:

$$H_{\rm st}{}^a=i\phi(g_1\psi_2{}^\dagger\gamma_4\gamma_5\psi_3+g_2\psi_3{}^\dagger\gamma_4\gamma_5\psi_1+g_3\psi_1{}^\dagger\gamma_4\gamma_5\psi_2)+\text{H.c.},\quad\text{(C1)}$$

where ϕ and ψ_i are, respectively, the fields of η_0' and a_i ($i=1,2,3$). The operator ϕ is Hermitian, but ψ_i is not. The three coupling constants g_1, g_2, g_3 can be arbitrary, provided that their product is *not* real; i.e.,

$$(g_1g_2g_3)^*\neq g_1g_2g_3.\quad\text{(C2)}$$

Let $H_{\rm st}{}^0$ be the usual strong interactions of all known hadrons; $H_{\rm st}{}^0$ is independent of ψ_i, and it contains, among other terms, the interaction between the nucleons and η_0'. We may write

$$H_{\rm st}{}^0=if\psi_N{}^\dagger\gamma_4\gamma_5\psi_N\phi+\cdots,\quad\text{(C3)}$$

where the nucleon field operator is

$$\psi_N=\begin{pmatrix}\psi_p\\ \psi_n\end{pmatrix},$$

and f is the appropriate coupling constant.

[29] For an explicit example with derivative couplings, see Ref. 9.

All known hadrons have $Q_K=0$; they also have their usual quantum numbers I, I_3, N and $Q=Q_J$. For the $a_i^\pm$, we assign $I=N=Q_J=0$ and $Q_K=\pm1$.

It can be verified that (C1) and (C3) conserve[30] $\mathbf{I}$, N, Q_J, and Q_K. The $H_{\rm st}{}^0$ is known to satisfy the $C_{\rm st}$ symmetry where $C_{\rm st}$ is given by Eq. (6.23). Under $C_{\rm st}$, all known hadrons transform in the usual way; e.g.,

$$C_{\rm st}\psi_N(x)C_{\rm st}{}^{-1}=\psi_N{}^\dagger(x),$$
$$C_{\rm st}\phi(x)C_{\rm st}{}^{-1}=\phi(x),$$

etc. The interaction $H_{\rm st}{}^a$ is also invariant under $C_{\rm st}$, provided $\psi_i(x)$ is unchanged under $C_{\rm st}$; i.e.,

$$C_{\rm st}\psi_i(x)C_{\rm st}{}^{-1}=\psi_i(x),\quad\text{(C4)}$$

where $i=1$, 2 or 3. Thus, the internal symmetry group $\mathcal{G}$ is (6.31). Both $H_{\rm st}{}^0$ and $H_{\rm st}{}^a$ satisfy $\mathcal{P}$ invariance, and therefore also $\mathcal{T}$ invariance. Furthermore, there is an operator $P_{\rm st}$ in the coset $\mathcal{G}\mathcal{P}$, which satisfies

$$P_{\rm st}\psi_i(\mathbf{r},t)P_{\rm st}{}^{-1}=\gamma_4\psi_i(-\mathbf{r},t),$$
$$P_{\rm st}\phi(\mathbf{r},t)P_{\rm st}{}^{-1}=-\phi(-\mathbf{r},t),$$

and

$$P_{\rm st}\psi_N(\mathbf{r},t)P_{\rm st}{}^{-1}=\gamma_4\psi_N(-\mathbf{r},t).\quad\text{(C5)}$$

The strong interaction $(H_{\rm st}{}^0+H_{\rm st}{}^a)$ is invariant under $C_{\rm st}$, $P_{\rm st}$ and $T_{\rm st}$ where $T_{\rm st}$ is given by Eq. (6.40).

The electromagnetic current of the $a_i^\pm$ is

$$J_\lambda{}^a=ie\sum_{i=1}^{3}\psi_i{}^\dagger\gamma_4\gamma_\lambda\psi_i,\quad\text{(C6)}$$

and the electromagnetic current $J_\lambda{}^0$ of all known hadrons is given by the usual expression

$$J_\lambda{}^0=ie\psi_p{}^\dagger\gamma_4\gamma_\lambda\psi_p+\cdots.\quad\text{(C7)}$$

It can be shown that the electromagnetic interaction $(J_\lambda{}^0+J_\lambda{}^a)A_\lambda$ is invariant[31] under C_γ, P_γ, and T_γ, where

$$C_\gamma\psi_N(x)C_\gamma{}^{-1}=\psi_N{}^\dagger(x),$$
$$C_\gamma\psi_i(x)C_\gamma{}^{-1}=\psi_i{}^\dagger(x),\quad\text{(C8)}$$

($i=1,2,3$), $P_\gamma=P_{\rm st}$, and

$$C_\gamma P_\gamma T_\gamma=C_{\rm st}P_{\rm st}T_{\rm st}=\mathcal{I}.$$

The charge-conjugation operator C_γ anticommutes with both Q_J and Q_K, but $C_{\rm st}$ anticommutes only with Q_J. Because of condition (C2), the strong interaction is not invariant under C_γ and T_γ. By using (C4), one finds that the electromagnetic interaction is not invariant under $C_{\rm st}$ and $T_{\rm st}$.

[30] We assume that the remaining part $\cdots$ in (C3) is invariant under $P_{\rm st}$, $T_{\rm st}$ and the group of transformations (6.31); we also assume that the remaining part $\cdots$ in (C7) is invariant under C_γ, P_γ, and T_γ.